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On the form of solutions of some functional equations

by M. Kuczma (Kraków)

In my previous paper [3] I have proved the following theorem:

If the function $f(x)$ is continuous and strictly increasing in an interval $(a, b\rangle$, and $f(x) > x$ in (a, b), $f(b) = b$, then the functional equation

$$\varphi(x)+\varphi[f(x)] = F(x) \tag{1}$$

possesses in the interval (a, b) at most one solution $\varphi(x)$, fulfilling for every $x \in (a, b)$ the condition

$$\lim_{n\to\infty}\{\varphi[f^n(x)]-\varphi[f^{n+1}(x)]\} = 0.\ ^{(1)} \tag{2}$$

In the first section of the present paper I prove that if this unique solution exists, it has to be of the form (12). In the second section I prove (under suitable conditions) that if equation (1) has a semimonotonic $\{f\}$ solution (2) then it has to be also of the form (12). If moreover the function $F(x)$ is semiconvex $\{f\}$, then such a solution necessarily exists.

In § 3 I consider the more general equation

$$\varphi[f(x)] = G\big(x, \varphi(x)\big). \tag{3}$$

I prove that under suitable assumptions equation (3) possesses exactly one solution which is continuous at the point $x = b$ such that

$$f(x) = x. \tag{4}$$

The uniqueness of such solution of equation (3) has been proved in [1]. There also has been stated the problem of finding when such solution really exists. Thus theorem IV of the present paper gives an answer to the above question.

§ 1. At first we shall prove the following:

Lemma I. *Let us suppose that the function $f(x)$ is continuous and strictly increasing in an interval $(a, b\rangle$ and that $f(x) > x$ in (a, b), $f(b) = b$.*

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(1) Throughout this paper the symbol $f^n(x)$ denotes the n-th iteration of the function $f(x)$.

(2) The notion of functions semimonotonic $\{f\}$ and semiconvex $\{f\}$ has been introduced in [3]. For the definition see also § 2 below.

If a function $\varphi(x)$ *satisfies equation* (1) *and for every* $x\epsilon(a, b)$ *fulfils the condition*

$$\lim_{n\to\infty}\varphi[f^n(x)] = 0, \tag{5}$$

then

$$\varphi(x) = \sum_{\nu=0}^{\infty}(-1)^{\nu}F[f^{\nu}(x)]. \tag{6}$$

Proof. Equation (1) can be written in the form

$$\varphi(x) = F(x)-\varphi[f(x)]. \tag{7}$$

Putting in relation (7) $f(x)$(3) in the place of x we obtain

$$\varphi[f(x)] = F[f(x)]-\varphi[f^2(x)]. \tag{8}$$

We have by (8) and (7)

$$\varphi(x) = F(x)-F[f(x)]+\varphi[f^2(x)]. \tag{9}$$

Putting in relation (8) $f(x)$ in the place of x we obtain

$$\varphi[f^2(x)] = F[f^2(x)]-\varphi[f^3(x)]. \tag{10}$$

We have by (10) and (9)

$$\varphi(x) = F(x)-F[f(x)]+F[f^2(x)]-\varphi[f^3(x)].$$

By induction one can obtain the relation

$$\varphi(x) = \sum_{\nu=0}^{n}(-1)^{\nu}F[f^{\nu}(x)]+(-1)^{n+1}\varphi[f^{n+1}(x)]. \tag{11}$$

Passing to the limit in relation (11) as $n\to\infty$ we obtain, according to (5), relation (6), which was to be proved.

THEOREM I. *Let the function* $f(x)$ *fulfil the hypotheses of the lemma* I. *If a function* $\varphi(x)$ *satisfies equation* (1) *and for every* $x\epsilon(a, b)$ *fulfils condition* (2), *then*

$$\varphi(x) = \tfrac{1}{2}\Big[F(x)-\sum_{\nu=0}^{\infty}(-1)^{\nu}\{F[f^{\nu+1}(x)]-F[f^{\nu}(x)]\}\Big]. \tag{12}$$

Proof. Let us put

$$\psi(x) \overset{\mathrm{df}}{=} \varphi[f(x)]-\varphi(x), \qquad x\epsilon(a, b). \tag{13}$$

(3) From the hypotheses of the lemma it follows that for $x\epsilon(a, b)$ also $f(x)\epsilon(a, b)$. (See [2]. Compare also the footnote on page 57.)

