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Remarks on some functional equations

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§ 1. In my preceding paper [4] I dealt with the functional equation

$$\varphi(x)+\varphi[f(x)] = F(x), \tag{1}$$

where $\varphi(x)$ denotes the required function and $f(x)$ and $F(x)$ are known functions. I proved the following theorem:

If the function $f(x)$ is continuous and strictly increasing in an interval: $\langle a, b\rangle$, and $f(x) > x$ in (a, b), $f(a) = a$, $f(b) = b$, then equation (1) possesses at most one solution that is continuous in the interval $(a, b\rangle$.

Let us denote by $f^n(x)$ the n-th iteration of the function $f(x)$:

$$f^0(x) = x, \quad f^{n+1}(x) = f[f^n(x)].$$

The above theorem may be generalized as follows:

Theorem I. *If the function $f(x)$ is continuous and strictly increasing in an interval $(a, b\rangle$, and $f(x) > x$ in (a, b), $f(b) = b$, then equation (1) possesses in the interval (a, b) at most one solution $\varphi(x)$, fulfilling the condition*

$$\lim_{n\to\infty}\{\varphi[f^n(x)]-\varphi[f^{n-1}(x)]\} = 0 \tag{2}$$

for every $x \in \langle\bar{x}, f(\bar{x})\rangle$, where $\bar{x}$ is a number from the interval (a, b).

Proof. Let $\varphi_1(x)$ and $\varphi_2(x)$ be two solutions of equation (1) fulfilling condition (2). The difference of these solutions

$$\varrho(x) = \varphi_1(x)-\varphi_2(x)$$

also fulfils condition (2) and satisfies the homogeneous equation

$$\varrho(x)+\varrho[f(x)] = 0. \tag{3}$$

Let us take an arbitrary $x \in \langle\bar{x}, f(\bar{x})\rangle$ and let us write

$$c = \varrho(x).$$

According to (3) we have (by induction):

$$\varrho[f^n(x)] = (-1)^n c,$$

whence
$$\varrho[f^n(x)]-\varrho[f^{n-1}(x)] = 2(-1)^n c.$$
Since $\varrho(x)$ fulfils condition (2), we must have $c = 0$. Then for every $x \in \langle \bar{x}, f(\bar{x})\rangle$ $\varrho(x) = 0$, and thus
$$\varphi_1(x) \equiv \varphi_2(x) \quad \text{for} \quad x \in \langle \bar{x}, f(\bar{x})\rangle.$$
Since (see [4]) every solution of equation (1) is unambiguously determined by its values from the interval $\langle \bar{x}, f(\bar{x})\rangle$, we have
$$\varphi_1(x) \equiv \varphi_2(x) \quad \text{for} \quad x \in (a, b),$$
which was to be proved.

Every function $\varphi(x)$ continuous in the interval $(a, b\rangle$ fulfils condition (2)(1). Therefore the theorem quoted at the beginning of this paper is a simple consequence of theorem I.

Condition (2) may seem a little artificial. Theorem I enables us, however, to prove some more natural conditions for the solutions of equation (1) to be unique. We shall prove the following:

THEOREM II. *Let x_0 be a number from the interval (a, b) and let us put:*
$$x_n \overset{\text{df}}{=} f^n(x_0), \qquad n = 0, 1, 2, \ldots$$
If the function $f(x)$ fulfils the hypotheses of theorem I, *and the function $F(x)$ fulfils the condition*

(4) $$\lim_{n\to\infty} \{F(x_n)-F(x_{n-1})\} = 0,$$

then equation (1) *possesses at most one solution monotonic in the interval $(b-\eta, b)$, where η is an arbitrary positive number.*

Proof. We have by relation (1):
$$F(x_n)-F(x_{n+1}) = \varphi(x_n)+\varphi(x_{n+1})-\varphi(x_{n+1})-\varphi(x_{n+2}) = \varphi(x_n)-\varphi(x_{n+2}),$$
whence by (4):

(5) $$\lim_{n\to\infty} \{\varphi(x_n)-\varphi(x_{n+2})\} = 0.$$

Since $\lim\limits_{n\to\infty} x_n = b$, there exists an index N such that $x_N \in (b-\eta, b)$. Let us take an arbitrary $x \in \langle x_N, x_{N+1})$ and let $\varphi(x)$ be a solution of equation (1) that is monotonic in $(b-\eta, b)$. The functions $f^n(x)$ are increasing, like $f(x)$. Hence we have
$$f^n(x_N) \leqslant f^n(x) \leqslant f^n(x_{N+1}),$$

(1) Under the assumptions of theorem I the sequence $f^n(x)$ is (for every $x \in (a, b)$) increasing and converges to b.

i. e.

