COLLOQUIUM MATHEMATICUM

VOL. VII 1960 FASC. 2

ON THE INTERSECTIONS OF TRANSFORMS OF LINEAR SETS

BY

ROY O. DAVIES (LEICESTER), J. M. MARSTRAND (BRISTOL) AND S. J. TAYLOR (BIRMINGHAM)

Introduction. A *translation* of the Euclidean line is a transformation $\varphi = \varphi_a$ of the form $\varphi_a(x) \equiv x+a$, where a is a real number; a *linear substitution* is a transformation φ of the form $\varphi(x) \equiv ax+b$, where a and b are real numbers and $a \neq 0$.

We consider the following properties of a linear set E:

(S_n) There exists a constant $\eta_n > 0$ such that for every set of n points $\{a_1, \ldots, a_n\}$ of diameter $< \eta_n$, the set $\prod_{s=1}^{n} \varphi_{a_s}(E)$ is non-empty.

(S_∞) For every positive integer n, the set E has property (S_n).

(S) There exists a constant $\eta > 0$ such that for every countable set of points $\{a_1, a_2, \ldots\}$ of diameter $< \eta$, the set $\prod_{s=1}^{\infty} \varphi_{a_s}(E)$ is non-empty.

(T) For every countable set of points $\{a_1, a_2, \ldots\}$, the set $\prod_{s=1}^{\infty} \varphi_{a_s}(E)$ is non-empty.

(T^*_∞) For every finite set of linear substitutions $\varphi_1, \ldots, \varphi_n$, the set $\prod_{s=1}^{n} \varphi_s(E)$ is non-empty.

(T^*) For every countable set of linear substitutions $\varphi_1, \varphi_2, \ldots$, the set $\prod_{s=1}^{n} \varphi_s(E)$ is non-empty.

Properties (S_n), (S) and (T) were defined by Marczewski (1), who posed the following problems:

Does there exist a perfect set of measure zero having property (S_3)?

Does there exist an F_σ set of measure zero having property (T)?

Recently Erdös and Kakutani [1] solved the first problem, and in fact constructed a bounded perfect set of measure zero having property (S_∞).

(1) [3]; [4]; this paper also referes to related earlier results.

Sets of Lebesgue measure zero can be classified by means of other measure functions. If $h(x)$ is an increasing real function and $h(0) = \lim_{x\to 0} h(x) = 0$ we call it a *measure function*, and we say that a set E is of *h-measure zero* if for every $\varepsilon > 0$ there exists a decomposition $E = E^1+E^2+\ldots$ such that $\sum_{i=1}^{\infty} h(\operatorname{diam} E^i) < \varepsilon$ (Hausdorff [2]). The Hausdorff dimension of E is the lower bound of values of α for which the Λ^α-measure of E is zero, where $\Lambda^\alpha(x) = x^\alpha$. The example of Erdös and Kakutani was of dimension one. We shall prove the existence, for every measure function $h(x)$, of linear sets E_1, E_2, E_3, E_4, all of h-measure zero, such that

E_1 *is an* F_σ *set having property* (T*),
E_2 *is a closed set having property* (T^*_∞),
E_3 *is a bounded* F_σ *set having property* (S), *and*
E_4 *is a bounded closed set having property* (S_∞).

Take

$$h(x) = \begin{cases} -(\log x)^{-1}, & 0 \leqslant x \leqslant \frac{1}{2}, \\ (\log 2)^{-1}, & x \geqslant \frac{1}{2}. \end{cases}$$

Each set then has dimension zero, and a fortiori Lebesgue measure zero. Since property (T*) implies property (T), the set E_1 gives a positive solution of Marczewski's second problem. Since a closed set has property (S_∞) only if the same is true of its perfect kernel (²), another solution of his first problem is provided by the perfect kernel of the set E_4.

The sets E_1, E_2, E_3 are as simple as possible, in the sense that (i) the only closed set having property (T*) (or even property (T)) is the whole line, (ii) no bounded set has property (T^*_∞), and (iii) a closed set having property (S) necessarily contains an interval and therefore has positive Lebesgue measure.

As Marczewski [4] showed, it is not difficult to construct for every measure function $h(x)$ a bounded G_δ set with property (S) and h-measure zero, and similarly one may construct a G_δ set with property (T*) and h-measure zero.

Our results generalize without difficulty to n-dimensional space.

Preliminaries. Let $h(x)$ be any measure function. Choose a sequence $\delta_0, \delta_1, \ldots$ of positive numbers, such that

$$(1) \qquad \delta_0 = 1, \quad \delta_\nu \leqslant \tfrac{1}{6}\delta_{\nu-1} \quad (\nu = 1, 2, \ldots),$$

and

$$(2) \qquad \lim_{\nu\to\infty}\left(\frac{6}{a\delta_{\nu-1}}+1\right) h(a\delta_\nu) = 0 \quad \text{for every rational } a > 0.$$

(²) For proof, see the remarks at the end of the paper.