The function $\psi(x)$ satisfies the functional equation

$$\psi(x)+\psi[f(x)] = F[f(x)]-F(x), \tag{14}$$

and moreover, by (2), fulfils the condition

$$\lim_{n\to\infty}\psi[f^n(x)] = 0. \tag{15}$$

Hence, on account of lemma I

$$\psi(x) = \sum_{\nu=0}^{\infty}(-1)^{\nu}\{F[f^{\nu+1}(x)]-F[f^{\nu}(x)]\}. \tag{16}$$

Now we have by (1) and (13)

$$\varphi(x)+\varphi[f(x)] = F(x), \qquad \varphi(x)-\varphi[f(x)] = -\psi(x),$$

whence

$$\varphi(x) = \tfrac{1}{2}[F(x)-\psi(x)].$$

Hence, according to (16), we obtain formula (12).

Remark. In my paper [2] I have proved that equation (1) with the function $f(x)$ fulfilling the hypotheses of lemma I has at most one solution $\varphi(x)$ that is continuous in the interval $(a, b\rangle$ and that this solution is given (provided it exists) by the formula

$$\varphi(x) = \tfrac{1}{2}F(b)+\sum_{\nu=0}^{\infty}(-1)^{\nu}\{F[f^{\nu}(x)]-F(b)\}. \tag{17}$$

Since every function $\varphi(x)$ continuous at the point $x = b$ fulfils condition (2)(4), formula (17) can be derived from formula (12).

§ 2. In [3] I have introduced the notion of functions semimonotonic $\{f\}$ and semiconvex $\{f\}$. We call a function $\varphi(x)$ *semiincreasing* $\{f\}$ or *semidecreasing* $\{f\}$ in (a, b) if for every $x\epsilon(a, b)$

$$\varphi[f(x)] \geqslant \varphi(x) \quad \text{or} \quad \varphi[f(x)] \leqslant \varphi(x),$$

respectively. Similarly, we call a function $\varphi(x)$ *semiconvex* $\{f\}$ or *semiconcave* $\{f\}$ in (a, b) if for every $x\epsilon(a, b)$

$$\varphi[f(x)] \leqslant \tfrac{1}{2}\{\varphi(x)+\varphi[f^2(x)]\} \quad \text{or} \quad \varphi[f(x)] \geqslant \tfrac{1}{2}\{\varphi(x)+\varphi[f^2(x)]\},$$

respectively.

(4) Under the assumptions of lemma I the sequence $f^n(x)$ is for every $x\epsilon(a, b)$ increasing and $\lim_{n\to\infty} f^n(x) = b$ (see [2]).

In [3] I have proved that under some conditions equation (1) possesses at most one solution which is semimonotonic $\{f\}$ in (a, b). Now I shall prove the following:

THEOREM II. *Let the function $f(x)$ fulfil the hypotheses of lemma I, and let the function $F(x)$ fulfil for every $x \in (a, b)$ the condition*

$$\lim_{n\to\infty} \{F[f^{n+1}(x)] - F[f^n(x)]\} = 0. \tag{18}$$

Then, if there exists a function $\varphi(x)$, semimonotonic $\{f\}$ in (a, b) and satisfying equation (1), it has to be of the form (12).

Proof. Let $\varphi(x)$ be a solution of equation (1) that is semimonotonic $\{f\}$ in (a, b). The function $\psi(x)$, defined by relation (13), satisfies equation (14). The function

$$G(x) \overset{\mathrm{df}}{=} F[f(x)] - F(x) \tag{19}$$

fulfils, according to (18), for every $x \in (a, b)$ the relation

$$\lim_{n\to\infty} G[f^n(x)] = 0. \tag{20}$$

We have by (14)

$$G[f^n(x)] = \psi[f^n(x)] + \psi[f^{n+1}(x)]. \tag{21}$$

The sequence $\psi[f^n(x)]$ has terms with a constant sign, because the function $\varphi(x)$ is semimonotonic $\{f\}$. Thus from (20) and (21) follows relation (15), and the proof runs further as the proof of theorem I.

