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On the characteristic exponents of the solutions of a system of ordinary differential equations

by C. Kluczny (Katowice)

In this paper we consider the asymptotic behaviour of the solutions of a system of ordinary differential equations, which can be written as a vectorial equation of the form

$$X' = AX + B(X, t) + C(X, t). \tag{i}$$

Here X denotes an n-dimensional vector, A is a constant $n\times n$-matrix, $B(X, t)$ and $C(X, t)$ are n-dimensional vector-valued functions continuous for $t \geqslant t_1$ and for arbitrary X.

It is supposed that

$$|B(X, t)| \leqslant |X|\chi(t), \tag{1}$$

$$|C(X, t)| \leqslant |X|^q w(t), \quad 0 \leqslant q < 1, \tag{2}$$

where

$$\chi(t) = \chi_1(t) + \chi_2(t), \quad \lim_{t\to+\infty} \chi_1(t) = 0, \quad \int_{t_1}^{\infty} \chi_2(t)\,dt < +\infty.$$

The fundamental notion we deal with is the characteristic exponent of the solution. In the following we shall write the characteristic exponent of the function $x(t)$ (or of the vector function $X(t)$) shortly as $\mathrm{Ce}\,x(t)$ ($\mathrm{Ce}\,X(t)$).

Some particular cases of (i) have been considered by P. M. Grobman [2], A. Wintner and P. Hartman [8], K. Tatarkiewicz [5], T. Peyovitch [4] and others. It is supposed in [2] and in [8] that $C(X, t) = 0$. In [5] the author assumes that $C(X, t) = W(t)$. Moreover, in [2] and in [5] the following assumption appears:

$$|B(\bar{X}, t) - B(\bar{\bar{X}}, t)| \leqslant |\bar{X} - \bar{\bar{X}}|\chi(t). \tag{3}$$

In [4] and in [5] it is supposed that the inequality $\mathrm{Ce}\,w(t) \leqslant s$ holds. In our paper this condition is replaced by the following (weaker) one:

$$\mathrm{Ce}\int_{t_1}^{t} w(\tau)\,d\tau \leqslant s \quad \text{ or } \quad \mathrm{Ce}\int_{t}^{\infty} w(\tau)\,d\tau \leqslant s.$$

In § 1 we present some well-known theorems concerning the $\mathrm{Ce}\varphi(t)$. Moreover we prove Theorem (1,3), which allows the introduction of some simple rules of operating on Ce. The same theorem permits us to generalize the assumption concerning the function $w(t)$ (see (2)). In § 2 we consider the equation $X' = F(X, t)$ and we obtain some theorems useful in the proof of Theorem C. This theorem, which is the main result of present paper, generalizes some results of the papers mentioned above. We discuss also in § 5 the equation

(ii) $$X' = AX+B(X, t).$$

If we replace (1) by (3), then there exists (see Theorem (5,2)) a one-one correspondence between the solutions of the equation $X' = AX$ and that of (ii). The corresponding solutions have the same characteristic exponents. The existence of such a correspondence has been proved in [2] under the additional assumption $B(0, t) = 0$ (see also Theorem (5,1) of this paper).

We apply the qualitative method of T. Ważewski founded on the notion of retract (see lemmas 1 and 2). Owing to this method the considerations in this paper are of geometrical character. This seems to be valuable. For example, Theorem (5,2) not only determines the quantity of the integrals with their $\mathrm{Ce} \leqslant r$ but at the same time informs about the structure of the family of all such integrals.

I am very much obliged to Prof. T. Ważewski for encouraging me to work on asymptotic problems. I wish also to express my thanks to Dr Z. Szmydt for her valuable remarks, which I have utilized in this paper.

§ 1. We now define the characteristic exponent of a function.

Definition (1,1). Let the function $\varphi(t)$ be continuous for $t \geqslant a$. $\mathrm{Ce}\varphi(t)$ (the characteristic exponent of $\varphi(t)$) is the greatest lower bound of such α that

$$\lim_{t\to+\infty} \varphi(t)e^{-\alpha t} = 0.$$

If there is no such α then $\mathrm{Ce}\varphi(t) \overset{\mathrm{df}}{=} +\infty$ [1].

Note that $\mathrm{Ce}\varphi(t)$ may be equal to $-\infty$. $\mathrm{Ce}\varphi(t)$ is uniquely determined if $\varphi(t)$ is defined and continuous for t sufficiently large. Furthermore the following relations hold:

(1,1) $$\mathrm{Ce}|\varphi(t)| = \mathrm{Ce}\varphi(t),$$

(1,2) $$\mathrm{Ce}\varphi(t) \leqslant \mathrm{Ce}\tau(t) \quad \text{if} \quad |\varphi(t)| \leqslant |\tau(t)|,$$

(1,3) $$\mathrm{Ce}\big(p\varphi(t)\big) = \mathrm{Ce}\varphi(t) \quad \text{if} \quad p \neq 0,$$

(1,4) $$\mathrm{Ce}\{p\} = 0 \text{ if } p \neq 0, \quad \mathrm{Ce}\{0\} = -\infty.$$

(1) $\mathrm{Ce}\varphi(t) = -\lambda$ where λ is the Liapounoff number of $\varphi(t)$ (see [3], p. 317).

Remark (1,1). $\mathrm{Ce}\varphi(t) \leqslant r < +\infty$ if and only if $\lim\limits_{t\to+\infty} \varphi(t)e^{-(r+\varepsilon)t} = 0$ for every $\varepsilon > 0$. Similarly $\mathrm{Ce}\varphi(t) \geqslant r > -\infty$ if and only if the function $\varphi(t)e^{-(r-\varepsilon)t}$ is unbounded in $[a, +\infty)$ for every $\varepsilon > 0$. In particular, if

$$\mathrm{Ce}\varphi(t) < 0 \text{ then } \lim_{t\to+\infty} \varphi(t) = 0 \text{ and } \int_a^\infty |\varphi(\tau)|\,d\tau < +\infty.$$

It follows easily from the results of Liapounoff (see [3]) that the following theorems are true:

THEOREM (1,1). *Suppose that $\varphi(t)$ is continuous for $t \geqslant a$ and $\varphi(t) \neq 0$. Then*

$$\mathrm{Ce}\varphi(t) = \limsup_{t\to+\infty} \frac{\ln|\varphi(t)|}{t}.$$

THEOREM (1,2). *Assume that $\mathrm{Ce}\varphi(t) = p$, $\mathrm{Ce}\tau(t) = q$. Then*

(a) $\mathrm{Ce}\big(\varphi(t)+\tau(t)\big) \leqslant \max(p, q)$.

If $\varphi(t)\tau(t) \geqslant 0$ or $p \neq q$ then $\mathrm{Ce}\big(\varphi(t)+\tau(t)\big) = \max(p, q)$. Furthermore the following formulas hold:

(b) $\mathrm{Ce}\big(\varphi(t)\,\tau(t)\big) \leqslant p+q$ [2],

(c) $\mathrm{Ce}\big(\varphi(t)\big)^k = kp$ *for* $k = 0, 1, 2, \ldots$ *and for arbitrary* $k > 0$ *if* $\varphi(t) \geqslant 0$,

(d) $\mathrm{Ce}\displaystyle\int_a^t \varphi(\tau)\,d\tau \leqslant p$ *whenever* $p \geqslant 0$,

(e) $\mathrm{Ce}\displaystyle\int_t^\infty \varphi(\tau)\,d\tau \leqslant p$ *if* $p < 0$.

In this paper we have to deal only with the $\mathrm{Ce} < +\infty$. Writing $\mathrm{Ce} \leqslant r$, we assume in the sequel that $r < +\infty$.

Now we are going to prove the following theorem:

THEOREM (1,3). *Let the functions $\beta(t)$, $\varphi(t)$ be continuous for $t \geqslant a$, $\beta(t) \geqslant 0$. Suppose that*

1° $\mathrm{Ce}\displaystyle\int_a^t \beta(\tau)\,d\tau \leqslant r$ *or* $\mathrm{Ce}\displaystyle\int_t^\infty \beta(\tau)\,d\tau \leqslant r$,

2° $\mathrm{Ce}\varphi(t) \leqslant \alpha$, $-\infty < \alpha$.

(2) If $p = +\infty$ ($p = -\infty$) and $q = -\infty$ ($q = +\infty$) then (b) holds when the right-hand member of the inequality is replaced by $+\infty$.

