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ANNALES
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IX (1960)

On the asymptotic behaviour of harmonic functions in the semi-space

by W. Bach (Kraków)

Let $f(x, y)$ be a continuous and bounded function in the open plane, say $|f(x, y)| \leqslant M$, and let $u(x, y, z)$ be a solution of the Dirichlet problem for the upper semi-space $z > 0$ with $f(x, y)$ as its boundary value. It is known [1] that in the class of functions $u(x, y, z)$ satisfying the condition

$$|u(x, y, z)| \leqslant M_1[|x|^a+|y|^a+|z|^a+1] \quad \text{for} \quad z \geqslant 0, \quad -\infty \leqslant x, y \leqslant \infty,$$

where M_1 and $a < 1$ are positive numbers, this solution is unique and is expressed by the formula

$$u(x, y, z) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta) \frac{dG}{dn} d\xi\, d\eta$$

where G is Green's function for the upper semi-space $z > 0$ i. e.

$$G = \frac{1}{pq} - \frac{1}{\overline{p}q}, \quad p = p(x, y, z), \quad \overline{p} = \overline{p}(x, y, -z), \quad q = q(\xi, \eta, \zeta)$$

and dG/dn is the inner normal derivative. Next $u(x, y, z)$ will denote the solution of the above-mentioned Dirichlet problem.

Let us consider the semi-line

$$(1) \qquad x = x_0+at, \quad y = y_0+bt, \quad z = z_0+ct, \quad t \in \langle 0, \infty),$$

a and b being the arbitrary real, c a positive number and (x_0, y_0, z_0) an arbitrary point in the semi-space $z \geqslant 0$. Then, we shall prove

THEOREM 1. *If* $(x, y, z) \underset{(1)}{\to} \infty$ (¹), *then neither* $\underline{\lim}\, u(x, y, z)$ *nor* $\overline{\lim}\, u(x, y, z)$ *depend on* (x_0, y_0, z_0).

(¹) The symbol $(x, y, z) \underset{(1)}{\to} \infty$ denotes that a point (x, y, z) tends to ∞ along the semi-line (1).

Proof. It is sufficient to prove that

$$u(x_0+at,\,y_0+bt,\,z_0+ct)-u(at,\,bt,\,ct)\to 0$$

as $t\to\infty$. Changing the coordinate system we may assume that $b=0$. Since

$$u(x,y,z)=\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(\xi,\eta)\frac{dG}{dn}\,d\xi\,d\eta$$

$$=\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{2zf(\xi,\eta)\,d\xi d\eta}{[(\xi-x)^2+(\eta-y)^2+z^2]^{3/2}}$$

we have

$$u(x_0+at,\,y_0,\,z_0+ct)-u(at,\,0,\,ct)$$

$$=\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{2(z_0+ct)f(\xi,\eta)\,d\xi d\eta}{[(x_0+at-\xi)^2+(y_0-\eta)^2+(z_0+ct)^2]^{3/2}}-$$

$$-\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{2ctf(\xi,\eta)\,d\xi d\eta}{[(at-\xi)^2+\eta^2+(ct)^2]^{3/2}}.$$

We introduce the notation

$$r_1=[(x_0+at-\xi)^2+(y_0-\eta)^2+(z_0+ct)^2]^{1/2},$$

$$r_2=[(at-\xi)^2+\eta^2+(ct)^2]^{1/2};$$

thus r_1 and r_2 are the lengths of the sides of the triangle whose vertexes are $(0,0,0)$, (x_0,y_0,z_0), $(\xi-at,\eta,-ct)$. Then

(2) $$|r_1-r_2|\leqslant\sqrt{x_0^2+y_0^2+z_0^2}.$$

If $t\to\infty$ then $r_2\to\infty$, the convergence being uniform with respect to ξ and η. Hence

$$\left|1-\frac{r_1}{r_2}\right|\leqslant\frac{\sqrt{x_0^2+y_0^2+z_0^2}}{r_2}<\frac12\quad\text{for}\quad t\geqslant T,\quad\text{or}$$

(3) $$\frac12\leqslant\frac{r_1}{r_2}\leqslant\frac32\quad\text{for}\quad t\geqslant T.$$