We shall also prove

THEOREM III. *If the hypotheses of theorem II are fulfilled and, moreover, the function $F(x)$ is semiconvex $\{f\}$ (semiconcave $\{f\}$) in (a, b), then equation (1) possesses exactly one solution that is semidecreasing $\{f\}$ (semi-increasing $\{f\}$) in (a, b).*

Proof. Let us suppose that the function $F(x)$ is semiconvex $\{f\}$ in (a, b) (in the case of the function $F(x)$ being semiconcave $\{f\}$ the proof runs similarly). For the proof of the theorem we need only to show that the series

$$\sum_{\nu=0}^{\infty} (-1)^\nu \{F[f^{\nu+1}(x)] - F[f^\nu(x)]\} \tag{22}$$

converges for every $x \in (a, b)$ and that the function $\varphi(x)$, defined by formula (12), is semidecreasing $\{f\}$ in (a, b).

To begin with, we remark that the function $F(x)$ is semiconvex $\{f\}$ if and only if the function $F[f(x)] - F(x)$ is semiincreasing $\{f\}$. Indeed, the function $F(x)$ is semiconvex $\{f\}$ if the inequality

$$F[f(x)] \leqslant \tfrac{1}{2}\{F(x) + F[f^2(x)]\} \tag{23}$$

holds for every $x \in (a, b)$. Similarly, the function $F[f(x)] - F(x)$ is semiincreasing $\{f\}$ if the inequality

$$F[f(x)] - F(x) \leqslant F[f^2(x)] - F[f(x)] \tag{24}$$

holds for every $x \in (a, b)$. Both inequalities (23) and (24) are equivalent.

Thus from the hypotheses of the theorem it follows that the function $F[f(x)] - F(x)$ is semiincreasing $\{f\}$ in (a, b). Consequently the sequence

$$F[f^{n+1}(x)] - F[f^n(x)]$$

is monotonic and by (18) converges to zero for every $x \in (a, b)$. Thus series (22) converges since it is alternating.

Further we have by (16)

$$\varphi[f(x)] - \varphi(x) = \psi(x) = \sum_{\nu=0}^{\infty} (-1)^\nu \{F[f^{\nu+1}(x)] - F[f^\nu(x)]\} = \sum_{\nu=0}^{\infty} (-1)^\nu G[f^\nu(x)],$$

where the function $G(x)$ is defined by formula (19). The sequence $G[f^\nu(x)]$ is (for every $x \in (a, b)$) increasing and converges to zero, and thus all its terms are non-positive. Consequently for $x \in (a, b)$

$$G(x) \leqslant \psi(x) \leqslant G(x) - G[f(x)] \leqslant 0.$$

Hence

$$\varphi[f(x)] - \varphi(x) \leqslant 0, \quad \text{i. e.} \quad \varphi[f(x)] \leqslant \varphi(x),$$

which means that the function $\varphi(x)$ is semidecreasing $\{f\}$ in (a, b). This completes the proof.

§ 3. In the present section we shall consider the functional equation

$$\varphi[f(x)] = G(x, \varphi(x)). \tag{3}$$

In the sequel we shall assume that

(i) The function $f(x)$ is continuous and strictly increasing in an interval $\langle a, b\rangle$, where a and b are two consecutive roots of equation (4), and $f(x) > x$ in (a, b).

(ii) The function $G(x, y)$ is continuous and has the continuous derivative $\partial G/\partial y \neq 0$ in a region Ω, normal with respect to the x-axis.

We shall denote by Ω_x the x-section of the set Ω:

$$\Omega_x = \underset{y}{E}\{(x, y) \in \Omega\},$$

and by Γ_x the set of values assumed by the function $G(x, y)$ for $y \in \Omega_x$. We shall also assume

(iii) $\Omega_x \neq 0$, $\Gamma_x = \Omega_{f(x)}$ for $x \in (a, b)$.

From hypothesis (ii) follows the existence of the unique function $H(x, y)$, inverse to the function $G(x, y)$ with respect to the variable y. The function $H(x, y)$ is defined and continuous in a region Ω':

$$\Omega' = \mathop{E}_{(x,y)} \{y \in \Gamma_x\},$$

normal with respect to the x-axis, and has the continuous derivative $\partial H/\partial y \neq 0$ in Ω'.