Then

(a) $$\mathrm{Ce}\int\limits_a^t \varphi(\tau)\beta(\tau)\,d\tau \leqslant a+r \quad \textit{if} \quad a+r \geqslant 0,$$

(b) $$\mathrm{Ce}\int\limits_t^\infty \varphi(\tau)\beta(\tau)\,d\tau \leqslant a+r \quad \textit{if} \quad a+r < 0.$$

Proof. Given any $\varepsilon > 0$ it follows from Remark (1,1) and from 2° that there is such a $b \geqslant a$ that $|\varphi(\tau)| \leqslant e^{(a+\varepsilon)\tau}$ for $\tau \geqslant b$. Therefore

(1,5) $$\Big|\int\limits_a^t \varphi(\tau)\beta(\tau)\,d\tau\Big| \leqslant M + \int\limits_b^t e^{(a+\varepsilon)\tau}\beta(\tau)\,d\tau.$$

Suppose that $\mathrm{Ce}\int\limits_a^t \beta(\tau)\,d\tau \leqslant r$. Integrating by parts shows that

(1,6) $$\int\limits_b^t e^{(a+\varepsilon)\tau}\beta(\tau)\,d\tau = e^{(a+\varepsilon)t}\int\limits_b^t \beta(\tau)\,d\tau - (a+\varepsilon)\int\limits_b^t \Big[e^{(a+\varepsilon)\tau}\int\limits_b^\tau \beta(s)\,ds\Big]d\tau.$$

If $\int\limits_b^\infty \beta(\tau)\,d\tau < +\infty$, then

(1,7) $$\int\limits_b^t e^{(a+\varepsilon)\tau}\beta(\tau)\,d\tau = N - e^{(a+\varepsilon)t}\int\limits_t^\infty \beta(\tau)\,d\tau + (a+\varepsilon)\int\limits_b^t \Big[e^{(a+\varepsilon)\tau}\int\limits_\tau^\infty \beta(s)\,ds\Big]\,d\tau.$$

From the previous remarks it follows that the characteristic exponents of the right-hand members of (1,6) and (1,7) are not greater than $a+\varepsilon+r$ provided that $a+r \geqslant 0$. Therefore in this case

$$\mathrm{Ce}\int\limits_a^t \varphi(\tau)\beta(\tau)\,d\tau \leqslant a+r+\varepsilon.$$

Hence (a) is proved. In order to prove (b) we take $\varepsilon > 0$ such that $a+r+\varepsilon < 0$. Owing to Remark (1,1) and 1° it follows from (1,6) and (1,7) that

$$\int\limits_t^\infty e^{(a+\varepsilon)\tau}\beta(\tau)\,d\tau < +\infty$$

and respectively

$$\int\limits_t^\infty e^{(a+\varepsilon)\tau}\beta(\tau)\,d\tau = -(a+\varepsilon)\int\limits_t^\infty \Big[e^{(a+\varepsilon)\tau}\int\limits_b^\tau \beta(s)\,ds\Big]\,d\tau - e^{(a+\varepsilon)t}\int\limits_b^t \beta(\tau)\,d\tau$$

or

$$\int\limits_t^\infty e^{(a+\varepsilon)\tau}\beta(\tau)\,d\tau = (a+\varepsilon)\int\limits_t^\infty \Big[e^{(a+\varepsilon)\tau}\int\limits_\tau^\infty \beta(s)\,ds\Big]\,d\tau + e^{(a+\varepsilon)t}\int\limits_t^\infty \beta(\tau)\,d\tau.$$

From these formulas and from the properties of the characteristic exponents given above we conclude that (b) holds.

Let us introduce the following definition:

Definition (1,2). Suppose we are given the vector function $\Phi(t) = \big(\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t)\big)$ continuous for $t \geqslant a$. We define the characteristic exponent of $\Phi(t)$ as that of $|\Phi(t)|$, i. e.

$$\mathrm{Ce}\,\Phi(t) \overset{\mathrm{df}}{=} \mathrm{Ce}\,|\Phi(t)|,$$

$$\text{where } |\Phi(t)| = \sqrt{\varphi_1^2(t)+\varphi_2^2(t)+\ldots+\varphi_n^2(t)}.$$

By (1,2), (1,3) we get

$$\mathrm{Ce}\,\Phi(t) = \mathrm{Ce}\sum_{i=1}^n |\varphi_i(t)| = \mathrm{Ce}\max_i |\varphi_i(t)|.$$

Remark that if $T = (t_{ij})$ is a non-singular $n\times n$ matrix and $\|T\| = \sum\limits_{i,j=1}^n |t_{ij}|$, then

$$\frac{|\Phi(t)|}{\|T^{-1}\|} \leqslant |T\Phi(t)| \leqslant \|T\|\,|\Phi(t)|.$$

Therefore $\mathrm{Ce}\big(T\Phi(t)\big) = \mathrm{Ce}\,\Phi(t)$.

Theorems of this section permit us to determine in a simple way the characteristic exponents of solutions of some differential equations. We now present some useful examples. Suppose that the functions $a(t), \beta(t), \gamma(t)$ are continuous for $t \geqslant t_1$, and that

1° $$\lim_{t\to+\infty}\frac{1}{t}\int\limits_{t_1}^t a(\tau)\,d\tau = a, \qquad -\infty < a < +\infty,$$

2° $$\mathrm{Ce}\int\limits_{t_1}^t |\beta(\tau)|\,d\tau \leqslant b \quad \text{or} \quad \mathrm{Ce}\int\limits_t^\infty |\beta(\tau)|\,d\tau \leqslant b,$$

3° $$\mathrm{Ce}\int\limits_{t_1}^t \gamma(\tau)\,d\tau \leqslant c \quad \text{or} \quad \mathrm{Ce}\int\limits_t^\infty \gamma(\tau)\,d\tau \leqslant c.$$

Example 1. Let us consider the scalar linear equation

(1,8) $$u' = a(t)u + \beta(t).$$

We write

$$\int\limits_{t_1}^t a(\tau)\,d\tau = ta(t).$$

By 1° we get $\lim\limits_{t\to+\infty} a(t) = a$. From Theorem (1,1) we conclude that $\mathrm{Ce}\,e^{ta(t)} = a$.

By 2° and Theorem (1,3) we get what follows:

$$\mathrm{Ce}\int_{t_1}^{t}|\beta(\tau)|e^{-\tau a(\tau)}d\tau \leqslant b-a \quad \text{whenever} \quad b-a\geqslant 0.$$

$$\mathrm{Ce}\int_{t}^{\infty}|\beta(\tau)|e^{-\tau a(\tau)}d\tau \leqslant b-a \quad \text{whenever} \quad b-a< 0.$$

The general solution of (1.8) is given by the following formula:

$$u = e^{ta(t)}\Big[C+\int_{t_1}^{t}\beta(\tau)e^{-\tau a(\tau)}d\tau\Big]. \tag{1,9}$$

If $b-a<0$, then the solution of (1,8) may also be expressed as follows:

$$u = e^{ta(t)}\Big[C_1-\int_{t}^{\infty}\beta(\tau)e^{-\tau a(\tau)}d\tau\Big]. \tag{1,10}$$

From theorems (1,2) and (1,3) we infer therefore that the following conditions hold:

(a) if $b\geqslant a$, then the Ce of the solutions of (1,8) are not greater than b,

(b) if $b<a$, then the Ce of the exceptional solution

$$u_0 = -e^{ta(t)}\int_{t}^{\infty}\beta(\tau)e^{-\tau a(\tau)}d\tau \tag{1,11}$$

does not exceed b. All the remaining solutions have their Ce equal to a.

Remark (1,2). Solution (1,11) is positive if $\beta(t)\leqslant 0$ and if for every $t^*>t_1$ the function $\beta(t)$ is not identically equal to zero in the interval $(t^*, +\infty)$.

Example 2. Let us consider the following equation:

$$u' = a(t)u+\beta(t)u^q, \quad 0\leqslant q<1\,(^3). \tag{1,12}$$

Suppose that $u(t)>0$ is the solution of (1,12) in the interval $[t_2, +\infty)$ $(t_2\geqslant t_1)$. Then the function

$$v(t) = [u(t)]^{1-q} \tag{1,13}$$

satisfies for $t\geqslant t_2$ the equation

$$v' = (1-q)a(t)v+(1-q)\beta(t)\,(^4). \tag{1,14}$$

(3) We write $u^q \overset{\mathrm{df}}{=} 1$, if $q=0$.

(4) The function $v(t)$ defined by (1,13) may not satisfy the equation (1,14) if one assumes that $u(t)\geqslant 0$.

One can easily infer from the definition of $v(t)$ and from the previous results (see Example 1), that the following conditions hold:

(a) if $b/(1-q)\geqslant a$, then every positive solution of (1,12) has its Ce not greater than $b/(1-q)$,

(b) if $b/(1-q)<a$, then every positive solution of (1,12) has its Ce equal to a except at most one, whose Ce is less than or equal to $b/(1-q)$. That exceptional solution is positive if $\beta(t)\leqslant 0$ and if for every t^* the function $\beta(t)$ is not identically equal to zero in $[t^*, +\infty)$. Note that (in case (b)) there always exist positive solutions. This follows from formula (1,10) as applied to (1,14) under the assumption $C_1>0$.