Then

$$|u(x_0+at,\,y_0,\,z_0+ct)-u(at,\,0,\,ct)|$$

$$=\left|\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{2z_0f(\xi,\eta)\,d\xi\,d\eta}{[(x_0+at-\xi)^2+(y_0-\eta)^2+(z_0+ct)^2]^{3/2}}+\right.$$

$$\left.+\frac{2ct}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1}{r_1^3}-\frac{1}{r_2^3}\right)f(\xi,\eta)\,d\xi\,d\eta\right|$$

$$\leqslant\frac{M}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{z_0\,d\xi\,d\eta}{[(x_0+at-\xi)^2+(y_0-\eta)^2+(z_0+ct)^2]^{3/2}}+$$

$$+\frac{Mct}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{|r_1^2-r_2^2|\,|r_1^2+r_1r_2+r_2^2|\,d\xi\,d\eta}{r_1^3r_2^3}.$$

The first integral on the right-hand side is equal to

$$\frac{M}{2\pi}\int_0^{2\pi}\int_0^{\infty}\frac{r\,dr\,d\varphi}{[r^2+(z_0+ct)^2]^{3/2}}=\frac{M}{2(z_0+ct)}$$

and tends to 0 when $t\to\infty$. In view of (2), (3) the second integral is

$$\leqslant\frac{M\sqrt{x_0^2+y_0^2+z_0^2}\,ct}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{r_2^2+\frac32r_2^2+\frac94r_2^2}{\frac18r_2^6}\,d\xi\,d\eta$$

$$=\frac{19M\sqrt{x_0^2+y_0^2+z_0^2}\,ct}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\xi d\eta}{[(at-\xi)^2+\eta^2+(ct)^2]^2}=\frac{19M\sqrt{x_0^2+y_0^2+z_0^2}}{ct}$$

and also tends to 0 as $t\to\infty$ q. e. d.

Let (r,φ) be the polar coordinates of a point (x,y) of the plane $z=0$.

THEOREM 2. *If the limit*

$$\lim_{r\to\infty}\int_0^{2\pi}f(r\cos\varphi,\,r\sin\varphi)\,d\varphi$$

exists, then

$$\lim_{z\to\infty}2\pi u(x,y,z)$$

also exists and they are both equal.

Proof. Since by theorem 1 neither $\overline{\lim}_{z\to\infty} u(x,y,z)$ nor $\underline{\lim}_{z\to\infty} u(x,y,z)$ depend on x, y, it is sufficient to prove that

$$\lim_{z\to\infty} u(0,0,z) = \frac{1}{2\pi} I \quad \text{where} \quad I = \lim_{r\to\infty} \int_0^{2\pi} f(r\cos\varphi, r\sin\varphi)\, d\varphi .$$

We choose an $\varepsilon > 0$. Then we can take $R > 0$ such that for $r > R$ we have

$$\left| \int_0^{2\pi} f(r\cos\varphi, r\sin\varphi)\, d\varphi - I \right| < \varepsilon . \tag{4}$$

Since we have

$$\lim_{z\to\infty} \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\infty} \frac{2zr}{(r^2+z^2)^{3/2}}\, dr\, d\varphi = 1,$$

$$\lim_{z\to\infty} \frac{1}{4\pi} \int_0^{2\pi} \int_0^{R} \frac{2zrf(r\cos\varphi, r\sin\varphi)\, dr d\varphi}{(r^2+z^2)^{3/2}} = 0,$$

$$u(0,0,z) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\infty} \frac{2zrf(r\cos\varphi, r\sin\varphi)\, dr d\varphi}{(r^2+z^2)^{3/2}}$$

$$= \frac{1}{4\pi} \int_0^{2\pi} \int_0^{R} \frac{2zrf(r\cos\varphi, r\sin\varphi)\, dr d\varphi}{(r^2+z^2)^{3/2}} +$$

$$+ \frac{1}{4\pi} \int_0^{2\pi} \int_R^{\infty} \frac{2zrf(r\cos\varphi, r\sin\varphi)\, dr d\varphi}{(r^2+z^2)^{3/2}},$$

we obtain by (4)

$$\frac{1}{2\pi}(I-\varepsilon) \leqslant \underline{\lim}_{z\to\infty} u(x,y,z) \leqslant \overline{\lim}_{z\to\infty} u(x,y,z) \leqslant \frac{1}{2\pi}(I+\varepsilon).$$

In view of $\varepsilon > 0$ being arbitrarily small we obtain our theorem.

