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ANNALES
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On the uniqueness of the non-negative solution of the homogeneous mixed problem for a system of partial differential equations

by A. Pliś (Kraków)

In this paper we shall deal with the following system of partial differential equations of the second order:

$$(1)\qquad \frac{\partial u_i}{\partial x}=\sum_{j=1}^{m}\sum_{k,q=1}^{n} r_{ijkq}(Z)\frac{\partial^2 u_j}{\partial y_k\partial y_q}+\sum_{j=1}^{m}\sum_{k=1}^{n} a_{ijk}(Z)\frac{\partial u_j}{\partial y_k}+\sum_{j=1}^{m} b_{ij}(Z)u_j$$

$$(i=1,\ldots,m;\ Z=(x,\ Y)=(x,y_1,y_2,\ldots,y_n)),$$

with vanishing initial and boundary data. It will be proved that $u_i\equiv 0$ $(i=1,\ldots,m)$ is the unique non-negative solution of that problem. The Cauchy problem for the partial differential equations of the first order $(r_{ijkq}\equiv 0)$ was treated also in [1]. For $m=1$, $r_{11kq}=\delta_k^q$, $a_{11k}\equiv 0$, $b_{11}\equiv 0$ system (1) reduces to the heat equation. In that particular case the restriction to the non-negative solutions is unnecessary. For the general case, however, it is essential as may be shown by a modification of the example given in [2].

Theorem T. *Let us assume that the coefficients r_{ijkq}, a_{ijk}, b_{ij} together with the derivatives $\partial r_{ijkq}/\partial y_k$, $\partial^2 r_{ijkq}/\partial y_k\partial y_q$, $\partial a_{ijk}/\partial y_k$ $(i,j=1,\ldots,m;$ $k,q=1,\ldots,n)$ are continuous*[1] *with respect to Y in the prism $R\,\{0<x<1,\ |y_k|\leqslant 1\ (k=1,\ldots,n)\}$ and the following inequalities are satisfied:*

$$(2)\qquad r_{ijkq}\geqslant 0\qquad (i,j=1,\ldots,m;\ k=1,\ldots,n)$$

on the lateral boundary S of R ($S=R-$interior of R), and for certain constant K $(K>0)$

$$(3)\qquad \frac{\partial^2 r_{ijkq}}{\partial y_k\partial y_q}\leqslant K,\qquad b_{ij}\leqslant K,\qquad \frac{\partial a_{ijk}}{\partial y_k}\geqslant -K$$

for $Z\in R$, $i,j=1,\ldots,m$; $k,q=1,\ldots,n$.

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(1) This condition has been introduced for conciseness; it may be replaced by a weaker one.

Under the above assumptions every solution $u_1(Z), \ldots, u_m(Z)$ of system (1) *of class C^2 in R, continuous in $\bar{R}$ satisfying the inequalities*

$$(4)\qquad u_i(Z) \geqslant 0 \quad (i = 1, \ldots, m) \quad \textit{for} \quad Z \in R$$

and the mixed conditions

$$(5)\qquad u_i(0, Y) = 0 \quad \textit{for} \quad |y_k| \leqslant 1, \quad k = 1, \ldots, n;\ i = 1, \ldots, m,$$

$$(6)\qquad u_i(x, Y) = 0 \quad \textit{for} \quad \max_{k=1,\ldots,n} |y_k| = 1, \quad i = 1, \ldots, m$$

vanishes identically in R.

Proof. Let us introduce a section C_ξ of prism R by a plane $x = \xi$ and sets P_ξ^j, Q_ξ^j bounding C_ξ, given by the relations

$$P_\xi^j \{x = \xi,\ |y_k| \leqslant 1\ (k = 1, \ldots, j-1, j+1, \ldots, n),\ y_j = -1\},$$

$$Q_\xi^j \{x = \xi,\ |y_k| \leqslant 1\ (k = 1, \ldots, j-1, j+1, \ldots, n),\ y_j = 1\}.$$

Now let us define auxiliary linear operations, associating with the functions of the variables $x, y_1, \ldots, y_n$ functions of the variable x, given by the formulas

$$H(f) = \overbrace{\int \ldots \int}^{n}_{C_x} f(x, y_1, \ldots, y_n)\, dy_1 \ldots dy_n,$$

$$(7)\qquad V_j(f) = \overbrace{\int \ldots \int}^{n-1}_{P_x^j} f(x, y_1, \ldots, y_n)\, dy_1 \ldots dy_{j-1} dy_{j+1} \ldots dy_n,$$

$$W_j(f) = \overbrace{\int \ldots \int}^{n-1}_{Q_x^j} f(x, y_1, \ldots, y_n)\, dy_1 \ldots dy_{j-1} dy_{j+1} \ldots dy_n\ (^2).$$

Let $u_1, \ldots, u_m$ be an arbitrary solution of (1) of class C^2 in R, continuous in $\bar{R}$ and satisfying conditions (4), (5), (6).