To do this, enumerate the positive rationals as $a_1, a_2, \ldots$, and define δ_ν by induction, taking it so small as to satisfy (1) and the conditions

$$(3) \qquad \left(\frac{6}{a_i\,\delta_{\nu-1}}+1\right) h(a_i\,\delta_\nu) < \frac{1}{\nu} \qquad (i = 1, 2, \ldots, \nu).$$

Since, by (3),

$$\left(\frac{6}{a_i\,\delta_{\nu-1}}+1\right) h(a_i\,\delta_\nu) < \varepsilon$$

provided $\nu \geqslant \max(i, \varepsilon^{-1})$, the sequence thus defined satisfies (2).

Let F_ν ($\nu = 1, 2, \ldots$) be a fixed set consisting of closed intervals of length δ_ν equally spaced out along the entire x-axis so that the distance between the end-points of adjacent intervals is $\frac{1}{6}\delta_{\nu-1}$. Define the closed sets $K_i = F_{2^{i-1}}\cdot F_{3.2^{i-1}}\cdot F_{5.2^{i-1}}\cdot \ldots$ ($i = 1, 2, \ldots$). Let $\psi_1, \psi_2, \ldots$ be an enumeration of all substitutions $\psi(x) \equiv ax+b$ for which a, b are rational and $a \neq 0$.

Each of the sets $E_1, \ldots, E_4$ which we construct will be a subset of the set

$$(4) \qquad \sum_{r=1}^{\infty}\sum_{i=1}^{\infty} \psi_r(K_i),$$

and we now show that they will be of h-measure zero by proving that *the set* (4) *is of h-measure zero.* It is enough to show that for any fixed r, i and any fixed interval I of length 1 the set $I\cdot\psi_r(K_i)$ is of h-measure zero. Now if $\psi_r(x) \equiv ax+b$, any set $\psi_r(F_\nu)$ consists of intervals of length $|a|\,\delta_\nu$ separated by gaps of length $\frac{1}{6}|a|\,\delta_{\nu-1}$. Hence the set $I\cdot\psi_r(F_\nu)$ consists of

$$m \leqslant \frac{6}{|a|\,\delta_{\nu-1}}+1$$

intervals $J_\nu^1, \ldots, J_\nu^m$, say, each of length $\leqslant |a|\,\delta_\nu$. Since

$$\sum_{\mu=1}^{m} h(\operatorname{diam} J_\nu^\mu) \leqslant \left(\frac{6}{|a|\,\delta_{\nu-1}}+1\right) h(|a|\,\delta_\nu) \to 0 \quad \text{as} \quad \nu\to\infty$$

by (2), and since for ν of the form $(2q-1)2^{i-1}$ we have $I\cdot\psi_r(K_i) \subset I\cdot\psi_r(F_\nu)$, our result follows.

The set E_1. We shall show that the F_σ set

$$E_1 = \sum_{r=1}^{\infty}\sum_{i=1}^{\infty} \psi_r(K_i)$$

has property (T*).

Let then $\varphi_1, \varphi_2, \ldots$ be any countable set of linear substitutions, and for each positive integer s choose from among $\psi_1, \psi_2, \ldots$ a substitution $\psi_{r(s)}$ such that $\chi_s = \varphi_s\psi_{r(s)}$ is of the form $ax+b$ where $1 \leqslant a \leqslant 2$. We have

$$(5)\qquad \prod_{s=1}^{\infty}\varphi_s(E_1) \supset \prod_{s=1}^{\infty}\varphi_s\psi_{r(s)}\Big(\sum_{i=1}^{\infty}K_i\Big) = \prod_{s=1}^{\infty}\chi_s\Big(\sum_{i=1}^{\infty}K_i\Big)$$
$$\supset \prod_{s=1}^{\infty}\chi_s(K_s) = [\chi_1(F_1)\cdot\chi_1(F_3)\cdot\chi_1(F_5)\ldots]\cdot[\chi_2(F_2)\cdot\chi_2(F_6)\ldots]\ldots$$
$$= \prod_{\nu=1}^{\infty}\lambda_\nu(F_\nu),$$

say, where $\lambda_\nu(x) \equiv a_\nu x + b_\nu$ with $1 \leqslant a_\nu \leqslant 2$.