Example 3. Suppose we are given the following equation:

$$v' = a(t)v+\beta(t)v^{\varrho}+\gamma(t)v^{r}, \quad 0\leqslant \varrho,\ r<1. \tag{1,15}$$

Suppose that

$$\beta(t)\geqslant 0, \quad \gamma(t)\geqslant 0, \quad b-a(1-\varrho)<0, \quad c-a(1-r)<0. \tag{1,15^0}$$

Assume that $v_0>0$, $t_0\geqslant t_1$. Then there exists a solution of (1,15), passing through (v_0, t_0), determined and positive for $t\geqslant t_0$. The last assertion is implied by the elementary properties of the solutions of a linear differential equation and by the obvious inequality $v'\leqslant\{|a(t)|+\beta(t)+ +\gamma(t)\}v$ for $v\geqslant 1$.

We shall prove that every positive solution of (1,15) has its Ce equal to a.

Let $v(t)$ be a positive solution of (1,15). Then $v'\geqslant a(t)v$ and there is a $K>0$ such that for $t\geqslant t_1$ we have $v(t)\geqslant Ke^{ta(t)}$ and $1/v(t)\leqslant e^{-ta(t)}/K$. Hence by Theorem (1,2) we conclude that

$$\mathrm{Ce}\frac{1}{v(t)}\leqslant -a, \qquad \mathrm{Ce}\, v^{\varrho-1}(t)\leqslant -a(1-\varrho).$$

Then from the third inequality of (1,15^0) and from Theorem (1,3) it follows that

$$\mathrm{Ce}\int_{t}^{\infty}\beta(\tau)v^{\varrho-1}(\tau)d\tau<0.$$

We therefore get the relation

$$\lim_{t\to+\infty}\frac{1}{t}\int_{t_1}^{t}\beta(\tau)v^{\varrho-1}(\tau)d\tau = 0. \tag{1,16}$$

Applying analogous arguments one shows that

$$\lim_{t\to+\infty}\frac{1}{t}\int_{t_1}^{t}\gamma(\tau)v^{r-1}(\tau)d\tau = 0. \tag{1,17}$$

By (1,16), (1,17), 1° and by Theorem (1,1) owing to the assumption that $v(t) > 0$ we obtain

$$\operatorname{Ce} v(t) = \limsup_{t\to+\infty} \frac{1}{t}\int_{t_1}^{t} \frac{v'(\tau)}{v(\tau)}\,d\tau$$

$$= \lim_{t\to+\infty} \frac{1}{t}\int_{t_1}^{t} \{\alpha(\tau)+\beta(\tau)v^{\varrho-1}(\tau)+\gamma(\tau)v^{r-1}(\tau)\}\,d\tau = a.$$

§ 2. Denote by X the n-dimensional vector $X = (x_1, x_2, \ldots, x_n)$ and by $F(X,t)$ the vector-valued functions $F(X,t) = \big(f_1(X,t), \ldots, f_n(X,t)\big)$. We shall consider the equation

$$\text{(I)} \qquad X' = F(X,t).$$

Let us introduce the following assumption:

ASSUMPTION K. 1° The vector function $F(X,t)$ is continuous in the set D: $t \geqslant t_1$, X arbitrary,

2° the scalar functions $\omega(t)$ and $w(t)$ are continuous for $t \geqslant t_1$, $w(t) \geqslant 0$,

$$3^\circ \lim_{t\to+\infty} \frac{1}{t}\int_{t_1}^{t} \omega(\tau)\,d\tau = \mu, \qquad -\infty < \mu < +\infty,$$

$$4^\circ \operatorname{Ce}\int_{t_1}^{t} w(\tau)\,d\tau \leqslant s \quad \text{or} \quad \operatorname{Ce}\int_{t}^{\infty} w(\tau)\,d\tau \leqslant s.$$

THEOREM A. *Let the Assumption* K *be satisfied. Suppose that there exists such a* q, $0 \leqslant q < 1$, *that the inequality*

5° $X\cdot F(X,t) \leqslant \omega(t)X^2 + w(t)|X|^{1+q}$

holds for $(X,t)\in D$. *Assume that* $X(t)$ *is an arbitrary solution of* (I). *Then*

$$\operatorname{Ce} X(t) \leqslant \max\left(\mu, \frac{s}{1-q}\right).$$

Proof. By 5° we get

$$\text{(2,1)} \qquad \frac{dX^2(t)}{dt} \leqslant 2\omega(t)X^2(t) + 2w(t)\left[X^2(t)\right]^p, \qquad p = \frac{1+q}{2}.$$

Denote by $u(t)$ a solution of the equation

$$\text{(2,2)} \qquad u' = 2\omega(t)u + 2w(t)u^p$$

such that $u(t_1) \overset{\text{df}}{=} u_1 > X^2(t_1)$. We have $u_1 > 0$, $0 < p < 1$ and $2w(t) \geqslant 0$. On the other hand, if $u > 0$ then $u' > 2\omega(t)u$ and for $u \geqslant 1$ the inequality $u' \leqslant 2\big(|\omega(t)| + w(t)\big)u$ holds. Therefore $u(t)$ exists and is positive for $t \geqslant t_1$. From (2,1) and (2,2) it follows that $X^2(t) \leqslant u(t)$. Hence $X(t)$ exists for $t \geqslant t_1$. By Theorem (1,2) we therefore get

$$\text{(2,3)} \qquad \operatorname{Ce}|X(t)| \leqslant \tfrac{1}{2}\operatorname{Ce} u(t).$$

Owing to the elementary properties of the solutions of (1,12) (see Example 2, § 1), from (2,2) and from Assumption K we obtain

$$\text{(2,4)} \qquad \operatorname{Ce} u(t) \leqslant \max\left\{2\mu, \frac{s}{1-p}\right\}.$$

Finally, with the help of the equality $s/(1-p) = 2s/(1-q)$ we find from (2,3), (2,4) and from Definition (1,2) that

$$\operatorname{Ce} X(t) \leqslant \max\left\{\mu, \frac{s}{1-q}\right\},$$

as was to be proved.

We give below a certain lemma needed in the proof of Theorem B.

LEMMA 1. *Assume that the positive function* $\varphi(t)$ *is continuously differentiable for* $t \geqslant t_1$. *Suppose that*

$$\text{(2,5)} \qquad 2XF\cdot(X,t) > \varphi'(t)$$

whenever $X^2 = \varphi(t)$ *and* $t \geqslant t_1$. *Then there exists at least one solution* $X_0(t)$ *of* (I) *determined for* $t \geqslant t_1$ *and satisfying inequality*

$$\text{(2,6)} \qquad X_0^2(t) < \varphi(t) \quad \textit{for} \quad t \geqslant t_1.$$

Proof. Define the function

$$\Phi(X,t) = X^2 - \varphi(t).$$

Let M and Ω be sets of points (X,t) defined as follows:

$$M\colon \quad \Phi(X,t) = 0, \quad t \geqslant t_1,$$

$$\Omega\colon \quad \Phi(X,t) < 0, \quad t \geqslant t_1.$$

Denote by $\dot\Phi_{(\mathrm{I})}(X,t)$ the derivative of $\Phi(X,t)$ along the solutions of (I) and let $(X,t)\in M$. It is easy to verify that

$$\dot\Phi_{(\mathrm{I})}(X,t) = 2XF\cdot(X,t) - \varphi'(t).$$

Owing to (2,5) we conclude therefore that $\dot\Phi_{(\mathrm{I})}(X,t) > 0$ for $(X,t)\in M$. This means that every point of M is a point of strict egress from Ω with respect to system (I) (see [7], p. 292-293).

Let $\tau > t_1$ and denote by Z a set of points (X,τ) such that $X^2 \leqslant \varphi(\tau)$.

It is easy to show that $Z\cdot M$ is not a retract of Z (see [7], p. 280). On the other hand $Z\cdot M$ is a retract of M. This can be proved as follows. To every point (X, t) of M there corresponds a unique point (X^*, t^*) of $Z\cdot M$ where

$$X^* = \frac{X}{|X|}\sqrt{\varphi(t)}\,, \qquad t^* = \tau.$$

We denote by T the transformation $(X, t)\to(X^*, t^*)$ defined above. It is continuous on M ($X\neq 0$ on M!) and maps M on $Z\cdot M$. We also have $TP = P$ whenever $P\epsilon Z\cdot M$. Hence $Z\cdot M$ is a retract of M.

Applying the topological principle of T. Ważewski ([7], Theorem 3 and [1], Theorem 3) we conclude that there exists at least one solution $X_0(t)$ of (I) such that $\big(X_0(t), t\big)\epsilon\Omega$ for $t\geqslant\tau$. Hence the first inequality of (2,6) holds for $t\geqslant\tau$. It holds as well for $t\geqslant t_1$ owing to the fact that the points of M are points of strict egress from Ω. Thus Lemma 1 is proved.

THEOREM B. *Suppose that Assumption* K *is satisfied. Assume that there is such a* $q\epsilon[0, 1)$ *that the inequality*

$5^{\circ\circ}$ $\quad X\cdot F(X, t) > \omega(t)X^2 - w(t)|X|^{1+q}$

holds for $(X, t)\epsilon D$, $X\neq 0$. *Moreover the constants* s, μ *(see* 3° *and* 4° *of* K*) and* q *satisfy the following condition*:

6° $\quad s-(1-q)\mu < 0.$

Furthermore we assume that the solutions of (I) *are determined for* $t\geqslant t_1$.

