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On certain estimations of coefficients of univalent analytic functions

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In this paper I occupy myself with the estimation of the coefficients of univalent analytic functions assuming certain hypotheses concerning the preceding coefficients. The first part of the paper will be devoted to the functions whose Laurent expansion is $z+\sum_{k=1}^{\infty} b_k z^{-k}$ in $|z|>1$. Making certain assumptions concerning the coefficients preceding b_n, I will prove that $|b_n| \leqslant 2/(n+1)$. The second part of the paper will be devoted to the functions whose Laurent expansion is $z+\sum_{k=2}^{\infty} a_k z^k$ in $|z|<1$. As before, making certain assumptions concerning the coefficients preceding a_n, I will prove that $|a_n| \leqslant 2/(n-1)$. The first theorems of this kind were obtained by W. Wolibner [1]. The theorems given in this paper are the generalizations of his results.

I. Theorem 1. *Let $f(z)$ be an analytic univalent function in $|z|>1$ whose Laurent expansion is*

$$f(z) = z + \sum_{k=1}^{S} b_{n_k} z^{-n_k} + \sum_{k=n_S+1}^{\infty} b_k z^{-k}. \tag{1}$$

Let $n_S = n$. If

$$f(z) \neq 0 \tag{2}$$

or

$$n \text{ is odd} \tag{3}$$

and if for every system $\alpha_1, \alpha_2, \ldots, \alpha_{S-1}$, where $\alpha_1, \alpha_2, \ldots, \alpha_{S-1}$ are non-negative integers, we have

$$n+1 \neq \sum_{k=1}^{S-1} \alpha_k (n_k+1), \tag{4}$$

then

$$|b_n| \leqslant 2/(n+1).$$

Proof. Because (2) or (3) holds, the function $(f(z))^{(n+1)/2}$ is regular in $|z|>1$. The Laurent expansion for $(f(z))^a$ is

$$(f(z))^a = z^a + \sum_{k=1}^{\infty} D_{M_k}^{(a)} z^{a-M_k} + \sum_{k=n+2}^{\infty} c_k^{(a)} z^{a-k}. \tag{5}$$

The exponents M_k have the form

$$M_k = \sum_{i=1}^{S} a_i(n_i+1), \tag{6}$$

where a_i are non-negative integers and

$$\sum_{i=1}^{S} a_i \neq 0. \tag{7}$$

The coefficients $D_{M_k}^{(a)}$ in (5) can be expressed by the formula

$$D_{M_k}^{(a)} = \sum \frac{a(a-1)\ldots\big(a-\sum_{k=1}^{S} a_k+1\big)}{a_1!\,a_2!\,\ldots\,a_S!} b_{n_1}^{a_1} b_{n_2}^{a_2} \ldots b_{n_S}^{a_S}, \tag{8}$$

where the summation is extended to all systems of integers $a_1, a_2, \ldots, a_S$ for which

$$\sum_{i=1}^{S} a_i(n_i+1) = M_k. \tag{9}$$

Hence

$$(f(z))^{(n+1)/2} = z^{(n+1)/2} + \sum_{k=1}^{\infty} D_{M_k}^{((n+1)/2)} z^{(n+1)/2-M_k} + \sum_{k=n+2}^{\infty} c_k^{((n+1)/2)} z^{(n+1)/2-k}. \tag{10}$$

The greatest exponent in the last sum of (10) is $(n+1)/2-n+2 = -(n+3)/2$. The greatest positive exponent, except $(n+1)/2$, in (10), has the form

$$R_1 = \frac{n+1}{2} - \sum_{i=1}^{S-1} a_i^{(1)}(n_i+1), \tag{11}$$

where

$$\sum_{i=1}^{S-1} a_i^{(1)}(n_i+1) \neq 0. \tag{12}$$

Let K_1 denote the coefficient at z^{R_1}. Now let us consider the Laurent expansion of $(f(z))^{R_1}$, which has the form of (5). In the first sum of these expansion the exponents have the following form (because (9) and (11)):

$$\frac{n+1}{2} - \sum_{i=1}^{S-1} a_i^{(1)}(n_i+1) - \sum_{i=1}^{S} a_i(n_i+1). \tag{13}$$

They are of the same form as the coefficients of the first sum of (10). The greater exponent in the last sum in the expansion of $(f(z))^{R_1}$ is less than or equal to $-(n+3)/2$.

Let us consider the function

$$W_1\big(\sqrt{f(z)}\big) = (f(z))^{(n+1)/2} - K_1(f(z))^{R_1}.$$

In the Laurent expansion of this function the greatest exponent, except $(n+1)/2$ is less than the corresponding exponent in (10).

Applying this operation, which we will denote by $\mathcal{R}$, to the function W_1 and to the next functions obtained in this way, we shall get, after a finite number of steps (at most $q = [(n+1)/2]$), the function

$$W_q\big(\sqrt{f(z)}\big) = z^{(n+1)/2} + \sum_{k=1}^{\infty} D_{M_k}^{*} z^{(n+1)/2-M_k} + \sum_{k=n+2}^{\infty} c_k^{*} z^{(n+1)/2-k} \tag{14}$$

for which the only positive exponent in its Laurent expansion is $(n+1)/2$.