In [1] the following theorem has been proved:

If hypotheses (i)-(iii) *are fulfilled, then for every* $x_0 \in (a, b)$ *and for every functon* $\bar{\varphi}(x)$ *which is continuous in the interval* $\langle x_0, f(x_0)\rangle$ *and fulfils the conditions*:

$$\bar{\varphi}(x) \in \Omega_x \quad \textit{for} \quad x \in \langle x_0, f(x_0)\rangle, \qquad \bar{\varphi}[f(x_0)] = G(x_0, \bar{\varphi}(x_0)),$$

there exists a function $\varphi(x)$, *defined and continuous in the interval* (a, b), *satisfying equation* (3) *and such that*

$$\varphi(x) = \bar{\varphi}(x) \quad \textit{for} \quad x \in \langle x_0, f(x_0)\rangle.$$

Let numbers c and d be roots of the equations

$$c = G(a, c) \quad \text{and} \quad d = G(b, d)$$

respectively, and let us suppose that the points (a, c) and (b, d) respectively belong to the region Ω. We shall define two functional sequences $\{g_n(x)\}$ and $\{h_n(x)\}$:

$$g_0(x) = c, \quad g_{n+1}(x) = G(f^{-1}(x), g_n[f^{-1}(x)]), \quad x \in \langle a, b).$$

$$h_0(x) = d, \quad h_{n+1}(x) = H(x, h_n[f(x)]), \quad x \in (a, b\rangle.$$

Lemma II. *Let us suppose that hypotheses* (i)-(iii) *are fulfilled. If the sequence* $g_n(x)$ $(h_n(x))$ *converges for* $x = x_0$, *then it converges also for* $x = f(x_0)$ *and* $x = f^{-1}(x_0)$ *and the function* $g(x) = \lim_{n\to\infty} g_n(x)$ $(h(x) = \lim_{n\to\infty} h_n(x))$ *satisfies equation* (3) *for all values* x *for which it is defined.*

The proof of this lemma is to be found in [1].

We shall now prove the following:

Theorem IV. *Let us suppose that hypotheses* (i)-(iii) *are fulfilled. If*

$$\left|\frac{\partial G}{\partial y}(b, d)\right| > 1, \tag{25}$$

then equation (3) *possesses exactly one solution* $\varphi(x)$ *that is continuous in the interval* $(a, b\rangle$ *and fulfils the condition* $\varphi(b) = d$. *This solution is given by the formula*

$$\varphi(x) = \lim_{n\to\infty} h_n(x).$$

Similarly, if

$$\left|\frac{\partial G}{\partial y}(a, c)\right| < 1,$$

then equation (3) *possesses exactly one solution* $\varphi(x)$ *that is continuous in the interval* $\langle a, b)$ *and fulfils the condition* $\varphi(a) = c$. *This solution is given by the formula*

$$\varphi(x) = \lim_{n\to\infty} g_n(x).$$

Proof. We shall prove only the first part of the theorem, the proof of the second being quite similar.

The uniqueness of the solution continuous at the point $x = b$ has been proved (under condition (25)) in [1]. Thus, according to lemma II, it is enough to prove that the sequence $h_n(x)$ converges in the whole interval $(a, b\rangle$ to a continuous function.

From condition (25) it follows that

$$\left|\frac{\partial H}{\partial y}(b, d)\right| < 1.$$

Then there exist positive numbers ε, η and $\vartheta < 1$ such that the rectangle

$$R: \quad |x-b| < \eta, \quad |y-d| < \varepsilon$$

is contained together with its closure $\bar{R}$ in $\Omega \cap \Omega'$ and that

$$|\partial H/\partial y| < \vartheta \tag{26}$$

for $(x, y) \in R$. The function $H(x, y)$, being continuous in Ω', is uniformly continuous in $\bar{R}$. Thus one can find $x_0 \in (b-\eta, b)$ such that

$$|H(x, y) - H(b, y)| < (1-\vartheta)\varepsilon/2 \quad \text{for} \quad x \in \langle x_0, b\rangle, \quad |y-d| < \varepsilon/2. \tag{27}$$

Hence

$$|H(x, y) - d| < \varepsilon/2 \quad \text{for} \quad x \in \langle x_0, b\rangle, \quad |y-d| < \varepsilon/2. \tag{28}$$