Our assumptions imply that the following conditions hold:

(a) *there exists at least one solution of* (I) *whose* Ce *is less than or equal to* $s/(1-q)$,

(b) *if* $X(t)$ *is the solution of* (I) *such that* $\operatorname{Ce}X(t) > s/(1-q)$ *then* $\operatorname{Ce}X(t)\geqslant\mu$.

Remark (2,1). Assume additionally that there is such a t^* that $w(t)\equiv 0$ for $t>t^*$. Then $s=-\infty$ and the following conditions hold:

(a) the solution of (I) whose Ce is equal to $-\infty$ is unique and it is identically equal to zero for $t>t^*$,

(b) all the remaining solutions have their Ce greater than or equal to μ.

Proof of Theorem B. Since Remark (2,1) will be proved next, we consider only the case where $w(t)$ is not identically equal to zero in any interval $[t, +\infty)$.

Let us consider the following equation:

$$u' = 2\omega(t)u - 2w(t)u^p, \qquad p = \frac{1+q}{2}. \tag{2,7}$$

It follows from Assumption K, from 6° and from the results previously

discussed (see Example 2, § 1), that equation (2,7) has a positive solution $\varphi_0(t)$ such that

$$\operatorname{Ce}\varphi_0(t)\leqslant\frac{s}{1-p} = \frac{2s}{1-q}. \tag{2,8}$$

By (2,7) and $5^{\circ\circ}$ we infer that $2X\cdot F(X, t) > \varphi_0'(t)$ if $X^2=\varphi_0(t)$ ($t\geqslant t_1$). From Lemma 1 we conclude therefore that there exists a solution $X_0(t)$ of (I) such that $X_0^2(t)<\varphi_0(t)$ for $t\epsilon[t_1, +\infty)$. This fact, together with (2,8), Theorem (1,2) and Definition (1,2), implies that $\operatorname{Ce}X_0(t)\leqslant s/(1-q)$.

Now we shall prove (b). Suppose that $\operatorname{Ce}X(t) > s/(1-q)$. It is thus seen that the inequality $X^2(t)\leqslant\varphi_0(t)$ is not satisfied for all $t\geqslant t_1$. Therefore there exists a $t_2\geqslant t_1$ such that $u_2\overset{\mathrm{df}}{=}X^2(t_2)>\varphi_0(t_2)$. Let $\varphi_1(t)$ be a solution of (2,7) issuing from the point (u_2, t_2). It is easy to show that $\varphi_1(t)>\varphi_0(t)$ for $t\geqslant t_2$.

Taking into account the discussion connected with Example 2. (§ 1) we infer from Assumption K that

$$\operatorname{Ce}\varphi_1(t) = 2\mu. \tag{2,9}$$

Condition $5^{\circ\circ}$ implies

$$\frac{dX^2(t)}{dt} > 2\omega(t)X^2(t) - 2w(t)|X^2(t)|^p. \tag{2,10}$$

From (2,7) and (2,10) it follows that $X^2(t)\geqslant\varphi_1(t)$ for $t\geqslant t_2$. Therefore by (2,9) we have $\operatorname{Ce}X(t) = \operatorname{Ce}|X(t)|\geqslant\mu$.

Proof of Remark (2,1). Suppose that $w(t)\equiv 0$ for $t\geqslant t^*$. By $5^{\circ\circ}$ we have

$$X\cdot F(X, t) > \omega(t)X^2 \tag{2,11}$$

whenever $X\neq 0$. But $F(X, t)$ is continuous, whence $F(0, t) = 0$ for $t\geqslant t^*$.

Let $X(t)$ be an arbitrary non-zero solution of (I) and $X^2(t_3)\overset{\mathrm{df}}{=}u_3>0$ ($t_3\geqslant t^*$). Suppose that $\varphi_2(t)$ satisfy the equation $u' = 2\omega(t)u$ and $\varphi_2(t_3) = u_3$. A reasoning similar to that presented above shows that $X^2(t)\geqslant\varphi_2(t)$ for $t\geqslant t_3$. It follows from the results discussed in Example 1 (§ 1) that $\operatorname{Ce}\varphi_2(t) = 2\mu$. Therefore $\operatorname{Ce}X(t)\geqslant\mu$.

In this way Remark (2,1) is proved and at the same time the proof of Theorem B is completed.

§ 3. This part of the paper deals with some properties of linear transformations.

Let A be an $n\times n$ square matrix with constant coefficients. It is known that there exists a non-singular matrix U such that the matrix $A^* = U^{-1}AU$ has the Jordan canonical form. By this we understand that

$$A^* = \left\| \begin{array}{c|c|c|c} D_1 & & & \\ \hline & D_2 & & \\ \hline & & \ddots & \\ \hline & & & D_r \end{array} \right\|$$

where D_i are square matrices having one of two forms:

$$\left\| \begin{array}{c|c|c|c} \varrho & & & \\ \hline \varepsilon_1 & \varrho & & \\ \hline & \varepsilon_2 & \ddots & \\ \hline & & \varepsilon_s & \varrho \end{array} \right\| \quad \text{or} \quad \left\| \begin{array}{c|c|c|c} \begin{matrix}\sigma, \tau \\ -\tau, \sigma\end{matrix} & & & \\ \hline \begin{matrix}\varepsilon_1, 0 \\ 0, \varepsilon_2\end{matrix} & \begin{matrix}\sigma, \tau \\ -\tau, \sigma\end{matrix} & \ddots & \\ \hline & \ddots & \ddots & \ddots \\ \hline & & \begin{matrix}\varepsilon_{2q-1}, 0 \\ 0, \varepsilon_{2q}\end{matrix} & \begin{matrix}\sigma, \tau \\ -\tau, \sigma\end{matrix} \end{array} \right\|$$

where the entries that have not been filled are all zeros. The number ϱ is a real characteristic root of A and $\sigma\pm i\tau$ are complex conjugate ones, ε_i are non-negative and may be made arbitrarily small provided that, U is suitably chosen. The sequence of characteristic roots of A is ordered in such a manner that the real parts of the roots of D_i do not decrease with respect to i.

Remark (3,1). Suppose that $\varepsilon_i < \varepsilon$ $(\varepsilon > 0)$ and that the different real parts of the characteristic roots are ordered in the following increasing sequence:

$$\varrho_1 < \varrho_2 < \ldots < \varrho_s < \varrho_{s+1} < \ldots < \varrho_k .$$

Let l_s denote the number of characteristic roots whose real parts do not exceed ϱ_s (each i-fold characteristic root is counted i times). Denote by Y_s the l_s-dimensional vector and by Z_s the $n-l_s$ dimensional vector. Writing $X = (Y_s, Z_s)$ we mean that the first l_s coordinates of X are identical with that of Y_s and the remaining $n-l_s$ with that of Z_s. Using this notation one can easily obtain the following inequalities, where A^* has the canonical Jordan form described above:

$$(Y_s, 0)A^*(Y_s, 0) < (\varrho_s+\varepsilon)Y_s^2 \quad \text{for} \quad Y_s \neq 0,$$
$$(0, Z_s)A^*(0, Z_s) > (\varrho_{s+1}-\varepsilon)Z_s^2 \quad \text{for} \quad Z_s \neq 0.$$

In particular we have

$$(\varrho_1-\varepsilon)X^2 < XA^*X < (\varrho_k+\varepsilon)X^2 \quad \text{for} \quad X\neq 0.$$

§ 4. Now we present our main result, which concerns the characteristic exponents of the solutions of the system

$$\text{(II)} \qquad X' = AX+B(X,t)+C(X,t).$$

To begin with we introduce the following assumption:

Assumption H. 1° The vector-valued functions $B(X,t)$, $C(X,t)$ are continuous in the set D: $t\geqslant t_1$, X arbitrary.

2° There exists a scalar function $\chi(t)$, continuous for $t\geqslant t_1$, such that

$$\chi(t) = \chi_1(t)+\chi_2(t), \quad \chi_1(t)\geqslant 0, \quad \chi_2(t)\geqslant 0,$$
$$\lim_{t\to+\infty}\chi_1(t) = 0, \quad \int_t^\infty \chi_2(\tau)\,d\tau < +\infty .$$

3° There exists a scalar function $w(t)\geqslant 0$ continuous for $t\geqslant t_1$ and a constant s $(-\infty\leqslant s<+\infty)$, such that

$$\mathrm{Ce}\int_{t_1}^{t} w(\tau)\,d\tau \leqslant s \quad \text{or} \quad \mathrm{Ce}\int_t^\infty w(\tau)\,d\tau \leqslant s .$$

4° Moreover, in the set D the following inequalities hold:

$$|B(X,t)|\leqslant |X|\chi(t), \quad |C(X,t)|\leqslant |X|^q w(t) \quad \text{where} \quad 0\leqslant q<1 .$$

Just as in section 3, denote by

$$\text{(4,1)} \qquad \varrho_1<\varrho_2<\ldots<\varrho_k$$

an increasing sequence of all real parts of the characteristic roots of the matrix A.