Consider the function

$$(8)\qquad g(x) = \sum_{i=1}^{m} H(u_i).$$

By (7), (1) we have

$$(9)\qquad \frac{dg(x)}{dx} = \sum_{i,j=1}^{m} \sum_{k,q=1}^{n} H\left(r_{ijkq} \frac{\partial^2 u_j}{\partial y_k \partial y_q}\right) + \sum_{i,j=1}^{m} \sum_{k=1}^{n} H\left(a_{ijk} \frac{\partial u_j}{\partial y_k}\right) + \sum_{i,j=1}^{m} H(b_{ij} u_j).$$

(²) For $n = 1$ we put $V_j(f) = f(x, -1)$, $W_j(f) = f(x, 1)$.

From the theorem on integration by parts we have

$$(10)\qquad H\left(r_{ijkq} \frac{\partial^2 u_j}{\partial y_k \partial y_q}\right) = W_k\left(r_{ijkq} \frac{\partial u_j}{\partial y_k}\right) - V_q\left(r_{ijkq} \frac{\partial u_j}{\partial y_k}\right) - H\left(\frac{\partial r_{ijkq}}{\partial y_q} \cdot \frac{\partial u_j}{\partial y_k}\right),$$

$$(11)\qquad H\left(\frac{\partial r_{ijkq}}{\partial y_q} \cdot \frac{\partial u_j}{\partial y_k}\right) = W_k\left(\frac{\partial r_{ijkq}}{\partial y_q} u_j\right) - V_k\left(\frac{\partial r_{ijkq}}{\partial y_q} u_j\right) - H\left(\frac{\partial^2 r_{ijkq}}{\partial y_k \partial y_q} u_j\right),$$

$$(12)\qquad H\left(a_{ijk} \frac{\partial u_j}{\partial y_k}\right) = W_k(a_{ijk} u_j) - V_k(a_{ijk} u_j) - H\left(\frac{\partial a_{ijk}}{\partial y_k} u_j\right).$$

By (6) we have

$$(13)\qquad u_j = 0 \quad \text{on} \quad P_x^k, Q_x^k \quad (j = 1, \ldots, m;\ k = 1, \ldots, n;\ 0 < x < 1),$$

$$(14)\qquad \frac{\partial u_j}{\partial y_q} = 0 \quad \text{on} \quad P_x^k, Q_x^k$$

$$(j = 1, \ldots, m;\ k = 1, \ldots, n;\ q = 1, \ldots, k-1, k+1, \ldots, n;\ 0 < x < 1).$$

From (6), (4) we obtain the inequalities

$$(15)\qquad \frac{\partial u_j}{\partial y_k} \geqslant 0 \quad \text{on} \quad P_x^k; \qquad \frac{\partial u_j}{\partial y_k} \leqslant 0 \quad \text{on} \quad Q_x^k$$

$$(j = 1, \ldots, m;\ k = 1, \ldots, n;\ 0 < x < 1).$$

From (10), (11), (13) it follows that

$$H\left(r_{ijkq} \frac{\partial^2 u_j}{\partial y_k \partial y_q}\right) = H\left(\frac{\partial^2 r_{ijkq}}{\partial y_k \partial y_q} u_j\right) + W_q\left(r_{ijkq} \frac{\partial u_j}{\partial y_k}\right) - V_q\left(r_{ijkq} \frac{\partial u_j}{\partial y_k}\right).$$

Hence by (14), (15), (2)

$$(16)\qquad H\left(r_{ijkq} \frac{\partial^2 u_j}{\partial y_k \partial y_q}\right) \leqslant H\left(\frac{\partial^2 r_{ijkq}}{\partial y_k \partial y_q} u_j\right).$$

