for the above function the point $p(0, 0)$. Let us describe a circle round this point with a radius ε. Then

$$\sigma(\varepsilon\cos\theta, \varepsilon\sin\theta) = -\varepsilon\sqrt{1+\tfrac{1}{2}\varepsilon\cos^3\theta}$$

attains the maximum value for $\theta = \pi$, viz.

$$\sigma(-\varepsilon, 0) = -\varepsilon\sqrt{1-\tfrac{1}{2}\varepsilon}.$$

Hence the gradient direction exists, whereas

$$g = \lim_{m\to p}\frac{\sigma(m)-\sigma(p)}{\overline{mp}} = \lim_{\varepsilon\to 0}\frac{-\varepsilon\sqrt{1-\frac{1}{2}\varepsilon}-0}{\varepsilon} = -1 < 0.$$

Remark 1. The generalized gradient indicates the direction of the maximum increase (or of the minimum decrease) of the function. Similarly the direction of the minimum increase (or that of the maximum decrease) could be introduced. If a gradient exists in the classical sense, then these two directions are in opposition. In general, it is not necessarily thus, and consequently there is some reason to speak about *two gradient directions*.

Remark 2. The above theorem holds true, as has been remarked by T. Ważewski, if the assumption 1 "the function is continuous in the neighbourhood of the point p" is replaced by a weaker one: "the function is defined in the neighbourhood of the point p". Then, evidently, the function σ need not attain its maximum in S_ν, but it has an upper limit in it, since the said function is bounded in the neighbourhood of the point p, owing to its possessing a differential at the point p. Then again, μ_ν would denote the upper limit of σ in S_ν, whilst by y_ν we should mean any of the points of S_ν with the following property:

There exists a sequence of points

$$q_{\lambda_\nu} \in S_\nu \quad (\lambda = 1, 2, \ldots)$$

convergent to y_ν, i. e. $\lim_{\lambda\to\infty} q_{\lambda_\nu} = y_\nu$ such as

$$\lim_{\lambda\to\infty}\sigma(q_{\lambda_\nu}) = \mu_\nu.$$

The further statement of the theorem remains unchanged. The above change of the theorem would involve some, almost formal, changes in the argument.

Reference

[1] S. Gołąb, *On the notion of gradient. I. Essentiality of regularity suppositions*, this volume, p. 1-4.

Reçu par la Rédaction le 23. 2. 1956

ANNALES
POLONICI MATHEMATICI
VIII (1960)

On the notion of gradient

III. Gradient as a limit value of a surface integral

by S. Gołąb and A. Pliś (Kraków)

§ 1. Let a scalar field σ be given. If the function σ has at a definite point p a total differential, then at that point a gradient can be formed

$$v = \operatorname{grad}\sigma \tag{1}$$

as a vector with components (see [1])

$$v_i = \frac{\partial\sigma}{\partial x_i} \quad (i = 1, 2, \ldots, n). \tag{2}$$

We know the integral theorem which, under certain assumptions both about the field σ and about the closed hypersurface S bounding a finite and regular region D of space, expresses the integral of gradient over the region D by a surface integral. This theorem, containing in its vector form the so-called Green theorem, is stated thus:

$$\int_D \operatorname{grad}\sigma = \int_S N\cdot\sigma, \tag{3}$$

where N denotes the unit normal vector to S with an outside orientation.

To the above theorem corresponds the "differential" form, viz.

$$\operatorname{grad}\sigma(p) = \lim_{S\to p}\frac{1}{V}\int_S N\cdot\sigma. \tag{4}$$

V denotes here the volume (n-dimensional measure) of the region D.

Now the said formula is not precisely stated(1). We are concerned on one hand with assumptions with respect to the field σ and on the other with those referring to S, and finally with the limiting convergence

(1) It will suffice to see what W. Rubinowicz has written on this subject in his book [2], p. 67-70, in order to realize that the corresponding theorem is not stated in a satisfactorily strict way.

"$S \to p$", i. e. with the question how to formulate clearly the manner in which the hypersurface S should "converge" to the point p.

The aim of this note is to formulate and prove, in a general and precise way, the theorem referring to formula (4), the weakest possible assumptions being made.