(6) $$x_{N+n} \leqslant f^n(x) \leqslant x_{N+1+n}.$$

Similarly

(7) $$x_{N+1+n} \leqslant f^{n+1}(x) \leqslant x_{N+2+n}.$$

The points $x_{N+n}, x_{N+n+2}, f^n(x), f^{n+1}(x)$ $(n = 0, 1, 2, \ldots)$ belong to the interval $(b-\eta, b)$. Then from (6) and (7) follows either

(8) $$\varphi(x_{N+n}) \leqslant \varphi[f^n(x)] \leqslant \varphi[f^{n+1}(x)] \leqslant \varphi(x_{N+n+2}),$$

or

(9) $$\varphi(x_{N+n}) \geqslant \varphi[f^n(x)] \geqslant \varphi[f^{n+1}(x)] \geqslant \varphi(x_{N+n+2}).$$

From (8) and (9) follows
$$|\varphi[f^n(x)]-\varphi[f^{n+1}(x)]| \leqslant |\varphi(x_{N+n})-\varphi(x_{N+n+2})|,$$
whence we have by (5):

(10) $$\lim_{n\to\infty} \{\varphi[f^{n+1}(x)]-\varphi[f^n(x)]\} = 0.$$

Since x has been an arbitrary point of the interval $\langle x_N, x_{N+1})$, relation (10) holds for every $x \in \langle x_N, x_{N+1})$. From theorem I it follows that there may exist at most one such solution.

Every function convex in the interval (a, b) is monotonic in an interval $(b-\eta, b)$. Thus we have

COROLLARY. *Under the hypotheses of theorem* II *equation* (1) *possesses at most one solution convex in the interval* (a, b).

§ 2. Now we shall consider a special case:
$$f(x) = x+1.$$
Then $f^n(x) = x+n$, $b = +\infty$, and a can be any number $-\infty \leqslant a < +\infty$.

In [2] and [3] the following theorem has been proved (2):

If the function $\psi(x)$ is concave in an interval (a, ∞) and fulfils the condition
$$\lim_{n\to\infty} \{\psi(n+1)-\psi(n)\} = 0,$$
then there exists exactly one (up to an additive constant) convex function $g(x)$ satisfying in the interval (a, ∞) the equation
$$g(x+1)-g(x) = \psi(x).$$

Now we shall prove the following:

(2) This theorem has in fact been proved for $a = 0$, but the proof is essentially the same when a is an arbitrary finite number. When $a = -\infty$ we must apply the theorem for intervals (a, ∞) and pass to the limit with $a \to -\infty$.

THEOREM III. *If the function $F(x)$ is convex and fulfils the condition*

(11) $$\lim_{n\to\infty}\{F(n+1)-F(n)\}=0,$$

then the equation

(12) $$\varphi(x)+\varphi(x+1)=F(x)$$

possesses exactly one monotonic solution in (a,b). This solution is given by the formula:

(13) $$\varphi(x)=g(\tfrac{1}{2}x)-g\big(\tfrac{1}{2}(x+1)\big),$$

where $g(x)$ is the convex solution of the equation

$$g(x+1)-g(x)=-F(2x).$$

Proof. The function $\varphi(x)$, given by formula (13), is decreasing, for the function $\dfrac{g\big(\frac{1}{2}(x+1)\big)-g(\frac{1}{2}x)}{\frac{1}{2}}$ is increasing as a differences quotient of a convex function. The function $\varphi(x)$ satisfies also equation (12):

$$\varphi(x)+\varphi(x+1)=g(\tfrac{1}{2}x)-g\big(\tfrac{1}{2}(x+1)\big)+g\big(\tfrac{1}{2}(x+1)\big)-g(\tfrac{1}{2}x+1)$$
$$=g(\tfrac{1}{2}x)-g(\tfrac{1}{2}x+1)=F(x).$$

From theorem II it follows that it is the unique solution of this kind.