Likewise by (13), (12)

$$(17)\qquad H\left(a_{ijk} \frac{\partial u_j}{\partial y_k}\right) = -H\left(\frac{\partial a_{ijk}}{\partial y_k} u_j\right).$$

From (9), (16), (17) we have

$$\frac{dg(x)}{dx} \leqslant \sum_{i,j=1}^{m} \sum_{k,q=1}^{n} H\left(\frac{\partial^2 r_{ijkq}}{\partial y_k \partial y_q} u_j\right) - \sum_{i,j=1}^{m} \sum_{k=1}^{n} H\left(\frac{\partial a_{ijk}}{\partial y_k} u_j\right) + \sum_{i,j=1}^{m} H(b_{ij} u_j)$$

and by (3), (8)

$$(18)\qquad \frac{dg(x)}{dx} \leqslant K(mn^2 + mn + m)\, g(x).$$

From (5), (8), (7) we obtain

$$g(0) = 0. \tag{19}$$

By the theorem on the ordinary differential inequality it follows from (18), (19) that

$$g(x) \leqslant 0 \quad \text{for} \quad 0 \leqslant x \leqslant 1.$$

By (8), (7), (4) it follows that $g(x) = 0$. Hence by (8)

$$u_i \equiv 0 \quad (i = 1, \ldots, m) \quad \text{for} \quad 0 < x < 1, \quad |y_k| \leqslant 1 \quad (k = 1, \ldots, n) \qquad \text{q. e. d.}$$

Remark. Applying Hadamard's lemma we may obtain an analogue of theorem 2 from [1] for the mixed problem for the following non-linear system of partial differential equations of the second order:

$$\frac{\partial u_i}{\partial x} = F_i\left(Z, U, \frac{\partial U}{\partial y_1}, \ldots, \frac{\partial U}{\partial y_n}, \frac{\partial^2 U}{\partial y_1 \partial y_1}, \ldots, \frac{\partial^2 U}{\partial y_1 \partial y_n}, \frac{\partial^2 U}{\partial y_2 \partial y_1}, \ldots, \frac{\partial^2 U}{\partial y_n \partial y_n}\right)$$
$$(i = 1, \ldots, m; \ U = (u_1, \ldots, u_m)).$$

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Sur la stabilité asymptotique des solutions d'un système d'équations différentielles

par Z. Opial (Kraków)

1. Considérons le système de deux équations différentielles

$$X' = f(X, t), \tag{1}$$

où X désigne le vecteur (x_1, x_2) et $f(X, t)$ est une fonction vectorielle $\big(f_1(x_1, x_2, t), f_2(x_1, x_2, t)\big)$ dont les composantes sont continues par rapport à (x_1, x_2, t) et de classe C^1 par rapport à (x_1, x_2) dans tout l'espace à trois dimensions (x_1, x_2, t).

Désignons par $X(t; X_0, t_0)$ la solution (unique en vertu des hypothèses précédentes) du système (1) qui passe par le point (X_0, t_0).

Soit K un ensemble fermé, borné et simplement connexe du plan (x_1, x_2). On dit (cf. [8], p. 83) que le système (1) est *relativement borné* dans l'ensemble K si pour tout $X_0 \epsilon K$ et tout t_0 on a $X(t; X_0, t_0) \epsilon K$ pour $t \geqslant t_0$.

On dit qu'une solution $X(t)$ du système (1) est *asymptotiquement stable dans l'ensemble* K si le système considéré est relativement borné dans cet ensemble et si pour tout $X_0 \epsilon K$ et tout t_0 on a la relation

$$\lim_{t \to +\infty} |X(t; X_0, t_0) - X(t)| = 0, \tag{2}$$

où $|X|$ désigne la longueur euclidienne du vecteur X.

De la relation (2) il vient que l'on a quels que soient les points $X_1, X_2 \epsilon K$ et t_0,

$$\lim_{t \to +\infty} |X(t; X_1, t_0) - X(t; X_2, t_0)| = 0, \tag{3}$$

ce qui signifie que toute solution du système (1) passant par un point de l'ensemble K est forcément asymptotiquement stable dans cet ensemble. Donc, au lieu de dire qu'une solution choisie du système envisagé est asymptotiquement stable dans K on peut dire tout simplement que c'est le système (1) lui-même qui est asymptotiquement stable dans cet ensemble.