§ 2. If we wanted, in proving formula (4), to apply the integral theorem (3), we should make a much stronger assumption with respect to the field σ. Then we could weaken the hypothesis in connection with the manner of the convergence "$S \to p$". Thus, assuming the regularity of the class C^1 for the hypersurface S (which involves the existence of the continuous gradient σ in the neighbourhood of p), we can apply, for the left side of formula (3), the mean value theorem for integrals and hence we obtain the equalities

$$m(D)\cdot(v_i)_{q_i} = \int_S N_i\cdot\sigma \tag{5}$$

where $m(D)$ denotes the measure of the region D, q_i $(i = 1, \ldots, n)$ denotes a point of the region D, N_i stands for the i-th component of the vector N. Hence, by virtue of the continuity of the field σ, we obtain formula (4), valid independently of the manner of the convergence "$S \to p$", provided the diameter of D tends to zero, so that the hypersurface becomes "slender" or of flat form (as a pancake).

Our concern, however, is to establish formula (4) with minimum assumptions with respect to the field σ, and then, as will be seen, the manner of the convergence "$S \to p$" is essential.

§ 3. THEOREM 1. *Suppose that the field σ has a gradient at the point p (or else σ has at the point p a differential in the Stolz-Fréchet sense) and is continuous in the neighbourhood of p, and that the sequence S_ν of closed hypersurfaces satisfies the following conditions:*

1. *The hypersurfaces S_ν contain the point p in their interior.*
2. *The hypersurfaces S_ν are measurable. Let $m(S_\nu)$ denote the $(n-1)$-dimensional measure of the surface S_ν; let further $m(D_\nu)$ denote the n-dimensional measure of the region D_ν bounded by S_ν.*
3. *The diameters d_ν of the regions D_ν tend to zero.*
4. *If by K_ν we denote the minimum hypersphere circumscribed over D_ν, and by F_ν its $(n-1)$-dimensional measure, and finally by V_ν the n-dimensional measure of the region bounded by K_ν, then there exist two positive constants α and β such that*
5. *$m(S_\nu)/F_\nu \leqslant \alpha$ and $V_\nu/m(D_\nu) \leqslant \beta$.*

Then we have the formula

$$\operatorname{grad}\sigma(p) = \lim_{\nu\to\infty}\frac{1}{m(D_\nu)}\int_{S_\nu} N_\nu\cdot\sigma, \tag{6}$$

where N_ν is a unity normal vector to S_ν with an outside orientation.

Remark. Assumptions 1, 2, 3 are clear; those of 4 and 5 as can easily be shown, are independent of each other. They put an essential restraint in the way of the convergence of S_ν to p. We intend, however, to show on examples the essentiality of these assumptions for the truth of formula (6).

Proof. Since

$$\operatorname{grad}(\sigma_1+\sigma_2) = \operatorname{grad}\sigma_1 + \operatorname{grad}\sigma_2$$

and since

$$\int_S N\cdot(\sigma_1+\sigma_2) = \int_S N\cdot\sigma_1 + \int_S N\cdot\sigma_2,$$

formula (6) will be proved if we prove it for properly chosen components

$$\sigma = \sigma_1+\sigma_2.$$

Let x_i^0 be coordinates of the point p. Let us put

$$\sigma_1 \overset{\mathrm{df}}{=} \sigma(p) + \sum_{i=1}^{n} v_i(x_i - x_i^0), \qquad \sigma_2 \overset{\mathrm{df}}{=} \sigma - \sigma_1. \tag{7}$$

Since, according to the hypothesis, the function σ has at the point p a differential in the Stolz-Fréchet sense, we may put

$$\sigma_2 = \varrho\cdot\varepsilon, \tag{8}$$

where

$$\varrho = \sqrt{\sum_{i=1}^{n}(x_i - x_i^0)^2}, \tag{9}$$

whereas

$$\varepsilon = \varepsilon(x_1, \ldots, x_n)$$

is a function with the property

$$\lim_{\varrho\to 0}\varepsilon(x_1, \ldots, x_n) = 0. \tag{10}$$

The function σ_1, as a linear one, is regular and for it, by virtue of the above note, formula (6) is satisfied. Thus it remains to prove the relation

$$\operatorname{grad}\sigma_2(p) = \lim_{\nu\to\infty}\frac{1}{m(D_\nu)}\int_{S_\nu} N_\nu\cdot\sigma_2. \tag{11}$$

But
$$\operatorname{grad}\sigma_2(p) = 0$$
and, it being so, the following equalities are still to be proved:

(12) $$\lim_{\nu\to\infty}\frac{1}{m(D_\nu)}\int_{S_\nu} N_{\nu i}\varrho\varepsilon = 0 \qquad (i = 1, \ldots, n),$$

where $N_{\nu i}$ are components of the vector N_ν.