Corollary. *If*

$$f(r\cos\varphi, r\sin\varphi) \underset{r\to\infty}{\to} g(\varphi)$$

almost everywhere, then

1° *there exists* $\lim_{z\to\infty} u(x,y,z)$,

2° *this limit does not depend on x, y and equals the integral mean of the function $g(\varphi)$ on $\langle 0, 2\pi\rangle$, i. e.*

$$\lim_{z\to\infty} u(x,y,z) = \frac{1}{2\pi} \int_0^{2\pi} g(\varphi)\, d\varphi .$$

Proof. Let r_n be an arbitrary sequence of positive numbers which tends to ∞. Since $f(r_n\cos\varphi, r_n\sin\varphi) \underset{n\to\infty}{\to} g(\varphi)$ almost everywhere, by the Lebesgue theorem $g(\varphi)$ is integrable L and the sequence

$$\int_0^{2\pi} f(r_n\cos\varphi, r_n\sin\varphi)\, d\varphi \to \int_0^{2\pi} g(\varphi)\, d\varphi .$$

Hence by our theorem 2

$$\lim_{z\to\infty} u(x,y,z) = \frac{1}{2\pi} \int_0^{2\pi} g(\varphi)\, d\varphi .$$

Although neither $\underline{\lim}_{z\to\infty} u(x,y,z)$ nor $\overline{\lim}_{z\to\infty} u(x,y,z)$ depend on x, y both these limits are in general distinct.

We shall construct an example of a bounded and continuous function for which $\underline{\lim}_{z\to\infty} u(x,y,z) \neq \overline{\lim}_{z\to\infty} u(x,y,z)$. Let $f(x,y) = h(x^2+y^2)$ where

$$h(t) = \begin{cases} 0 & \text{for} \quad R_{2i} \leqslant t \leqslant R_{2i+1}-1, \\ t+R_{2i+1}-1 & \text{for} \quad R_{2i+1}-1 \leqslant t \leqslant R_{2i+1}, \\ 1 & \text{for} \quad R_{2i+1} \leqslant t \leqslant R_{2i+2}-1, \\ -t+R_{2i} & \text{for} \quad R_{2i+2}-1 \leqslant t \leqslant R_{2i+2} \end{cases} \qquad i = 0, 1, 2,$$

and where $R_0 = 0$, $R_{i+1} = 1000R_i+1$ $(i = 1, 2, 3, \ldots)$.

Put $z_n = 10R_n$. Since

$$u(0,0,z_n) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{R_n} \frac{2z_nrf(r\cos\varphi, r\sin\varphi)\, dr\, d\varphi}{(r^2+z_n^2)^{3/2}} +$$

$$+ \frac{1}{4\pi} \int_0^{2\pi} \int_{R_n}^{R_{n+1}-1} \frac{2z_nrf(r\cos\varphi, r\sin\varphi)\, dr\, d\varphi}{(r^2+z_n^2)^{3/2}} +$$

$$+ \frac{1}{4\pi} \int_0^{2\pi} \int_{R_{n+1}-1}^{\infty} \frac{2z_nrf(r\cos\varphi, r\sin\varphi)\, dr\, d\varphi}{(r^2+z_n^2)^{3/2}}$$

and the sum of the first and the third integrals is $\leqslant \frac{1}{10}$ for every n, and since the second integral is equal to 0 if n is an even number and is $\geqslant \frac{1}{5}$ if n is an odd number, we have

$$\underline{\lim}_{z\to\infty} u(x, y, z) \leqslant \tfrac{1}{10}, \quad \overline{\lim}_{z\to\infty} u(x, y, z) \geqslant \tfrac{3}{10}.$$

Hence

$$\underline{\lim}_{z\to\infty} u(x, y, z) \neq \overline{\lim}_{z\to\infty} u(x, y, z) \qquad \text{q. e. d.}$$

M. Tsuji has defined in [2] that $u^*(\theta, \varphi)$ is a harmonic function on the unit sphere if

$$\frac{1}{\sin\theta}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u^*}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\cdot\frac{\partial^2 u^*}{\partial\varphi^2} \equiv 0.$$

Now we shall prove

THEOREM 3. *If*

$$f(r\cos\varphi, r\sin\varphi) \to g(\varphi)$$

uniformly in $\langle 0, 2\pi\rangle$ *when* $r\to\infty$, *then*

1° *there exists* $\lim u(x, y, z)$, *when* $(x, y, z) \underset{(1)}{\to} \infty$ *which depends on* φ *and* θ.