Write

$$v_0 = \frac{s}{1-q}.$$

Now form the sequence

$$\text{(4,2)} \qquad v_0<v_1<\ldots<v_p$$

containing v_0 and all members of (4,1) greater than v_0. Moreover, let n_i be the number of characteristic roots of A whose real parts are not greater than v_i (each k-fold characteristic root is counted k times). Denote by S_i $(i = 0,1,\ldots,p)$ the family of the solutions of (II) whose Ce do not exceed v_i and by $S_i(\tau)$ a set of points $X\in E^n$ such that there exists a solution of (II) issuing from (X,τ) and belonging to S_i.

We now introduce some definitions.

Definition (4,1). The set $Z \subset E^n$ is said to be *of type* I_k ($k = 1, 2, \ldots, n-1$) if there exists a k-dimensional subspace $E^k \subset E^n$ such that each hyperplane parallel to a certain complementary space E^{n-k} [5] has at least one point common with Z.

We say that Z is *of type* I_0 if it is non-void, and *of type* I_n if $Z = E^n$.

Definition (4,2). The set Z of type I_k ($k = 1, 2, \ldots, n-1$) is called a *surface of type* I_k if there exists a homeomorfic correspondence between the hyperplanes parallel to the complementary space E^{n-k} and the points of intersection of Z with those hyperplanes.

Theorem C. *Suppose that Assumption* H *holds. Let $X(t)$ be an arbitrary solution of* (II) *and* $\mathrm{Ce}\,X(t) = r$.

Then the following conditions hold:

(a) $r \leqslant v_0$ *or* $r = v_i$, *where i is one of the integers* $1, 2, \ldots, p$ [6].

(b) *there exists such a number t_0 that for $\tau > t_0$ each of the sets $S_i(\tau)$* ($i = 0, 1, \ldots, p$) *is of type* I_{n_i}.

First we prove two lemmas needed for the proof of Theorem C. Those lemmas concern the system

$$\text{(I)} \qquad X' = F(X, t) \qquad (X = (Y, Z))$$

It is supposed that $F(X, t)$ satisfies 1° of K.

Lemma 2. *Suppose that the function $g(u, t) > 0$ for $u > 0$ has continuous derivatives of the first order for $t \geqslant t_0 \geqslant t_1$ and for arbitrary u. Assume that the inequality*

$$(*) \quad 2(0, Z)\cdot F(Y, Z, t) - 2(Y, 0)\cdot F(Y, Z, t)g'_u(Y^2, t) - g'_t(Y^2, t) > 0$$

holds on the surface $\mathfrak{M}$ defined by the equation $Z^2 = g(Y^2, t)$ and for $t \geqslant t_0$. Then, for each $\tau > t_0$ and for arbitrary Y_0 there exists such a solution $X(t) = \big(Y(t), Z(t)\big)$ of (I) *that $Y(\tau) = Y_0$, and $Z^2(t) < g\big(Y^2(t), t\big)$ for $t \geqslant \tau$.*

Proof. We write $(Y, Z, t) = P$ and $\psi(P) = Z^2 - g(Y^2, t)$. Denote by ω such a set of points P that $\psi(P) < 0$. Let us find the derivative $\dot{\psi}_{(\mathrm{I})}(P)$ for $P \in \mathfrak{M}$ along the solution of (I). We obtain

$$\dot{\psi}_{(\mathrm{I})}(P) = 2Z\cdot Z' - 2g'_u(Y^2, t)\, YY' - g'_t(Y^2, t).$$

Owing to $(*)$ and from the relations $Y\cdot Y' = (Y, 0)\cdot F(P)$, $Z\cdot Z' = (0, Z)\cdot F(P)$ we get the inequality $\dot{\psi}_{(\mathrm{I})}(P) > 0$ for $P \in \mathfrak{M}$. This means that each point of $\mathfrak{M}$ is a point of strict egress from ω. Using Ważewski's results, just as has been done in the proof of Lemma 1, we conclude that for every $\tau > t_0$ and every Y_0 there exists at least one Z_0 such that at least one solution of (I) issuing from the point (Y_0, Z_0, τ) remains in ω for $t \geqslant \tau$.

Taking into account the definition of ω we find lemma 2 proved.

Lemma 3. *Let us suppose that the solutions originating on the hyperplane $t = t_1$ are determined for $t \geqslant t_1$. Suppose that there exist numbers λ and μ such that*

$$5^\circ \qquad \frac{s}{1-q} < \lambda < \mu.$$

We assume moreover that

$$6^\circ \qquad (Y, 0)\cdot F(X, t) \leqslant \lambda Y^2 + |Y|\,|X|\,\chi(t) + |Y|\,|X|^q w(t),$$

$$7^\circ \qquad (0, Z)\cdot F(X, t) > \mu Z^2 - |Z|\,|X|\,\chi(t) - |Z|\,|X|^q w(t) \qquad \textit{for} \quad Z \neq 0,$$

where $\chi(t)$, $w(t)$ satisfy the assumptions 2°-3° of H *and* $0 \leqslant q < 1$.

We then assert what follows:

(a) *there is such a number t_0 that for $\tau > t_0$ and every Y_0 there exists at least one solution $X(t) = \big(Y(t), Z(t)\big)$ of* (I) *such that $Y(\tau) = Y_0$ and* $\mathrm{Ce}\,X(t) \leqslant \lambda$,

(b) *if the* Ce *of any other solution of* (I) *is greater than λ, then it is equal to μ at least.*

Proof. By 5° there exists a σ satisfying the inequality

$$(4,3) \qquad \frac{s}{1-q} < \sigma < \lambda.$$

Put

$$(4,4) \qquad k_1(t) = 10\chi_1(t), \qquad k_2(t) = 10\{\chi_2(t) + w(t)e^{-(1-q)\sigma t}\}.$$

By assumption 2° of H we get

$$(4,5) \qquad \lim_{t\to+\infty} k_1(t) = 0$$

and by 3° of H, (4,3) and Theorem (1,3)

$$(4,6) \qquad \mathrm{Ce}\int_t^\infty w(\tau)e^{-(1-q)\sigma\tau}\,d\tau < 0.$$

Hence using once more assumption 2° of H concerning $\chi_2(t)$ we have

$$(4,7) \qquad \int_t^\infty k_2(\tau)\,d\tau < +\infty.$$

[5] We say that E^{n-k} is a *complementary space to* E^k if E^n is a direct sum of E^k and E^{n-k}. The hyperplane is parallel to E^k if it is obtained by a translation of E^k along a suitable vector belonging to E^{n-k}.

[6] If $s = -\infty$ then $v_0 = -\infty$. In that case $r \leqslant v_0$ means that $r = -\infty$.

Now let us write

$$h(t) = e^{\int_t^\infty k_2(\tau)d\tau} \tag{4,8}$$

$$g(u, t) = h^2(t)(u + e^{2\sigma t}) \tag{4,9}$$

and let $t_2 \geqslant t_1$ be chosen in such a manner that

$$k_1(t) \leqslant \mu - \lambda, \qquad h(t) \leqslant 2 \quad \text{for} \quad t \geqslant t_2. \tag{4,10}$$

Denote by ω the set of points satisfying the inequality

$$\omega\colon \quad Z^2 < g(Y^2, t), \qquad t \geqslant t_2 \tag{4,11}$$

and by $\mathfrak{M}$ the boundary of ω

$$\mathfrak{M}\colon \quad Z^2 = g(Y^2, t), \qquad t \geqslant t_2. \tag{4,12}$$

We shall prove that $g(u, t)$ satisfies the assumptions of Lemma 2. In order to show that assumption $(*)$ (of Lemma 2) is satisfied we put

$$\Gamma(P) = 2[(0, Z)\cdot F(P) - (Y, 0)\cdot F(P) g_u'(Y^2, t)] - g_t'(Y^2, t).$$

We have to prove that $\Gamma(P) > 0$ on $\mathfrak{M}$.