Now we have
$$|N_{\nu i}| \leqslant 1 \qquad (i = 1, \ldots, n)$$
and therefore, since
$$\Big|\int_{S_\nu} N_{\nu i}\varrho\varepsilon\Big| \leqslant \int_{S_\nu}|N_{\nu i}|\varrho|\varepsilon| \leqslant \int_{S_\nu}\varrho|\varepsilon|,$$
it suffices to prove the formula

(13) $$\lim_{\nu\to\infty}\frac{1}{m(D_\nu)}\int_{S_\nu}\varrho|\varepsilon| = 0.$$

Thus let us evaluate the integral $\int_{S_\nu}\varrho|\varepsilon|$. Let us take the number $\eta > 0$ previously given. Let us take an index ν as great as to make
$$|\varepsilon| < \eta \quad \text{for} \quad (x_1, \ldots, x_n)\in S_\nu.$$
It may be attained by virtue of relation (10) as well as by assumption 3. In that case we shall have

(14) $$\int_{S_\nu}\varrho|\varepsilon| < \eta\int_{S_\nu}\varrho$$

for sufficiently great ν. Now it suffices to show that there exists for ν sufficiently great a constant M such that the following inequality is satisfied

(15) $$\frac{1}{m(D_\nu)}\int_{S_\nu}\varrho < M.$$

In order to evaluate the integral $\int_{S_\nu}\varrho$ let us denote by r_ν the radius of the hypersphere K_ν. We have on the hypersurface S_ν
$$\varrho \leqslant 2r_\nu.$$
Thus
$$I_\nu = \int_{S_\nu}\varrho \leqslant 2r_\nu\int_{S_\nu} = 2r_\nu m(S_\nu) \leqslant 2r_\nu\cdot F_\nu\cdot\alpha.$$

Hence we have further

(16) $$\frac{1}{m(D_\nu)}\int_{S_\nu}\varrho = \frac{I_\nu}{m(D_\nu)} \leqslant \frac{2r_\nu\cdot F_\nu\cdot\alpha}{m(D_\nu)} \leqslant \frac{2r_\nu\cdot F_\nu\cdot\alpha\cdot\beta}{V_\nu}.$$

It is known, however, that between F_ν and V_ν exists the relation

(17) $$V_\nu = F_\nu\cdot r_\nu/n.$$

(16) and (17) finally give

(18) $$\frac{1}{m(D_\nu)}\int_{S_\nu}\varrho \leqslant 2n\alpha\beta = M,$$

which was to be shown. Thus our theorem has been proved.

§ 4. From the above theorem a certain conclusion can be drawn, namely for the particular case of the hypersurfaces S_ν being **convex**. Let us denote, in this case, by s_ν the width of the region D_ν, i. e. the minimum distance between two hyperplanes parallel and tangent with respect to S_ν. Let the ratio d_ν/s_ν be named the *coefficient of slenderness* of the region D_ν. We have the following

THEOREM 2. *Under the same assumptions with respect to σ as in the preceding theorem if the sequence S_ν satisfies conditions* 1, 2, 3 *of that theorem and if instead of* 4 *and* 5, *it satisfies the conditions*:

4*. *S_ν are convex,*

5*. *the coefficients of slenderness of S_ν are jointly bounded i. e. there exists a positive constant γ such that*
$$d_\nu/s_\nu \leqslant \gamma,$$
then formula (6) *is satisfied.*

Proof. It suffices to show that the suppositions of this theorem involve the suppositions of theorem 1.

Now, assumption 4 is already satisfied by virtue of assumption 4* itself. Since S_ν is a convex hypersurface, the sphere K_ν circumscribed over it has a greater measure than that of S_ν, or equal to it, i. e.
$$F_\nu \geqslant m(S_\nu),$$
and thus we have inequality 4 with a constant $\alpha = 1$. We shall prove that inequality 5 also holds. As a matter of fact, it does. Let k_ν denote a hypersphere inscribed in S_ν with a maximum radius. Then
$$m(D_\nu) \geqslant m(k_\nu),$$

and the radius r_ν^* of the sphere k_ν is

$$r_\nu^* = \tfrac{1}{2}s_\nu .$$

Similarly, for the sphere K_ν we have

$$r_\nu = \tfrac{1}{2}d_\nu .$$

Hence

$$\frac{V_\nu}{m(D_\nu)} \leqslant \frac{V_\nu}{m(k_\nu)} = \frac{[\Gamma(\frac{1}{2})]^n}{\Gamma(\frac{1}{2}n+1)}\cdot\left(\frac{d_\nu}{2}\right)^n\cdot\frac{\Gamma(\frac{1}{2}n+1)}{[\Gamma(\frac{1}{2})]^n}\cdot\left(\frac{2}{s_\nu}\right)^n = \left(\frac{d_\nu}{s_\nu}\right)^n \leqslant \gamma^n$$

and it suffices to put $\beta = \gamma^n$ in order to have inequality 5 satisfied. Thus the proof of the theorem is finished.