2° $u^*(\theta, \varphi) = \lim_{r\to\infty} u(x_0+r\cos\theta\cos\varphi,\ y_0+r\sin\theta\cos\varphi,\ z_0+r\sin\varphi)$ *is a harmonic function on the unit sphere.*

Proof. Let us consider the function $f_1(x, y)$ defined as follows:

$$f_1(r\cos\varphi, r\sin\varphi) = g(\varphi) \quad \text{for every } r > 0,$$

$$f_1(0, 0) = 0.$$

Let us write

$$\overline{u}(x, y, z) = \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(\xi, \eta)\frac{dG}{dn}\,d\xi\,d\eta.$$

It is easy to see that

$$\overline{u}(x, y, z) - u(x, y, z) \to 0 \quad \text{as} \quad (x, y, z) \underset{(1)}{\to} \infty,$$

so that it is sufficient to prove 1° and 2° for $\overline{u}(x, y, z)$.

If we prove that $\overline{u}(\varrho x, \varrho y, \varrho z) = \overline{u}(x, y, z)$ for every $\varrho > 0$, then 1° and 2° will be proved.

It is sufficient to prove (changing the coordinate system) that

$$\overline{u}(\varrho x, 0, \varrho z) = \overline{u}(x, 0, y) \quad \text{for every} \quad \varrho > 0.$$

Let us consider the angular regions T_i: $\left\{0 < r < \infty,\ \frac{\pi i}{n} \leqslant \varphi \leqslant \frac{\pi(i+1)}{n}\right\}$, $i = 0, 1, 2, \ldots, 2n-1$ and let us write

$$u_i(x, y, z) = \frac{1}{4\pi}\iint_{T_i}\frac{2z\,d\xi\,d\eta}{[(\xi-x)^2+(\eta-y)^2+z^2]^{3/2}},$$

$$M_i = \sup_{T_i} f_1(x, y), \qquad m_i = \inf_{T_i} f_1(x, y).$$

Let ε be an arbitrary positive number. We can find a number N such that $M_i - m_i < \varepsilon$ for $i = 0, 1, 2, \ldots, 2n-1$, $n \geqslant N$. Hence $\sum_{i=0}^{2n-1}(M_i - m_i)\times$ $\times u_i(x, y, z) \leqslant \varepsilon\sum_{i=0}^{2n-1} u_i(x, y, z) = \varepsilon$ and

$$\sum_{i=0}^{2n-1} m_i u_i(x, y, z) \leqslant \overline{u}(x, y, z) \leqslant \sum_{i=0}^{2n-1} M_i u_i(x, y, z);$$

then it will be sufficient to prove that $u_i(\varrho x, 0, \varrho z) = u_i(x, 0, z)$, $\varrho > 0$, $i = 0, 1, 2, \ldots, 2n-1$.

Let us introduce the polar coordinate system $r = \overline{PQ}$, $\gamma = OPQ$ where $P = P(x, 0, 0)$, $Q = Q(\overline{x}, \overline{y}, 0)$, $O = O(0, 0, 0)$. The Jacobi determinant of this transformation is equal to r. Hence for $0 \leqslant i \leqslant n-1$ we have

$$u_i(x, y, z) = \frac{1}{4\pi}\iint_{\sum_{k=i}^{n-1} T_k}\frac{2z\,d\xi\,d\eta}{[(\xi-x)^2+(\eta-y)^2+z^2]^{3/2}} -$$

$$-\frac{1}{4\pi}\iint_{\sum_{k=i+1}^{n-1} T_k}\frac{2z\,d\xi\,d\eta}{[(\xi-x)^2+(\eta-y)^2+z^2]^{3/2}} = \int_{\pi i/n}^{\pi}\int_{PR_i}^{\infty}\frac{2rz\,dr\,d\gamma}{[r^2+z^2]^{3/2}} -$$