It easily follows from (4,9) and assumptions 6° and 7° that the inequality

$$\Gamma(P) > 2\{\mu Z^2 - |Z|\,|X|\chi(t) - |Z|\,|X|^q w(t) - h^2(t)[\lambda Y^2 + |Y|\,|X|\chi(t) +$$
$$+ |Y|\,|X|^q w(t)] - h(t)h'(t)(Y^2 + e^{2\sigma t}) - h^2(t)\sigma e^{2\sigma t}\}$$

holds when $(Y, Z, t)\in D$ and $Z \neq 0$. But (4,8) implies that $h(t) \geqslant 1$. Hence on $\mathfrak{M}$ we have (see (4,12) and (4,9))

$$|Z| > |Y|, \; |Z| \geqslant e^{\sigma t}, \; |X| \leqslant |Y| + |Z| < 2|Z|, \; h^2(t)Y^2 = Z^2 - h^2(t)e^{2\sigma t}.$$

From the last inequality and from (4,3) we conclude that

$$\mu Z^2 - \lambda h^2(t)Y^2 - \sigma h^2(t)e^{2\sigma t} = \mu Z^2 - \lambda Z^2 + (\lambda - \sigma)h^2(t)e^{2\sigma t} > (\mu - \lambda)Z^2.$$

From the previous inequalities and (4,10), (4,4) we have

$$-|Z|\,|X|\chi(t) - h^2(t)|Y|\,|X|\chi(t) \geqslant -2Z^2\big(1 + h^2(t)\big)\chi(t)$$
$$\geqslant -10Z^2[\chi_1(t) + \chi_2(t)] = -k_1(t)Z^2 - 10Z^2\chi_2(t) \geqslant -(\mu - \lambda)Z^2 - 10Z^2\chi_2(t)$$

and similarly

$$-|Z|\,|X|^q w(t) - h^2(t)|Y|\,|X|^q w(t) \geqslant -2|Z|^{1+q}w(t)\big(1 + h^2(t)\big)$$
$$\geqslant -10w(t)|Z|^{1+q} \geqslant -10w(t)Z^2 e^{-(1-q)\sigma t}.$$

Summing the above inequalities and using the relation

$$h(t)h'(t)(Y^2 + e^{2\sigma t}) = \frac{Z^2}{h(t)}h'(t)$$

we get

$$\frac{1}{2}\Gamma(P) > \frac{Z^2}{h(t)}[-k_2(t)h(t) - h'(t)].$$

The last inequality together with (4,8) implies that for $P\in\mathfrak{M}$ we have $\Gamma(P) > 0$.

As follows from Lemma 2, part (a) of Lemma 3 will be completely proved if we show that the Ce of any solution of (I) which remains in ω for $t \geqslant t_2$ is equal to λ at most. Solutions of that kind satisfy the following inequality:

$$Z^2(t) < h^2(t)\{Y^2(t) + e^{2\sigma t}\}. \tag{4,13}$$

By (4,10) ($h(t) \leqslant 2$) we have

$$|Z| < 2(|Y| + e^{\sigma t}), \; |X| < 3|Y| + 2e^{\sigma t} \quad \text{for} \quad (Y, Z, t)\in\omega. \tag{4,14}$$

Therefore by assumption 6° we obtain

$$(Y, 0)\cdot F(P) \leqslant \lambda Y^2 + |Y|(3|Y| + 2e^{\sigma t})\chi(t) + + w(t)|Y|(3|Y| + 2e^{\sigma t})^q. \tag{4,15}$$

Write

$$\tfrac{1}{2}f(u, t) = \big(\lambda + 3\chi(t)\big)u + 6w(t)u^p + \big(2e^{\sigma t}\chi(t) + 4w(t)e^{q\sigma t}\big)u^{1/2}, \quad p = \frac{1+q}{2}. \tag{4,16}$$

From (4,15) and (4,16) and from the inequality $0 \leqslant q < 1$ we get

$$2(Y, 0)\cdot F(P) \leqslant f(Y^2, t)\,(^7).$$

Hence, if the solution $X(t) = \big(Y(t), Z(t)\big)$ remains in ω for $t \geqslant t_2$ then the following inequality is satisfied:

$$\frac{dY^2(t)}{dt} \leqslant f\big(Y^2(t), t\big). \tag{4,17}$$

Now let us consider the scalar equation

$$u' = f(u, t). \tag{4,18}$$

(7) Here we make use of the inequality $(a+b)^q \leqslant 2(a^q + b^q)$ provided that $a > 0$, $b > 0$, $0 \leqslant q < 1$.

Equation (4,18) has been investigated in Example 3 (§ 1). By (4,3) we have $s+\sigma q<\sigma$. Using the same notation as in Example 3 we infer on the basis of Theorems (1,2), (1,3) (see § 1) and owing to assumptions 2° and 3° of H that

$$a=2\lambda,\quad b=s,\quad c=\sigma<\lambda,\quad \varrho=p=\frac{1+q}{2},\quad r=1/2\,(^8).$$

Hence by (4,3) we have

$$b-a(1-\varrho)=s-2\lambda\frac{1-q}{2}<0,\qquad c-a(1-r)=\sigma-\lambda<0.$$

It has been shown in § 1 that every positive solution of (4,18) has its Ce equal to 2λ.

If $\varphi(t)$ is a maximum solution of (4,18) such that $Y^2(t_2)<\varphi(t_2)$ then, owing to (4,17), we have $Y^2(t)<\varphi(t)$ for $t\geqslant t_2$ (⁹). This means that $\mathrm{Ce}|Y(t)|\leqslant\lambda$. Inequalities (4,14) and the inequality $\sigma<\lambda$ imply that $\mathrm{Ce}|Z(t)|\leqslant\lambda$. Therefore $\mathrm{Ce}|X(t)|\leqslant\lambda$. This completes the proof of part (a) of Lemma 3.

Now we are going to prove part (b) of Lemma 3. Suppose there exists a solution $X(t)$ of (I) such that $\mathrm{Ce}\,X(t)>\lambda$. $X(t)$ cannot be contained in ω. Since every point $P\in\mathfrak{M}$ is a point of strict egress from ω, there exists such a $t_3\geqslant t_2$ that for $t\geqslant t_3$ we have

$$Z^2(t)>h^2(t)\{Y^2(t)+e^{2\sigma t}\}. \tag{4,19}$$

Since $h(t)\geqslant 1$, (4,19) implies inequality

$$|Y(t)|<|Z(t)|\quad\text{for}\quad t\geqslant t_3. \tag{4,20}$$

(⁸) To prove that $c=\sigma$ we have to show that

(i) $$\mathrm{Ce}\Big\{\int_{t_1}^{t}2e^{\sigma\tau}\chi(\tau)\,d\tau+\int_{t_1}^{t}4w(\tau)e^{q\sigma\tau}\,d\tau\Big\}\leqslant\sigma$$

or

(ii) $$\mathrm{Ce}\Big\{\int_{t}^{\infty}2e^{\sigma\tau}\chi(\tau)\,d\tau+\int_{t}^{\infty}4w(\tau)e^{q\sigma\tau}\,d\tau\Big\}\leqslant\sigma.$$

There are three cases: $s+\sigma q<\sigma<0$, $s+\sigma q<0\leqslant\sigma$, $0\leqslant s+\sigma q<\sigma$. It follows from Theorems (1,2) and (1,3) that in the first case (ii) is true, and in the other cases (i) holds. In the second case we have to note that $\mathrm{Ce}\int_{t_1}^{t}4w(\tau)e^{q\sigma\tau}\,d\tau\leqslant 0$.

(⁹) One can replace the term "maximum solution" by "solution" according to the uniqueness property.

Now let us write system (I) in the form

$$Y'=F_1(Y,Z,t),\qquad Z'=F_2(Y,Z,t)\qquad (F=(F_1,F_2))$$

and write $K(Z,t)=F_2\big(Y(t),Z,t\big)$. Let the set R of points (Z,t) be defined by the inequalities

$$R:\quad |Z|>|Y(t)|,\quad t\geqslant t_3.$$

Now put, in 7°, $X=\big(Y(t),Z\big)$. Owing to the inequality $|X|<2|Z|$ we infer that the inequality

$$Z\cdot K(Z,t)>\big(\mu-2\chi(t)\big)Z^2-2|Z|^{1+q}w(t) \tag{4,21}$$

holds for $(Z,t)\in R$.

$Z(t)$ is a solution of the following differential equation

$$Z'=K(Z,t). \tag{4,22}$$

By (4,20) the curve $\big(Z(t),t\big)$ is contained in R for $t\geqslant t_3$. From (4,19) and (4,3) it follows that

$$\mathrm{Ce}\,Z(t)\geqslant\sigma>\frac{s}{1-q}. \tag{4,23}$$

Applying Theorem B to (4,22), according to (4,21) and (4,23) we conclude that $\mathrm{Ce}\,Z(t)\geqslant\mu$. Therefore $\mathrm{Ce}\,X(t)\geqslant\mu$. This completes the proof of Lemma 3.

Proof of Theorem C. Let $X(t)$ be an arbitrary solution of (II) and let $r=\mathrm{Ce}\,X(t)$. Write

$$\varepsilon_0=\frac{1}{3}\min_i(v_i-v_{i-1})\qquad (i=1,2,\ldots,p). \tag{4,24}$$

First we prove part (a) of Theorem C. In order to do so it suffices to show that given any $\varepsilon>0$

$$0<\varepsilon\leqslant\varepsilon_0 \tag{4,25}$$

the inequality

$$r\leqslant v_p+\varepsilon \tag{4,26}$$

holds, and in the case where $v_0<\varrho_k$ the following alternatives are satisfied:

$$r\leqslant v_i+\varepsilon\quad\text{or}\quad r\geqslant v_{i+1}-\varepsilon\qquad (i=0,1,\ldots,p-1). \tag{4,27}$$

Let ε satisfy (4,25) and let the non-singular linear transformation T_ε $(X^*=T_\varepsilon X)$, change equation (II) into the following one:

$$X^{*\prime}=A^*X^*+B^*(X^*,t)+C^*(X^*,t). \tag{II*}$$

The matrix A^* is in the Jordan canonical form (see § 3) with its underdiagonal numbers not exceeding ε. If $X^*(t) = T_s X(t)$ and $X(t)$ is a solution of (II), then $X^*(t)$ is a solution of (II*) and $\mathrm{Ce}\, X(t) = \mathrm{Ce}\, X^*(t)$.