§ 5. Now we will give two examples illustrating the essentiality of suppositions 4 and 5 of theorem 1. Those examples are constructed for $n = 2$ in order to avoid some lengthy computations. Analogous counter-examples can be constructed for $n \geqslant 3$.

EXAMPLE 1. Let be given a scalar field

$$\sigma(x_1, x_2) = \sqrt{x_1^2+x_2^2}\cdot\sqrt[3]{x_2}. \tag{19}$$

This function is everywhere continuous, and at the point $(0, 0)$ possesses a differential in the Stolz-Fréchet sense (equal to zero). Moreover, the field σ has a gradient at the point $(0, 0)$, which is a null vector. As a sequence S_ν we will take the sequence of ellipses

$$x_1 = \frac{1}{\nu}\cos t, \qquad x_2 = \frac{1}{\nu^2}\sin t \qquad (\nu = 1, 2, \ldots).$$

The unit normal vector N_ν has the following components:

$$N_\nu\left\{\frac{\cos t}{\sqrt{\cos^2 t+\nu^2\sin^2 t}}, \frac{\nu\sin t}{\sqrt{\cos^2 t+\nu^2\sin^2 t}}\right\}.$$

Further

$$ds = \frac{1}{\nu^2}\sqrt{\cos^2 t+\nu^2\sin^2 t}\,dt .$$

Let us put

$$I_2 \overset{\mathrm{df}}{=} \int_0^{2\pi}\frac{\nu\sin t}{\sqrt{\cos^2 t+\nu^2\sin^2 t}}\cdot\sigma\,ds .$$

From this it follows that

$$I_2 = \int_0^{2\pi}\frac{\sin t\cdot\sigma}{\nu}\,dt = \frac{1}{\nu}\int_0^{2\pi}\sin t\sqrt[3]{\frac{1}{\nu^2}\sin t}\sqrt{\frac{1}{\nu^2}\cos^2 t+\frac{1}{\nu^4}\sin^2 t}\,dt$$

$$= \frac{1}{\nu^{8/3}}\int_0^{2\pi}(\sin t)^{4/3}\cdot\sqrt{\cos^2 t+\frac{1}{\nu^2}\sin^2 t}\,dt .$$

It is obvious that the function under the sign of integral is not negative, and we will diminish it (as well as the integral) replacing the coefficient $\sqrt{\cos^2 t+\frac{1}{\nu^2}\sin^2 t}$ by $|\cos t|$. Let us write

$$I_0 \overset{\mathrm{df}}{=} \int_0^{2\pi}|\cos t|\sqrt[3]{\sin^4 t}\,dt .$$

In this case

$$I_2 > \frac{I_0}{\nu^{8/3}} .$$

In our example we have

$$m(D_\nu) = \pi\frac{1}{\nu}\cdot\frac{1}{\nu^2} = \frac{\pi}{\nu^3},$$

thus

$$\frac{I_2}{m(D_\nu)} > \frac{I_0}{\pi}\nu^{1/3} .$$

Since $I_0 > 0$, $\nu^{1/3}\to\infty$ if $\nu\to\infty$, we have $I_2/m(D_\nu)\to\infty$, while $I_2/m(D_\nu)\to 0$ if formula (6) holds true. Thus in this example the sequence S_ν does not satisfy supposition 5, although it satisfies all the other assumptions. The statement of theorem (6), as we have established, does not hold.