The function $\varphi(x)$ given by formula (13) is evidently also the unique convex solution of equation (12) if such a one exists. But it does not necessarily exist, as is illustrated by the following example:

Let us define the function $g''(x)$ in the interval $\langle 0,\infty)$ as follows:

$$g''(n)=\frac{1}{2^n},\qquad g''(n+\tfrac{1}{2})=\frac{1}{2^{n+2}},\qquad n=0,1,2,\ldots$$

$g''(x)$ linear in the intervals $\langle n, n+\frac{1}{2}\rangle$ and $\langle n+\frac{1}{2}, n+1\rangle$, continuous in $\langle 0,\infty)$.

The function $g''(x)$ thus defined fulfils the following conditions:

$$g''(x)>0 \quad\text{for}\quad x\in(0,\infty),$$
$$g''(x+1)-g''(x)<0 \quad\text{for}\quad x\in(0,\infty),$$

(14) $g''\big(\frac{1}{2}(x+1)\big)-g''(\frac{1}{2}x)$ has not a constant sign in $(0,\infty)$.

Further we put

$$g'(x)=-\int_x^\infty g''(t)\,dt,\qquad x\in(0,\infty),$$

$$g(x)=-\int_x^\infty g'(u)\,du,\qquad x\in(0,\infty).$$

The function $g(x)$ is convex and decreasing in $(0,\infty)$, for $g''(x)>0$ and $g'(x)<0$ in $(0,\infty)$. We shall show that

(15) $$\lim_{n\to\infty}\{g(n+1)-g(n)\}=0.$$

We have

$$0\geqslant g(n+1)-g(n)=\int_n^{n+1} g'(u)\,du\geqslant g'(n)=-\int_n^\infty g''(t)\,dt.$$

For $x\in\langle n, n+1\rangle$ is $g''(x)\leqslant 1/2^n$. Then

$$\int_n^\infty g''(t)\,dt\leqslant\sum_{k=n}^\infty\frac{1}{2^k}=\frac{1}{2^{n-1}}.$$

Hence

$$0\geqslant g(n+1)-g(n)\geqslant-\frac{1}{2^{n-1}},$$

whence follows relation (15).

Thus if we put $F(x)\overset{\mathrm{df}}{=}g(x+1)-g(x)$, then we have by (15)

$$\lim_{n\to\infty}F(n)=0,$$

and since the function $F(x)$ is monotonic (as a differences quotient of a convex function),

$$\lim_{x\to\infty}F(x)=0.$$

Moreover, the function $F(x)$ is concave, for $F''(x)=g''(x+1)-g''(x)<0$. Then the only monotonic solution of the equation

$$\varphi(x)+\varphi(x+1)=-F(\tfrac{1}{2}x)$$

is

$$\varphi(x)=g(\tfrac{1}{2}x)-g\big(\tfrac{1}{2}(x+1)\big),$$

which is not convex, since, according to (14), $\varphi''(x)=\frac{1}{4}\{g''(\frac{1}{2}x)-g''\big(\frac{1}{2}(x+1)\big)\}$ changes the sign.

§ 3. Theorem III is a generalization of the following theorem of A. Mayer ([5]):

A convex function $\Phi(x)>0$ for $x>0$, satisfying for $x>0$ the equation

(16) $$\Phi(x+1)=1/x\Phi(x)$$

has to be of the form

(17) $$\Phi(x)=\frac{1}{\sqrt{2}}\cdot\frac{\Gamma(\frac{1}{2}x)}{\Gamma\big(\frac{1}{2}(x+1)\big)},$$

where $\Gamma(x)$ is the Euler function.

In fact, if a convex function $\Phi(x) > 0$ satisfies equation (16), then the function $\varphi(x) \overset{\mathrm{df}}{=} \ln\Phi(x)$ is monotonic (for sufficiently large x), and satisfies the equation

$$\varphi(x)+\varphi(x+1) = -\ln x.$$

Thus is has to be of the form

$$\varphi(x) = g(\tfrac{1}{2}x)-g\big(\tfrac{1}{2}(x+1)\big),$$

where $g(x)$ is the convex solution of the equation

$$g(x+1)-g(x) = \ln x+\ln 2.$$

Hence (see [2], [3])

$$g(x) = \ln\Gamma(x)+\tfrac{1}{2}x\ln 2+C,$$

and

$$\varphi(x) = \ln\frac{1}{\sqrt{2}}\cdot\frac{\Gamma(\frac{1}{2}x)}{\Gamma\big(\frac{1}{2}(x+1)\big)},$$

which proves the theorem of A. Mayer.