It is easy to see that equation (II*) satisfies the assumptions of Theorem C. In particular, there are the functions $\chi^*(t), w^*(t)$, which play the same role with respect to B^* and C^* as $\chi(t)$, $w(t)$ with respect to B and C. q and s have the same values, whence the sequence (4,2) is not changed.

Now we prove (4,26) and afterwards (4,27).

Owing to Remark (3,1) we have

$$X^*\cdot X^{*\prime} \leqslant \big(\varrho_k+\varepsilon+\chi^*(t)\big)X^{*2}+w^*(t)\,|X^*|^{1+q}.$$

This means that (II*) satisfies the assumptions of Theorem A. We therefore get

$$\mathrm{Ce}\, X^*(t) \leqslant \max\{\varrho_k+\varepsilon, v_0\}. \tag{4,29}$$

Since $v_p = \varrho_k$ or $v_p = v_0$, (4,28) and (4,29) imply (4,26).

Suppose that $v_0 < \varrho_k$. We shall prove first the following particular case of (4,27): if $v_0 < \varrho_1$ then

$$r \leqslant v_0 \quad \text{or} \quad r \geqslant v_1-\varepsilon. \tag{4,30}$$

On account of (4,25) and (4,24) we have $v_0 < \varrho_1-\varepsilon$ and $\varrho_1 = v_1$. Owing to Remark (3,1) we have

$$X^*\cdot X^{*\prime} > \big(\varrho_1-\varepsilon-\chi^*(t)\big)X^{*2}-w^*(t)\,|X^*|^{1+q} \quad \text{for} \quad X^* \neq 0.$$

This inequality shows that (4,30) is a consequence of Theorem B.

In the remaining cases of (4,27) we have $n_i \neq 0$ and $n_i \neq n$; thus we can put $X = (Y_i, Z_i)$, $Y_i = (x_1, x_2, \ldots, x_{n_i})$, $Z_i = (x_{n_i+1}, \ldots, x_n)$.

The integer i assumes the values

$$\begin{cases} 1, 2, \ldots, p-1 & \text{if} \quad v_0 < \varrho_1, \\ 0, 1, \ldots, p-1 & \text{if} \quad \varrho_1 \leqslant v_0 < \varrho_k. \end{cases} \tag{4,31}$$

Let j denote a fixed value of i. Matrix A may be written as follows:

$$\begin{Vmatrix} A_j^1 & \\ & A_j^2 \end{Vmatrix}$$

where A_j^1 is an $n_j\times n_j$ matrix. Similarly we write equation (II*) in the form

$$Y_j' = A_j^1 Y_j+B_j^1(Y_j, Z_j, t)+C_j^1(Y_j, Z_j, t),$$
$$Z_j' = A_j^2 Z_j+B_j^2(Y_j, Z_j, t)+C_j^2(Y_j, Z_j, t),$$

or shortly $(Y_j', Z_j') = F(Y_j, Z_j, t)$. Here by $B_j^1(P)$ and $C_j^1(P)$ we denote the sequences of first n_j coordinates of vectors $B^*(P)$ and $C^*(P)$ respectively and

$$B^*(P) = \big(B_j^1(P), B_j^2(P)\big), \qquad C^*(P) = \big(C_j^1(P), C_j^2(P)\big).$$

From the assumptions of Theorem C and from Remark (3,1) it follows that

$$\begin{cases} (Y_j, 0)\cdot F(Y_j, Z_j, t) \leqslant (v_j+\varepsilon)\,Y_j^2+|Y_j|\,|X^*|\,\chi^*(t)+ \\ \qquad\qquad +|Y_j|\,|X^*|^q w^*(t), \\ (0, Z_j)\cdot F(Y_j, Z_j, t) > (v_{j+1}-\varepsilon)Z_j^2-|Z_j|\,|X^*|\,\chi^*(t)-|Z_j|\,|X^*|^q w^*(t). \end{cases} \tag{4,32}$$

Inequalities (4,32) are satisfied in D, the second of them whenever $Z_j \neq 0$. By (4,24), (4,25), and (4,31) we have

$$\frac{s}{1-q} = v_0 < v_j+\varepsilon < v_{j+1}-\varepsilon.$$

Hence (II*) satisfies the assumptions of Lemma 3 ($\lambda = v_j+\varepsilon$, $\mu = v_{j+1}-\varepsilon$). We therefore obtain $r \leqslant v_j+\varepsilon$ or $r \geqslant v_{j+1}-\varepsilon$. j may be chosen arbitrarily from the set (4,31). Hence the alternatives (4,27) are satisfied and part (a) of Theorem C is completely proved.

Remark (4,1). Theorem B (§ 2) implies that in the case where $v_0 < \varrho_1$ there exists at least one solution of (II*) for which the inequality $r \leqslant v_0$ holds. In particular if $w(t) \equiv 0$ then (II*) possesses a trivial solution and the Ce of any other one is equal to some member of the sequence: $\varrho_1, \varrho_2, \ldots, \varrho_k$.

The proof of part (b) of Theorem C results from the remarks given below.

A non-singular linear transformation maps a set of type I_i on a set of the same type. It is sufficient therefore to prove (b) for (II*) assuming that $\varepsilon_i \leqslant \varepsilon_0$.

It follows from part (a) of Theorem C and from (4,24) that the condition $r \leqslant v_i$ is equivalent to $r \leqslant v_i+\varepsilon_0$ for $i = 0, 1, \ldots, p$. Hence $S_i(\tau)$ is a set of such X-es that there exists a solution passing through (X, τ) whose Ce does not exceed $v_i+\varepsilon_0$.

If $n_i \neq n$ and $n_i \neq 0$ then we have one of the cases of (4,31); thus X may be written as (Y_i, Z_i) and (II*) satisfies the assumptions of Lemma 3. From part (a) of the assertion of that lemma as applied to equation (II*) ($\lambda = v_i+\varepsilon_0$) it follows that there is such a t_{2i} that for $\tau \geqslant t_{2i}$ and for each Y_i there exists at least one Z_i such that $(Y_i, Z_i, \tau) \in S_i(\tau)$.

If $n_i = 0$ then $i = 0$ and $v_0 < \varrho_1$, and from Remark (4,1) we conclude that $S_0(\tau)$ is of type I_0 for $\tau \geqslant t_1$.

If $n_i = n$ then $i = p$, and by (4,26) we infer that $S_p(\tau)$ is of type I_n for $\tau \geqslant t_1$.

It follows from these remarks that assertion (b) holds whenever we put $t_0 = \max_i t_{2i}$. Thus Theorem C is proved.

§ 5. We shall consider in this section the system

$$\text{(III)} \qquad X' = AX + B(X, t).$$

ASSUMPTION H_1. 1° $B(X, t)$ satisfies assumption 1° of H.

2° There exists a function $\chi(t)$ satisfying assumption 2° of H.

3° $B(0, t) = 0$ for $t \geqslant t_1$,

4° $|B(\overline{X}, t) - B(\overline{\overline{X}}, t)| \leqslant |\overline{X} - \overline{\overline{X}}|\,\chi(t)$.

Grobman (see [2] Theorem 5, p. 141, and corollary p. 142) has proved that if Assumption H_1 is satisfied, then each set $S_i(\tau)$ is an n_i-dimensional set; n_i — denotes here the number of characteristic roots whose real parts are not greater than ϱ_i (each r-fold characteristic root is counted r times). This result may be formulated in the following slightly generalized form:

THEOREM (5,1). *If $B(X, t)$ satisfies Assumption H_1 then there exists such a t_0 that the following conditions hold:*

1° *each of the sets $S_i(\tau)$ $(i = 1, 2, \ldots, k-1)$ is a surface of type I_{n_i} for $\tau \geqslant t_0$.*

2° *for every $\tau > t_0$ there exists a continuous and one-one transformation T mapping the hyperplane $t = \tau$ on itself and such that the solution of* (III) *issuing from the point (X, τ) and the solution of the corresponding linear equation*

$$\text{(5,1)} \qquad X' = AX$$

issuing from (TX, τ) have equal characteristic exponents.

Proof. Following Grobman let us consider the integral equation

$$\text{(5,2)} \qquad X(t) = W(t)C^i + \int_{\tau}^{t} W_{n_i}(t-s)B\big(X(s), s\big)\,ds + \int_{+\infty}^{t} W_{n-n_i}(t-s)B\big(X(s), s\big)\,ds$$

where $W(t)$ is a so called normal matrix of (5,1)(10).