In the third summand of (14), the greatest exponent is $-(n+3)/2$, as in (10). Hence, the coefficient at $z^{-(n+1)/2}$ is D_{n+1}^{*}.

We shall show that

$$D_{n+1}^{*} = D_{n+1}^{((n+1)/2)}. \tag{15}$$

It is sufficient to prove that the coefficient at $z^{-(n+1)/2}$ remains unchanged by operation $\mathcal{R}$. Suppose, on the contrary, that there exists an exponent of the expansion of $(f(z))^{R_1}$ which is equal to $-(n+1)/2$. Because of (12), we have in (13) $a_S = 0$. Hence, by (13), $n+1 = \sum_{i=1}^{S-1} a_i(n_i+1)$, contrary to (4).

By (4) and (6), the system of $a_1, a_2, \ldots, a_S$ corresponding to $M_k = n+1$ is $a_k = 0$, for $k < S$, and $a_S = 1$. Hence, by (8), we have

$$D_{n+1}^{((n+1)/2)} = \frac{n+1}{2} b_n. \tag{16}$$

Function (14) is a polynomial of the square root of a univalent function. Applying the generalized Bieberbach area theorem (G. M. Goluzin [2]; M. Biernacki [3], p. 5) to function (14) we obtain, by (16),

$$-(n+1)+(n+1)\left|\frac{n+1}{2}b_n\right|^2 \leqslant 0,$$

that is

$$|b_n| \leqslant 2/(n+1).$$

REMARK 1. *The hypotheses of theorem 1 hold if $n_k = pk-1$, $k = 1, 2, \ldots, S-1$, where p is an integer and if p is not a divisor of $n+1$.*

REMARK 2. *The hypotheses of theorem 1 also hold if 1° $n_k = pk-1$, where p is an integer and $n_k \leqslant (n-1)/2$, where $k = 1, 2, \ldots, S-1$, 2° p is not a divisor of $n-n_k$ for $n_k > (n-1)/2$, 3° p is not a divisor of $n+1$.*

II. Now let us consider the univalent analytic functions $g(z)$ which are regular in the unit circle $|z| < 1$ and have the following Laurent expansion:

$$g(z) = z+\sum_{i=1}^{L} a_{n_i} z^{n_i} + \sum_{i=n_L+1}^{\infty} a_i z^i. \tag{17}$$

Let us denote n_L by n. Suppose that $a_n \neq 0$.

THEOREM 2. *If for every system $\alpha_1, \alpha_2, \ldots, \alpha_{L-1}$, where α_i, $i = 1, 2, \ldots, L-1$, are non-negative integers,*

$$n-1 \neq \sum_{i=1}^{L-1} \alpha_i(n_i-1) \tag{18}$$

then

$$|a_n| \leqslant 2/(n-1).$$

Proof. The function $g(z)$ may be written in the form

$$g(z) = \frac{1}{f(1/z)} = \frac{z}{1+\sum_{k=1}^{\infty} b_{m_k} z^{m_k+1}}, \tag{19}$$

where $f(z)$ is a regular univalent function and $f(z) \neq 0$ for $|z| > 1$. The exponents of the Taylor expansion of $g(z)/z$ have the form $\sum \alpha_k(m_k+1)$, where α_k are non-negative integers. This follows from (19). Hence, for every $i = 1, 2, \ldots, n$, there exists a system of non-negative integers $\alpha_1, \alpha_2, \ldots$, such that

$$n_i - 1 = \sum \alpha_k(m_k+1). \tag{20}$$

By (18) and (19), we have for every system of non-negative integers $\alpha_1, \alpha_2, \ldots$

$$n-1 \neq \sum \alpha_k(m_k+1), \quad \text{where} \quad m_k+1 < n-1. \tag{21}$$

Because $a_n \neq 0$, the coefficient at z^{n-1} in the Laurent of $zf(1/z)$ cannot be equal to 0. We define the integer m_S by the formula

$$m_S+1 = n-1. \tag{22}$$

Because $f(z) \neq 0$ for $|z| > 1$, and by (20), the theorem 1 holds for m_S and $f(z)$. Hence

$$|b_{m_S}| \leqslant 2/(m_S+1). \tag{23}$$

By (20) and (22), we have

$$n-1 = \sum_{k=1}^{S} \alpha_k(m_k+1) = \sum_{k=1}^{S-1} \alpha_k(m_k+1)+\alpha_S(m_S+1).$$

From (18) it follows that $\alpha_S \neq 0$, and from (22) that $\alpha_S = 1$ and $\alpha_k = 0$ if $k < S$. Hence, from (19) it follows that $a_n = b_{m_S}$. Then formula (23) may be written as

$$|a_n| \leqslant 2/(n-1).$$

REMARK 3. *The hypotheses of theorem 2 hold if $n_i = pi+r$, $i = 1, 2, \ldots, L$, where p is an integer, r is a non-negative integer and*

$$r > -p+1, \quad 1 \leqslant L < (p+r-1)/d \tag{24}$$

where d is the greatest common divisor of p and $r-1$.