§ 4. J. Anastassiadis [1] introduces the notion of functions semimonotonic ω and semiconvex ω. We call a function $\psi(x)$ *semiincreasing* ω (ω — a fixed positive number) in an interval I if for every x such that $x\in I$ and $x+\omega\in I$

$$\psi(x+\omega) \geqslant \psi(x).$$

Analogically we call a function $\psi(x)$ *semidecreasing* ω if

$$\psi(x+\omega) \leqslant \psi(x), \quad x\in I, \quad x+\omega\in I.$$

We call a function $\psi(x)$ *semiconvex* ω if

$$\psi(x+\omega) \leqslant \tfrac{1}{2}[\psi(x)+\psi(x+2\omega)], \quad x\in I, \quad x+\omega\in I, \quad x+2\omega\in I.$$

J. Anastassiadis proves that the only function semidecreasing 1 or semiconvex 1, positive for $x > 0$, and satisfying equation (16) is the function $\Phi(x)$ given by formula (17).

It can be proved that if the function $F(x)$ is monotonic (in the ordinary sense) and fulfils condition (11), then equation (12) possesses at most one solution semimonotonic 1 or semiconvex 1 in (a, b). Below we shall show a more general theorem.

The notion of functions semimonotonic ω and semiconvex ω may be generalized in the following manner:

Let $f(x)$ be a positive function, $f(x) > x$ in an interval I. We call a function $\psi(x)$ *semiincreasing* $\{f\}$ *in* I if

$$\psi[f(x)] \geqslant \psi(x) \quad \text{for} \quad x, f(x)\in I.$$

We call a function $\psi(x)$ *semidecreasing* $\{f\}$ *in* I if

$$\psi[f(x)] \leqslant \psi(x) \quad \text{for} \quad x, f(x)\in I.$$

We call a function $\psi(x)$ *semiconvex* $\{f\}$ *in* I if

$$\psi[f(x)] \leqslant \tfrac{1}{2}\big[\psi(x)+\psi\big(f^2(x)\big)\big] \quad \text{for} \quad x, f(x), f^2(x)\in I.$$

We shall prove the following

THEOREM IV. *If the function $f(x)$ fulfils the hypotheses of theorem* I, *and the function $F(x)$ is monotonic (in the ordinary sense) and fulfils condition* (4), *then equation* (1) *possesses at most one solution semimonotonic $\{f\}$ in the interval $(b-\eta, b)$, where η is an arbitrary positive number.*

Proof. It follows from the monotonity of the function $F(x)$ and from condition (4) that

$$\lim_{n\to\infty}\{F[f^n(x)]-F[f^{n+1}(x)]\} = 0 \tag{18}$$

for every $x\in(a, b)$. Let $\varphi(x)$ be an arbitrary solution of equation (1) that is semimonotonic $\{f\}$ in $(b-\eta, b)$. From (18) it follows that

$$\lim_{n\to\infty}\{\varphi[f^n(x)]-\varphi[f^{n+2}(x)]\} = 0 \tag{19}$$

for every $x\in(a, b)$. Let us fix an arbitrary $x\in(a, b)$. There exists an index N such that $f^n(x)\in(b-\eta, b)$ for $n \geqslant N$. Since $\varphi(x)$ is semimonotonic $\{f\}$ in $(b-\eta, b)$

$$|\varphi[f^n(x)]-\varphi[f^{n+1}(x)]| \leqslant |\varphi[f^n(x)]-\varphi[f^{n+2}(x)]| \quad \text{for} \quad n \geqslant N,$$

whence, according to (19),

$$\lim_{n\to\infty}\{\varphi[f^n(x)]-\varphi[f^{n+1}(x)]\} = 0. \tag{20}$$

Since x has been arbitrarily fixed, relation (20) holds for every $x\in(a, b)$. Then, on account of theorem I, equation (1) possesses at most one solution semimonotonic $\{f\}$ in $(b-\eta, b)$.