(10) A matrix $W(t)$ is called a *normal matrix* of (5,1) if its columns give us a system of n linearly independent solutions of (5,1) and, moreover, if the characteristic exponents of the first l_1 columns are equal to ϱ_1, of the next l_2 — to ϱ_2 and so on, and the characteristic exponents of the last l_k columns are equal to ϱ_k. l_i denotes here the number of characteristic roots of A with real parts equal to ϱ_i (each r-fold root is counted r times).

The matrices $W_{n_i}(t)$ and $W_{n-n_i}(t)$ are obtained from $W(t)$; the former by replacing the last $n-n_i$ columns by zeros and the latter by replacing the first n_i columns by zeros. C^i is an n-dimensional vector of the form $C^i = (c_1, c_2, \ldots, c_{n_i}, 0, 0, \ldots, 0)$.

Grobman has proved (see [2], Lemma 3, p. 132 and p. 136) that for every $i = 1, 2, \ldots, k$ and for every vector C^i there exists exactly one solution of (5,2) with its Ce equal to or less than ϱ_i which depends continuously on C^i. According to the notation adopted in this paper we put $C^i = (Y_i, 0)$, $Y_i = (c_1, c_2, \ldots, c_{n_i})$ and denote by $X(Y_i, t)$ the solution of (5,2) which corresponds to C^i. Then we have $\mathrm{Ce}\,X(Y_i, t) \leqslant \varrho_i$ for every Y_i.

It suffices to prove Theorem (5,1) in the case where A is in the Jordan canonical form. In this case the matrix $e^{A(t-\tau)}$ is a normal one with respect to (5,1). Let us replace in (5,2) $W(t)$ by $e^{A(t-\tau)}$. From (5,2) it follows that for $i < k$ $X(Y_i, \tau)$ has the form $X(Y_i, \tau) = (Y_i, 0) + \big(0, d^i(Y_i)\big)$. d^i is an $n-n_i$ dimensional vector function which depends continuously on Y_i. Hence

$$\text{(5,3)} \qquad X(Y_i, \tau) = \big(Y_i, d^i(Y_i)\big).$$

This means that $S_i(\tau)$ $(i = 1, 2, \ldots, k-1)$ may be considered as a diagram of a continuous vector function defined on the subspace E^{n_i}. Therefore $S_i(\tau)$ is a surface of type I_{n_i}.

This completes the proof of part 1° of Theorem (5,1).

In order to prove part 2° let us define by induction the transformations $T^{k-1}, T^{k-2}, \ldots, T^1$. Put

$$T^{k-1}X = X - \big(0, d^{k-1}(Y_{k-1})\big).$$

Owing to (5,3) we have $T^{k-1}X = (Y_{k-1}, 0)$ for $X \in S_{k-1}(\tau)$ and $T^{k-1}S_{k-1}(\tau) = E^{n_{k-1}}$. Since $S_i(\tau) \subset S_j(\tau)$ for $i < j$, we get

$$T^{k-1}X = \big(Y_{k-2}, h^{k-2}(Y_{k-2}), 0\big) \quad \text{for} \quad X \in S_{k-2}(\tau),\ k > 2,$$

where h^{k-2} is an $n_{k-1} - n_{k-2}$ dimensional continuous vector function composed of the first $n_{k-1} - n_{k-2}$ coordinates of $d^{k-2}(Y_{k-2})$.

Now let us suppose that T^i $(i > 1)$ is already defined and that it satisfies the following conditions: $T^i S_j(\tau) = E^{n_j}$ for $i \leqslant j \leqslant k-1$ and $T^i X = \big(Y_{i-1}, h^{i-1}(Y_{i-1}), 0\big)$ for $X \in S_{i-1}(\tau)$, where $h^{i-1}(Y_{i-1})$ is a continuous $n_i - n_{i-1}$ dimensional vector function. The transformation T^{i-1} is defined as follows:

$$T^{i-1}X = T^i X - \big(0, h^{i-1}(Y_{i-1}), 0\big)$$

(zeros are the first n_{i-1} and the last $n - n_i$ coordinates of that vector). It is clear that $T^{i-1}S_j(\tau) = E^{n_j}$ for $i-1 \leqslant j \leqslant k-1$ and if $i > 2$ then

$T^{i-1}X = \big(Y_{i-2}, h^{i-2}(Y_{i-2}), 0\big)$ for $X \in S_{i-2}(\tau)$, and if $i = 2$ then $T^1(0) = 0$. In this way we shall finally obtain a non-singular transformation $T_1 \overset{\mathrm{df}}{=} T$ which maps the hyperplane $t = \tau$ on itself. Each of the sets $S_i(\tau)$ is mapped by T on E^{n_i} $(i = 1, 2, \ldots, k-1)$, and $(0, 0, \ldots, 0)$ is a fixed point of T.

Suppose that $X(t)$ is a solution of (III) and that $\mathrm{Ce}\, X(t) = \varrho_i$. Since $X(\tau) \in S_i(\tau)$ and $X(\tau) \notin S_j(\tau)$ for $j < i$, we have $TX(\tau) \in E^{n_i}$ and $TX(\tau) \notin E^{n_j}$ for $j < i$. The solution of (5,1) issuing from $\big(TX(\tau), \tau\big)$ is of the form $e^{A(t-\tau)}(Y_i, 0)$ where $Y_i \in E^{n_i}$ and $Y_i \notin E^{n_j}$ for $j < i$. Hence its characteristic exponent is equal to ϱ_i.

This completes the proof of part 2° of Theorem (5,1).

The following theorem is a generalization of Theorem (5,1):

THEOREM (5,2). *If the function $B(X, t)$ satisfies assumptions* 1°, 2° and 4° of H_1 *and if there exists a constant s such that*

$$(5,4) \qquad \mathrm{Ce} \int_{t_1}^{t} |B(0, \tau)|\, d\tau \leqslant s \quad \text{or} \quad \mathrm{Ce} \int_{t}^{\infty} |B(0, \tau)|\, d\tau \leqslant s,$$

then there exists a number t_0 such that for $\tau > t_0$ the following conditions hold:

1° *Each of the sets $S_i(\tau)$ is a surface of type I_{n_i} (except the cases where $S_i(\tau)$ is a single point or covers the whole space).*

2° *There exists a one-one continuous transformation T mapping the hyperplane $t = \tau$ on itself and such that the solution of* (III) *issuing from (X, τ) and the solution of the linear equation* (5,1) *issuing from (TX, τ) have characteristic exponents which either are equal to each other or do not exceed v_0.*

Proof. Equation (III) is a particular case of (II) and it satisfies Assumption H where $q = 0$. Moreover $w(t) = |B(0, t)|$ and owing to Theorem C there exists a solution $X_0(t)$ of (III) such that its Ce is not greater than v_0. Let $X(t)$ be an arbitrary solution of (III). Now put

$$(5,5) \qquad X(t) = X_0(t) + \xi(t).$$

The vector-valued function $\xi(t)$ is a solution of the equation

$$(5,6) \qquad X' = AX + B^*(X, t),$$

where

$$B^*(X, t) = B\big(X + X_0(t), t\big) - B\big(X_0(t), t\big).$$

We have $B^*(0, t) \equiv 0$. Hence (5,6) satisfies the assumptions of Theorem (5,1). Therefore relation (5,5) establishes a one-one correspondence between the solutions of (III) and that of the linear equation (5,1).

From Theorem (1,2) and from Definition (1,2) it follows that

$$\mathrm{Ce}\, X(t) = \mathrm{Ce}\, |X(t)| = \mathrm{Ce}\, |X_0(t) + \xi(t)| \leqslant \max\{\mathrm{Ce}\, |X_0(t)|, \mathrm{Ce}\, |\xi(t)|\}$$
$$\text{and } \mathrm{Ce}\, X(t) = \mathrm{Ce}\, \xi(t) \text{ if } \mathrm{Ce}\, \xi(t) > \mathrm{Ce}\, X_0(t).$$

The remarks given above immediately imply Theorem (5,2).

Finally let us consider the linear equation

$$(\mathrm{L}) \qquad X' = \big(A + V(t)\big)X + W(t)$$

where the matrix $V(t)$ and vector function $W(t)$ are continuous for $t \geqslant t_1$ and satisfy the following conditions:

$$V(t) = V_1(t) + V_2(t), \quad \lim_{t \to +\infty} V_1(t) = 0, \quad \int_t^\infty \|V_2(\tau)\|\, d\tau < +\infty,$$

$$\mathrm{Ce} \int_{t_1}^{t} |W(\tau)|\, d\tau \leqslant s \quad \text{or} \quad \mathrm{Ce} \int_{t}^{\infty} |W(\tau)|\, d\tau \leqslant s.$$

Equation (L) is a particular case of (III) considered above $(B(X, t) = V(t)X + W(t))$. From Theorem (5,2) it follows in particular that (L) possesses an n_i-parameter family of solutions whose characteristic exponents do not exceed v_i $(i = 0, 1, \ldots, p)$.

This result is a generalization of some results of T. Peyovitch [4].

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