This is analogous to the theorem of W. Wolibner ([1], p. 126).

Proof. We shall show that (18) holds. Suppose, on the contrary, that $n-1 = \sum_{i=1}^{L-1} \alpha_i(n_i-1)$, that is

$$Lp+r-1 = Ap+a(r-1), \tag{25}$$

where

$$A = \sum_{i=1}^{L-1} i\alpha_i \quad \text{and} \quad a = \sum_{i=1}^{L-1} \alpha_i. \tag{26}$$

Formula (25) may be written as

$$(L-A)p = (a-1)(r-1), \tag{27}$$

whence p/d is a divisor of $a-1$. For $a > 1$ it is not possible; in this case we obtain $a \geqslant 1+p/d$ and $A > a$ and finally, by (27),

$$L > (p+r-1)/d$$

contrary to (24). If $a = 1$, then by (27) we obtain $L = A$, contrary to (26).

Remark 4. *The hypotheses of theorem 2 hold if $n_i = pi+1$, $i = 1, 2, \ldots, L-1$, where p is an integer and p is not a divisor of $n-1$.*

The function which possesses this property has, up to $(n-1)$-th term, the same Taylor expansion as a p-symetric function.

Remark 5. *The hypotheses of theorem 2 hold if 1° $n_i = ip+1$, where p is an integer and $n_i \leqslant (n-1)/2$, 2° p is not a divisor of $n-n_i$ for $n_i > (n-1)/2$ and 3° p is not a divisor of $n-1$.*

III. The above estimations cannot be improved.

The hypotheses of theorem 1 are satisfied by the coefficient b_n of the function $f(z) = z(1+z^{-n-1})^{2/(n+1)}$, because $b_k = 0$ if $k < n$; but $b_n = 2/(n+1)$.

The hypotheses of theorem 2 are satisfied by the coefficient a_n of the function $g(z) = z(1+z^{n-1})^{2/(n-1)}$, because $a_k = 0$ if $k < n$; but $a_n = 2/(n-1)$.

References

[1] W. Wolibner, *Sur les coefficients des fonctions analytiques univalentes à l'extérieur d'un cercle*, Studia Mathematica 11 (1949), p. 125-132.

[2] G. M. Goluzin, *Über p-valente Funktionen*, Récueil Mathématique 8 (1940), p. 277-284.

[3] M. Biernacki, *Sur les fonctions en moyenne multivalentes*, Bulletin des Sciences Mathématiques 70 (1946), p. 51-76.

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Propriétés des intégrales d'une équation de l'hydrodynamique d'un fluide visqueux

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1. Introduction. Les équations du mouvement d'un fluide visqueux incompressible ont la forme

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{W}\times\boldsymbol{v} = -\boldsymbol{F} - \nu \operatorname{rot}\boldsymbol{W}, \tag{1}$$

où $\boldsymbol{W} = \operatorname{rot}\boldsymbol{v}$, $\boldsymbol{v}$ désigne la vitesse du fluide, ν est le coefficient de viscosité cinématique, $\boldsymbol{F}$ le vecteur des forces extérieures.

D'après la transformation

$$\operatorname{rot}\frac{\partial \boldsymbol{v}}{\partial t} + \operatorname{rot}[\boldsymbol{W}\times\boldsymbol{v}] = -\operatorname{rot}\boldsymbol{F} - \nu\operatorname{rot}(\operatorname{rot}\boldsymbol{W})$$

et d'après l'équation de continuité $\operatorname{div}\boldsymbol{v} = 0$, nous obtenons les équations (1) sous la forme

$$\frac{d\boldsymbol{W}}{dt} = (\boldsymbol{W}\cdot\nabla)\boldsymbol{v} + \nu\Delta\boldsymbol{W} + \operatorname{rot}\boldsymbol{F}, \tag{2}$$

où ∇ désigne l'opérateur de Hamilton.

Les équations (2) deviennent plus simples pour le mouvement plan, puisque dans ce cas $v_z = 0$, v_x et v_y ne dépendent pas de la coordonnée z, ce qui permet d'introduire, grâce à la supposition $\operatorname{div}\boldsymbol{v} = 0$, la fonction du courant $\psi(x, y, t)$ définie par les égalités

$$v_x = \frac{\partial\psi}{\partial y}, \quad v_y = -\frac{\partial\psi}{\partial x} \tag{3}$$

et de remplacer les équations (2) par une seule équation

$$\nu\Delta(\Delta\psi) - \frac{\partial(\Delta\psi)}{\partial t} = \frac{\partial\psi}{\partial y}\cdot\frac{\partial(\Delta\psi)}{\partial x} - \frac{\partial\psi}{\partial x}\cdot\frac{\partial(\Delta\psi)}{\partial y} + \Phi, \tag{4}$$

où la fonction $\Phi(x, y, t) = \operatorname{rot}_z\boldsymbol{F}$ est connue.