We shall prove also

THEOREM V. *Under the hypotheses of theorem* IV *equation* (1) *possesses at most one solution semiconvex $\{f\}$ in (a, b).*

Proof. Let $\varphi(x)$ be a solution of equation (1) semiconvex $\{f\}$ in (a, b). From the hypotheses of the theorem follows that relation (19) holds for every $x\in(a, b)$. Let us fix an arbitrary $x\in(a, b)$. We shall show that the sequence $\varphi[f^n(x)]$ is monotonic for sufficiently large n. Let us suppose that there exists an index N such that

$$\varphi[f^{N+1}(x)] > \varphi[f^N(x)]. \tag{21}$$

Since the function $\varphi(x)$ is semiconvex $\{f\}$

(22) $$2\varphi[f^{N+1}(x)] \leqslant \varphi[f^N(x)] + \varphi[f^{N+2}(x)].$$

Subtracting $2\varphi[f^{N+1}(x)]$ from both sides of inequality (22) we obtain

$$0 \leqslant \varphi[f^N(x)] - 2\varphi[f^{N+1}(x)] + \varphi[f^{N+2}(x)],$$

whence, by (21)

$$0 < \varphi[f^{N+1}(x)] - \varphi[f^N(x)] \leqslant \varphi[f^{N+2}(x)] - \varphi[f^{N+1}(x)],$$

i. e.

$$\varphi[f^{N+2}(x)] > \varphi[f^{N+1}(x)].$$

By induction we obtain

$$\varphi[f^{n+1}(x)] > \varphi[f^n(x)] \quad \text{for} \quad n \geqslant N,$$

which proves that the sequence $\varphi[f^n(x)]$ is increasing for large n. On the other hand, if relation (21) does not hold for any N, it means that

$$\varphi[f^{n+1}(x)] \leqslant \varphi[f^n(x)]$$

for every n, which proves that the sequence $\varphi[f^n(x)]$ is decreasing.

Thus we have shown that the sequence $\varphi[f^n(x)]$ is monotonic for large n. Hence it follows that the inequality

$$|\varphi[f^n(x)] - \varphi[f^{n+1}(x)]| \leqslant |\varphi[f^n(x)] - \varphi[f^{n+2}(x)]|$$

holds for sufficiently large n, and hence, by (19), follows relation (20). And since x has been arbitrary, relation (20) holds for every $x \in (a, b)$. Thus from theorem I it follows that there may exist at most one solution of equation (1) semiconvex $\{f\}$ in (a, b), which was to be proved.

References

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Remarque sur la note de A. B. Turowicz sur l'approximation des racines de nombres positifs

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Dans la note de A. B. Turowicz „Sur l'approximation des racines de nombres positifs" (1) on trouve la formule

(1) $$x_{k+1} = \frac{(n-1)x_k^{n+1} + (n+1)Ax_k}{(n+1)x_k^n + (n-1)A}, \quad x_0 > 0,$$

pour l'approximation de la racine $\sqrt[n]{A}$ d'un nombre positif $A > 0$. A. B. Turowicz démontre que la suite (1) est monotone et converge vers $\sqrt[n]{A}$. Il donne aussi une évaluation de la rapidité de la convergence $|x_k - \sqrt[n]{A}| \to 0$.

Nous allons montrer dans cette note que l'idée essentielle utilisée par A. B. Turowicz pour établir la relation

(2) $$\operatorname{sign}(x_k - x_{k+1}) = \operatorname{sign}(x_0 - \sqrt[n]{A}) \quad \text{pour} \quad k = 0, 1, 2, \ldots,$$

et la convergence

(3) $$x_k \to \sqrt[n]{A}$$

peut facilement être appliquée au cas plus général de la suite $\{x_k\}$, définie par les formules suivantes:

(4) $$x_k = \frac{x_{k-1}\{\psi(x_{k-1}) - \varphi(x_{k-1}^n - A)\}}{\psi(x_{k-1}) + \varphi(x_{k-1}^n - A)}, \quad x_0 > 0,$$

où les fonctions $\psi(x)$ et $\varphi(u)$ sont continues et satisfont aux relations suivantes:

(5) $$\psi(x) + \varphi(x^n - A) > 0 \quad \text{pour} \quad x \geqslant 0,$$

(6) $$\psi(x) - \frac{x+\sqrt[n]{A}}{x-\sqrt[n]{A}}\varphi(x^n - A) > 0 \quad \text{pour} \quad x^n \neq A,$$

(7) $$\varphi(u) = -\varphi(-u), \quad \varphi(u) \text{ est monotone.}$$

(1) Ce volume p. 265-269.