EXAMPLE 2. We take the same scalar field as in the preceding example, whereas the sequence S_ν we determine otherwise. The enclosed figure represents the form of a polygonal line which makes up S_ν. The polygonal (closed) line links up the following sequence of points

$$A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2, A_3, \ldots, D_{\nu-1}, A_\nu, E, \bar{E}, \bar{A}_\nu, \ldots,$$
$$\bar{A}_3, \bar{D}_2, \ldots, \bar{B}_1, \bar{A}_1, A_1,$$

where every particular point has the following coordinates

$$
\begin{array}{lll}
A_j\left(\alpha_j, \dfrac{1}{\nu}\right), & \bar{A}_j\left(-\alpha_j, \dfrac{1}{\nu}\right), & j = 1, \ldots, \nu, \\
B_j\left(\alpha_j, -\dfrac{1}{\nu}+\dfrac{1}{\nu^2}\right), & \bar{B}_j\left(-\alpha_j, -\dfrac{1}{\nu}+\dfrac{1}{\nu^2}\right) & \\
C_j\left(\beta_j, -\dfrac{1}{\nu}+\dfrac{1}{\nu^2}\right), & \bar{C}_j\left(-\beta_j, -\dfrac{1}{\nu}+\dfrac{1}{\nu^2}\right) & j = 1, \ldots, \nu-1, \\
D_j\left(\beta_j, \dfrac{1}{\nu}\right), & \bar{D}_j\left(-\beta_j, \dfrac{1}{\nu}\right) & \\
E\left(\dfrac{1}{\nu}, -\dfrac{1}{\nu}\right), & \bar{E}\left(-\dfrac{1}{\nu}, -\dfrac{1}{\nu}\right). &
\end{array}
$$

The sequences are chosen thus:

$$0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \ldots < \beta_{\nu-1} < \alpha_\nu = \frac{1}{\nu},$$

and the lengths of the intervals are to be so chosen that

$$\sum_{j=1}^{\nu-1} (\beta_j - \alpha_j) > \frac{1}{\nu} - \frac{1}{\nu^2}.$$

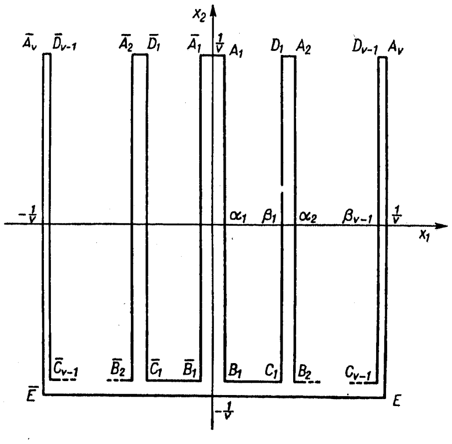

Fig. 1

Let us evaluate $m(D_\nu)$. Obviously

$$m(D_\nu) < 2\cdot\frac{1}{\nu}\cdot\frac{1}{\nu^2} + 2\cdot\frac{1}{\nu}\cdot\frac{1}{\nu^2} = \frac{4}{\nu^3}. \tag{20}$$

Denoting by N_ν the unit normal vector to S_ν and with an outside orientation let us notice the following. With respect to the properties of the function σ expressed by

$$\sigma(-x_1, x_2) = \sigma(x_1, x_2), \qquad \sigma(x_1, -x_2) = -\sigma(x_1, x_2) \tag{21}$$

the vectors N_ν in the corresponding points (symmetrical to the axis x_2) of the segments A_jB_j and $\bar{A}_j\bar{B}_j$ will be in opposition to each other. Consequently

$$\int_{A_jB_j} N_\nu\cdot\sigma\, ds = -\int_{\bar{A}_j\bar{B}_j} N_\nu\cdot\sigma\, ds.$$

For the same reasons we shall have

$$\int_{C_jD_j} N_\nu\cdot\sigma\, ds = -\int_{\bar{C}_j\bar{D}_j} N_\nu\cdot\sigma\, ds.$$

Consequently the integral

$$\int_{S_\nu} N_\nu\cdot\sigma\, ds$$

is reduced to the sum of integrals along the horizontal segments of the line S_ν, i. e.

$$I_\nu = \int_{S_\nu} N_\nu\cdot\sigma\, ds = \sum_{j=1}^{\nu-1}\Big\{\int_{B_jC_j} + \int_{\bar{B}_j\bar{C}_j} + \int_{D_jA_{j+1}} + \int_{\bar{D}_j\bar{A}_{j+1}}\Big\} + \int_{E\bar{E}}. \tag{22}$$