Now a set $\lambda_{\nu-1}(F_{\nu-1})$ consists of closed intervals of length $\geqslant \delta_{\nu-1}$ and a set $\lambda_\nu(F_\nu)$ consists of closed intervals of length $\leqslant 2\delta_\nu \leqslant \frac{1}{3}\delta_{\nu-1}$ (by (1)), separated by gaps of length $\leqslant \frac{1}{3}\delta_{\nu-1}$. There is therefore at least one complete interval of $\lambda_\nu(F_\nu)$ in each interval of $\lambda_{\nu-1}(F_{\nu-1})$, whence the set $\prod_{\nu=1}^{\infty}\lambda_\nu(F_\nu)$ is non-empty. By (5) the set $\prod_{s=1}^{\infty}\varphi_i(E_1)$ is also non-empty and thus E_1 has property (T*).

The set E_2. For each $m\,(=1, 2, \ldots)$ choose closed intervals I_m, I'_m such that

(i) the set $\sum_{r=1}^{m}\psi_r(I_m)$ lies entirely at a distance $\geqslant m$ from the origin, and

(ii) I'_m is of length $\geqslant 1$ and is contained in all intervals $\chi(I_m)$ for χ of the form $ax+b$ where $1 \leqslant a \leqslant 2$ and $0 \leqslant b \leqslant 1$. We may for instance take as I_m, I'_m the respective intervals $[k, 2k+2]$, $[2k+1, 2k+2]$ if k is sufficiently large. We shall show that the set

$$E_2 = \sum_{m=1}^{\infty}\sum_{r=1}^{\infty}\psi_r(K_r I_m),$$

which is closed by (i), has property (T^*_∞).

Let then $\varphi_1, \ldots, \varphi_n$ be any finite set of linear substitutions and choose from among $\psi_1, \psi_2, \ldots$ *distinct* substitutions $\psi_{r(1)}, \ldots, \psi_{r(n)}$ such that for each value of $s\,(=1, \ldots, n)$, $\chi_s = \varphi_s\psi_{r(s)}$ is of the form $ax+b$ where $1 \leqslant a \leqslant 2$ and $0 \leqslant b \leqslant 1$. If $R = \max[r(1), \ldots, r(n)]$, we have

$$(6)\qquad \prod_{s=1}^{n}\varphi_s(E_2) \supset \prod_{s=1}^{n}\varphi_s\Big\{\sum_{r=1}^{R}\psi_r(K_r\cdot I_R)\Big\} \supset \prod_{s=1}^{n}\varphi_s\psi_{r(s)}(K_{r(s)}\cdot I_R)$$
$$= \prod_{s=1}^{n}\chi_r(K_{r(s)}\cdot I_R) \supset \prod_{r=1}^{\infty}\chi'_r(K_r\cdot I_R),$$

where we put $\chi'_r = \chi_s$ if there exists s such that $r = r(s)$ and we put $\chi'_r(x) \equiv x$ otherwise. Now

$$(7)\qquad \prod_{r=1}^{\infty}\chi'_r(K_r\cdot I_R) = \Big\{\prod_{r=1}^{\infty}\chi'_r(I_R)\Big\}\cdot\Big\{\prod_{r=1}^{\infty}\chi'_r(K_r)\Big\}$$
$$\supset I'_R\cdot\prod_{r=1}^{\infty}\chi'_r(K_r) = I'_R\cdot\prod_{\nu=1}^{\infty}\lambda_\nu(F_\nu),$$

say, where $\lambda_\nu(x) \equiv a_\nu x + b_\nu$ with $1 \leqslant a_\nu \leqslant 2$.

Since $\delta_1 \leqslant \frac{1}{6}$ (by (1)), and I'_R has length $\geqslant 1$, there is at least one complete interval of $\lambda_1(F_1)$ in I'_R, and at least one complete interval of $\lambda_\nu(F_\nu)$ in each interval of $\lambda_{\nu-1}(F_{\nu-1})$, whence the set $I'_R\cdot\prod_{\nu=1}^{\infty}\lambda_\nu(F_\nu)$ is non-empty. By (6) and (7) the set $\prod_{s=1}^{n}\varphi_s(E_2)$ is also non-empty, and thus E_2 has property (T^*_∞).

The set E_3. Let I denote a closed interval of length 2. We shall show that the bounded F_σ set

$$E_3 = \sum_{i=1}^{\infty}K_i\cdot I$$

has property (S), with constant $\eta = 1$.

Let then $\{a_1, a_2, \ldots\}$ be any countable set of points of diameter < 1. We have

$$(8)\qquad \prod_{s=1}^{\infty}\varphi_{a_s}(E_3) \supset \prod_{s=1}^{\infty}\varphi_{a_s}(K_s\cdot I) = \Big\{\prod_{s=1}^{\infty}\varphi_{a_s}(I)\Big\}\cdot\Big\{\prod_{s=1}^{\infty}\varphi_{a_s}(K_s)\Big\}.$$

Since $\prod_{s=1}^{\infty}\varphi_{a_s}(I)$ contains (as is easily seen) a closed interval of length $\geqslant 1$, the same argument as before shows that the set on the right of (8) is non-empty. By (8) the set $\prod_{s=1}^{\infty}\varphi_{a_s}(E_3)$ is also non-empty, and thus E_3 has property (S).