In fact, we have by (26) and (27)

$$|H(x,y)-d| = |H(x,y)-H(b,d)| \leqslant |H(x,y)-H(b,y)|+|H(b,y)-$$
$$-H(b,d)|<(1-\vartheta)\varepsilon/2+|H_y(b,d+\Theta(y-d))|\,|y-d|<(1-\vartheta)\varepsilon/2+\vartheta\varepsilon/2=\varepsilon/2.$$

Following (28) one can easily prove by induction that

(29) $$|h_n(x)-d| < \varepsilon/2 \quad \text{for} \quad x\in\langle x_0, b\rangle, \quad n=0,1,2,\ldots$$

Now we put

$$r_n(x) \stackrel{\mathrm{df}}{=} h_{n+1}(x)-h_n(x), \quad n=0,1,2,\ldots$$

We shall show that

(30) $$|r_n(x)| < \vartheta^n \varepsilon/2 \quad \text{for} \quad x\in\langle x_0, b\rangle, \quad n=0,1,2,\ldots$$

For $n=0$ we have by (29)

$$|r_0(x)| < \varepsilon/2.$$

Let us suppose that (30) holds for a certain $n \geqslant 0$. We have

$$r_{n+1}(x) = h_{n+2}(x)-h_{n+1}(x) = H\big(x, h_{n+1}[f(x)]\big)-H\big(x, h_n[f(x)]\big)$$
$$= H_y\big(x, h_n[f(x)]+\Theta(x)r_n[f(x)]\big)r_n[f(x)].$$

For $x\in\langle x_0, b\rangle$ also $f(x)\in\langle x_0, b\rangle$. Thus we have by (29) and the inductive hypothesis

$$|h_n[f(x)]+\Theta(x)r_n[f(x)]-d| \leqslant |h_n[f(x)]-d|+|r_n[f(x)]| < \varepsilon,$$

whence, according to (26), we have for $x\in\langle x_0, b\rangle$:

$$\big|H_y\big(x, h_n[f(x)]+\Theta(x)r_n[f(x)]\big)\big| < \vartheta$$

and

$$|r_{n+1}(x)| < \vartheta|r_n[f(x)]|.$$

Hence, it follows by induction that inequality (30) holds for $x\in\langle x_0, b\rangle$, $n=0,1,2,\ldots$

Consequently the series $\sum_{n=0}^{\infty} r_n(x)$ uniformly converges in the interval $\langle x_0, b\rangle$. Thus also the sequence $h_n(x)$ uniformly converges in the interval $\langle x_0, b\rangle$ and its limit

$$h(x) = \lim_{n\to\infty} h_n(x)$$

is a continuous function in the interval $\langle x_0, b\rangle$. Moreover, by (29), the graphs of the functions $h_n(x)$ pass for $x\in\langle x_0, b\rangle$ through the rectangle R, and so

$$h(x)\in\Omega_x \quad \text{for} \quad x\in\langle x_0, b\rangle.$$

On account of lemma II the sequence $h_n(x)$ converges in the whole interval (a, b) and the function $h(x)$ satisfies equation (3). Hence it follows that there exists a function $\varphi(x)$, continuous in the interval (a, b), satisfying equation (3) and such that

(31) $$\varphi(x) = h(x) \quad \text{for} \quad x\in\langle x_0, f(x_0)\rangle.$$

Since every solution of equation (3) is in the interval (a, b) uniquely determined by its values from the interval $\langle x_0, f(x_0))$, from (31) follows

$$\varphi(x) = h(x) \quad \text{for} \quad x\in(a, b),$$

which proves that the function $h(x)$ is continuous in $(a, b\rangle$. Moreover, from the evident relations

$$h_n(b) = d, \quad n=0,1,2,\ldots$$

follows

$$h(b) = d.$$

This completes the proof.

References

[1] J. Kordylewski and M. Kuczma, *On the functional equation $F(x, \varphi(x), \varphi[f(x)]) = 0$*, Ann. Pol. Math. 7 (1959), p. 21-32.

[2] M. Kuczma, *On the functional equation $\varphi(x)+\varphi[f(x)] = F(x)$*, Ann. Pol. Math. 6 (1959), p. 281-287.

[3] — *Remarks on some functional equations*, Ann. Pol. Math. 8 (1960), p. 277-284.

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