Let us denote by $I_{\nu1}, I_{\nu2}$ the components of the vector I_ν. We will calculate $I_{\nu2}$ ($I_{\nu1}$ equals zero). The vectors $\sigma\cdot N_\nu$ are vertical at all points of the segments which appear on the right side of the formula (22); moreover, the second component of the said vectors is positive, which results from the second property (21) of the function σ and from the principle that N_ν is directed to the outside of S_ν. From this it follows that, denoting by (λ, μ) the components of the vector $\sigma\cdot N_\nu$, we have

$$I_{\nu2} = \int_{S_\nu} \mu\, ds > \int_{E\bar{E}} \mu\, ds. \tag{23}$$

But μ at the segment $E\bar{E}$ is

$$\mu = \sqrt{x_1^2 + \frac{1}{\nu^2}}\cdot\sqrt[3]{\frac{1}{\nu}}.$$

Hence

$$\int\limits_{E\bar{E}} \mu\, ds = 2\int\limits_0^{1/\nu} \sqrt{x_1^2+\frac{1}{\nu^2}}\sqrt[3]{\frac{1}{\nu}}\, dx_1 > 2\int\limits_0^{1/\nu} \sqrt{x_1^2}\sqrt[3]{\frac{1}{\nu}}\, dx_1$$

$$= 2\sqrt[3]{\frac{1}{\nu}}\int\limits_0^{1/\nu} x_1\, dx_1 = 2\sqrt[3]{\frac{1}{\nu}}\left(\frac{x_1^2}{2}\right)_0^{1/\nu} = \sqrt[3]{\frac{1}{\nu}}\cdot\frac{1}{\nu^2} = \frac{1}{\nu^{7/3}}.$$

Thus

$$I_{\nu 2} > \frac{1}{\nu^{7/3}}.$$

Therefore with respect to inequality (20) we have

$$\frac{I_{\nu 2}}{m(D_\nu)} > \frac{\nu^3}{4\nu^{7/3}} = \frac{\nu^{2/3}}{4}$$

and thence it is obvious that

$$\frac{I_{\nu 2}}{m(D_\nu)} \to \infty \quad \text{for} \quad \nu \to \infty,$$

while, if formula (6) were true, $I_{\nu 2}/m(D_\nu)$ would tend to zero. In this example the sequence S_ν satisfies all suppositions of theorem 1 except supposition 4. It may be shown that here $m(S_\nu)/F_\nu \to \infty$ as $\nu \to \infty$.

References

[1] S. Gołąb, *On the notion of gradient. I. Essentiality of regularity suppositions*, this volume, p. 1-4.

[2] W. Rubinowicz, *Wektory i tensory*, Warszawa-Wrocław 1950.

Reçu par la Rédaction le 23. 2. 1956

ANNALES
POLONICI MATHEMATICI
VIII (1960)

Sur l'équation différentielle ordinaire du premier ordre dont le second membre satisfait aux conditions de Carathéodory

par Z. Opial (Kraków)

1. Supposons que la fonction $f(x,y)$, définie dans un rectangle R: $a \leqslant x \leqslant b$, $c \leqslant y \leqslant d$, vérifie les conditions de Carathéodory (cf. [2], p. 665), c'est-à-dire

(i) pour tout $y \in \langle c, d\rangle$ $f(x,y)$ est une fonction mesurable par rapport à x;

(ii) pour tout $x \in \langle a, b\rangle$ $f(x,y)$ est continue par rapport à y;

(iii) il existe une fonction mesurable $M(x)$ telle que l'on ait dans le rectangle R:

$$|f(x,y)| \leqslant M(x) \quad \text{et} \quad \int\limits_a^b M(x)\,dx < +\infty. \tag{1}$$

Envisageons l'équation différentielle

$$y' = f(x,y). \tag{2}$$

On dit qu'une fonction absolument continue $y(x)$ est une solution de l'équation (2) dans un intervalle $(\alpha, \beta) \subset \langle a, b\rangle$, si la relation

$$y'(x) = f\big(x, y(x)\big)$$

est vérifiée en tout point de cet intervalle, sauf peut-être aux points d'un ensemble de mesure nulle. On sait (cf. [2], p. 665-674, [5], p. 140-146) que pour tout point (x_0, y_0) appartenant à l'intérieur du rectangle R il existe au moins une solution de l'équation (2), définie dans un voisinage suffisamment petit de x_0 et égale à y_0 au point x_0.

2. Toute solution $y(x)$ de l'équation (2), étant une fonction absolument continue, est dérivable presque partout dans l'intervalle où elle est définie. L'ensemble des points où la dérivée $y'(x)$ n'existe pas est donc de mesure nulle. Mais est-il possible que l'ensemble des points x en lesquels une au moins des intégrales de l'équation (2) n'est pas dérivable soit de mesure positive ou même identique à tout l'intervalle $\langle a, b\rangle$?