The set E_4. It is enough for us to construct for each $n \geqslant 2$ a bounded closed set D_n with property (S_n), since if we then place sets similar to $D_2, D_3, D_4, \ldots$ in the respective intervals $(0, \frac{1}{2})$, $(\frac{1}{2}, \frac{3}{4})$, $(\frac{3}{4}, \frac{7}{8})$, ... and add the point $\{1\}$, we obtain a bounded closed set E_4 having property (S_∞). We shall show that if I is a closed interval of length 2, the bounded closed set

$$D_n = \sum_{i=1}^{n}K_i\cdot I$$

has property (S_n), with constant $\eta_n = 1$.

Let then $\{a_1, \ldots, a_n\}$ be any set of n points of diameter < 1. We have

$$\prod_{s=1}^{n} \varphi_{a_s}(D_n) \supset \prod_{s=1}^{n} \varphi_{a_s}(K_s \cdot I) = \Big\{\prod_{s=1}^{n} \varphi_{a_s}(I)\Big\}\Big\{\prod_{s=1}^{n} \varphi_{a_s}(K_s)\Big\}. \tag{9}$$

For the same reasons as before, the set on the right of (9) is non-empty, and D_n has property (S_n).

Remarks. I. *No countable set D has property* (S_2). For given any constant $\eta > 0$ there exists a real number a satisfying $0 < a < \eta$ and not equal to the distance between any two points of D, whence $\varphi_0(D)\varphi_a(D) = 0$.

II. *If a set E has property* (S_∞) *then for every n there exists a constant $\eta_n' > 0$ such that for every set of n points $\{a_1, \ldots, a_n\}$ of diameter $< \eta_n'$, the set $\prod_{s=1}^{n}\varphi_{a_s}(E)$ has property* (S_2).

Proof. Let E have property (S_∞). Then it has property (S_{2n}), with constant η_{2n}, and we shall show that we may take $\eta_n' = \frac{1}{2}\eta_{2n}$. Let $\{a_1, \ldots, a_n\}$ be any set of n points of diameter $< \frac{1}{2}\eta_{2n}$, and $\{a, b\}$ any pair of points of diameter $< \frac{1}{2}\eta_{2n}$. Then diameter of the set $\{a+a_1, \ldots, a+a_n, b+a_1, \ldots, b+a_n\}$ is less than η_{2n}, whence the set

$$\varphi_a\Big\{\prod_{s=1}^{n} \varphi_{a_s}(E)\Big\} \cdot \varphi_b\Big\{\prod_{s=1}^{n} \varphi_{a_s}(E)\Big\} = \Big\{\prod_{s=1}^{n} \varphi_{a+a_s}(E)\Big\} \cdot \Big\{\prod_{s=1}^{n} \varphi_{b+a_s}(E)\Big\}$$

is non-empty. Thus the set $\prod_{s=1}^{n}\varphi_{a_s}(E)$ has property (S_2) with constant $\frac{1}{2}\eta_{2n}$.

Similarly, for sets having properties (S), (T), (T_∞^*) or (T^*) the non-empty intersections in fact have property (S_2) (possibly with an altered value of the constant η, in the case of property (S)).

From I and II one immediately deduces III, from which follows IV and then V:

III. *The intersections in question are non-countable.* Consequently, if E is analytic these intersections contain perfect subsets and are of the power of the continuum. In particular, this is true of our sets $E_1, \ldots, E_4$.

IV. *If a set E has any of the properties* (S_∞), (S), (T), (T_∞^*) *or* (T^*) *and if D is countable, then $E-D$ has the same property.*

V. *A closed set has any of these properties only if its perfect kernel has the same property.*

Generalizations. We could generalize our results, replacing linear substitutions by a wider class of transformations, for example those of the form $\varphi(x) \equiv (ax+b)/(cx+d)$, where $ad \neq bc$, and using the same methods. We have not however determined how far it is possible to go in this direction.

REFERENCES

[1] P. Erdös and S. Kakutani, *On a perfect set*, Colloquium Mathematicum 4 (1957), p. 195-196.

[2] F. Hausdorff, *Dimension und äußeres Maß*, Mathematische Annalen 79 (1919), p. 157-179.

[3] E. Marczewski, **P 125.** Colloquium Mathematicum 3 (1954), p. 75.

[4] — *O przesunięciach zbiorów i o pewnym twierdzeniu Steinhausa* (*On translations of sets and on a certain theorem of Steinhaus*), Prace Matematyczne 1 (1955), p. 256-263 (in Polish).

Reçu par la Rédaction le 28. 2. 1959