$$-\int_{\pi(i+1)/n}^{\pi}\int_{PR_{i+1}}^{\infty}\frac{2rz\,dr\,d\gamma}{[r^2+z^2]^{3/2}}$$

where $PR_i = \dfrac{x\sin(\pi i/n)}{\sin(\gamma+\pi i/n)}$ (fig. 1). Hence

$$u_i(x,0,z) = \frac{1}{2\pi}\int\limits_{\varphi_i}^{\pi} \frac{z\,d\gamma}{\left[\dfrac{x^2\sin^2(\pi i/n)}{\sin^2[\gamma+\pi(i+1)/n]}+z^2\right]^{1/2}} - $$

$$-\frac{1}{2\pi}\int\limits_{\varphi_{i+1}}^{\pi} \frac{z\,d\gamma}{\left[\dfrac{x^2\sin^2(\pi i/n)}{\sin^2(\gamma+\pi(i+1)/n)}+z^2\right]^{1/2}}$$

so that we have $u_i(\varrho x, 0, \varrho z) = u_i(x, 0, z)$ for $i = 0, 1, \ldots, n-1$. Similarly $u_i(\varrho x, 0, \varrho z) = u_i(x, 0, z)$ for $i = n, n+1, \ldots, 2n-1$, q. e. d.

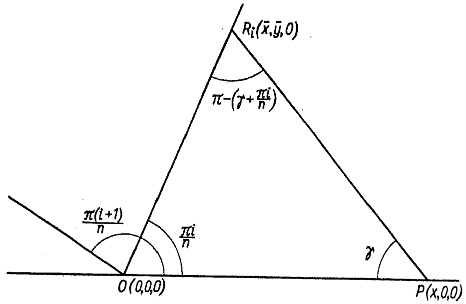

Fig. 1

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ANNALES
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Sur un théorème de C. E. Langenhop et G. Seifert

par Z. Opial (Kraków)

1. Considérons l'équation différentielle non-linéaire du second ordre

$$x''+f(x)x'+g(x) = p(t), \tag{1}$$

équivalente, comme on le sait, au système de deux équations différentielles du premier ordre

$$x' = y-F(x), \quad y' = -g(x)+p(t), \tag{2}$$

où $F(0) = 0$ et $F'(x) = f(x)$.

Désignons par λ_0 la racine (unique) de l'équation transcendante

$$\ln\lambda = (\pi+2\operatorname{arc\,ctg}\sqrt{4\lambda-1})/\sqrt{4\lambda-1}. \tag{3}$$

Il est facile de vérifier que c'est un nombre positif un peu plus grand que 3.

Relativement aux fonctions $f(x)$ et $g(x)$, définies et continues dans tout l'intervalle $(-\infty, +\infty)$, admettons l'hypothèse suivante:

Hypothèse (M). Il existe deux nombres finis a et b $(a<b)$ tels que dans l'intervalle $\langle a, b\rangle$ la fonction $g(x)$ soit de classe C^1, strictement croissante, et

$$f(x) \geqslant m > 0, \quad 0 \leqslant g'(x) \leqslant l < \lambda_0 m^2 \quad (a \leqslant x \leqslant b). \tag{4}$$

Supposons de plus que $p(t)$ soit une fonction continue dans tout l'intervalle $(-\infty, +\infty)$.

Evidemment, à toute solution $x(t)$ de l'équation (1) correspond une solution $\big(x(t), y(t)\big)$ du système (2) avec $y(t) = x'(t)+F\big(x(t)\big)$. Désignons par P la bande du plan (x, y) déterminée par les inégalités $a \leqslant x \leqslant b$, $|y| < +\infty$. Convenons de dire qu'une solution $x(t)$ de l'équation (1) ou, ce qui revient au même, que la solution correspondante $\big(x(t), y(t)\big)$ du système (2) est *P-bornée* dans un intervalle (t_1, t_2), si $a \leqslant x(t) \leqslant b$ quel que soit $t \in (t_1, t_2)$.

Dans la présente note je me propose d'établir d'abord deux théorèmes généraux sur l'unicité et la stabilité des solutions de l'équation (1), P-bornées dans l'intervalle $(-\infty, +\infty)$, et d'en déduire ensuite quelques