parallèle au même axe. Supposons que la vitesse angulaire de rotation et la vitesse de translation soient constantes et égales à l'unité.

Au cours de ce mouvement, envisagé dans un voisinage suffisamment petit de la position initiale du plan Π, chaque point $(\lambda, 0, 0)$ décrit une hélice

$$H_\lambda: \quad x = \lambda\cos\tau, \quad y = \lambda\sin\tau, \quad z = \tau \quad \left(0 \leqslant \tau \leqslant \frac{\pi}{2}\right)$$

tandis que chacune des courbes C_λ décrit une surface

$$S_\lambda: \quad x = \frac{\lambda}{\cos t}\cdot\cos\tau, \quad y = \frac{\lambda}{\cos t}\cdot\sin\tau,$$

$$z = t - \operatorname{tg} t + \tau \quad \left(-\frac{\pi}{2} < t < \frac{\pi}{2},\ 0 \leqslant \tau \leqslant \frac{\pi}{2}\right).$$

En posant $u = t+\tau$ et $\sigma = -\operatorname{tg} t$, on peut écrire les équations de la surface S_λ sous la forme

$$x = \lambda(\cos u - \sigma\sin u), \quad y = \lambda(\sin u + \sigma\cos u), \quad z = u+\sigma.$$

Il en résulte que S_λ est une partie de la surface réglée engendrée par les tangentes à l'hélice H_λ.

Par chaque point du domaine Ω: $x>0$, $y>0$ et $-\infty < z < +\infty$ passe une et une seule des surfaces S_λ. Par conséquent, tout point de ce domaine est situé sur une génératrice d'une surface S_λ. L'ensemble de ces droites définit dans le domaine envisagé un système d'équations différentielles

$$\frac{dx}{dy} = f_1(y, x, z), \quad \frac{dz}{dy} = f_2(y, x, z), \tag{2}$$

où les fonctions $f_1(y, x, z)$ et $f_2(y, x, z)$ peuvent être choisies de façon qu'elles soient continues.

Par tout point de Ω passe une intégrale rectiligne du système (2), à savoir une génératrice d'une des surfaces S_λ. Mais, en outre, chacune des hélices H_λ constitue aussi une courbe intégrale de ce système.

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ON RANDOM VARIABLES WHOSE QUOTIENT FOLLOWS THE CAUCHY LAW

BY

I. KOTLARSKI (WARSAW)

1. Introduction. Let the random variable X with frequency function $f(x)$ have the following property: the quotient of two independent random variables having the same frequency function $f(x)$ follows the Cauchy law, with the frequency function

$$g(y) = \frac{1}{\pi}\cdot\frac{1}{1+y^2} \quad (-\infty < y < +\infty). \tag{1}$$

Denote by $\mathfrak{X}$ the set of the random variables X having the above-mentioned property. It has been supposed that only the variable with the normal frequency

$$f(x) = \frac{1}{\sqrt{\pi}} e^{-x^2} \quad (-\infty < x < +\infty) \tag{2}$$

belongs to $\mathfrak{X}$. Laha [6] proved this conjecture to be false, showing that the variable with the frequency

$$f(x) = \frac{\sqrt{2}}{\pi}\cdot\frac{1}{1+x^4} \quad (-\infty < x < +\infty) \tag{3}$$

also belongs to $\mathfrak{X}$.

In this paper we shall give a method of construction of an arbitrary number of random variables belonging to $\mathfrak{X}$, and some conditions necessary and sufficient for belonging to $\mathfrak{X}$.

I am greatly indebted to professor M. Fisz for his suggestions and remarks.

2. The Mellin transform of a symmetrical random variable. It is known that for X to belong to $\mathfrak{X}$ it is necessary and sufficient that its frequency $f(x)$ satisfy the integral equation

$$\int_{-\infty}^{+\infty} f(yx)f(x)\,|x|\,dx = \frac{1}{\pi}\cdot\frac{1}{1+y^2} \quad (-\infty < y < +\infty) \tag{4}$$

(see [3], formula (2.9.16′)).

Representing the function $f(x)$ as a sum of its even part and its odd part, and substituting the sum into (4) we see that its odd part vanishes, and the frequency $f(x)$ satisfies the condition

$$f(-x) = f(x) \quad (-\infty < x < +\infty). \tag{5}$$

We shall solve the problem by using the Mellin transform of the symmetrical random variable X (see [2, 4, 8]); it is defined by the formula

$$h(z) = E\{|X|^z\} = \int_{-\infty}^{+\infty} |x|^z f(x)\,dx \tag{6}$$

where z is a complex variable. The function $h(z)$ is analytical in a strip S containing the imaginary axis and parallel to it. The inverse transform is expressed by the formula

$$f(x) = \frac{1}{4\pi i}\int_{c-i\infty}^{c+i\infty} |x|^{-z-1} h(z)\,dz \tag{7}$$

for every x where $f(x)$ is continuous, the path of integration being the imaginary axis or any line parallel to it and lying within S.

It should be noted that if $h(z)$ is the Mellin transform of a random variable X, then the function $\varphi(t) = h(it)$ is the characteristic function of the variable $U = \ln|X|$, since according to (6) we have

$$h(it) = E\{|X|^{it}\} = E\{e^{iUt}\} = \varphi(t). \tag{8}$$

Conversely, if $\varphi(t)$ is the characteristic function of the random variable U, then $h(it) = \varphi(t)$ defines the Mellin transform of a symmetrical variable X, for which $|X| = e^U$, on the imaginary axis.

If we have n independent symmetrical random variables $X_1, X_2, \ldots, X_n$, with their Mellin transforms $h_1(z), h_2(z), \ldots, h_n(z)$, then for every system $k_1, k_2, \ldots, k_n = \pm 1$, and for every real $a \neq 0$, the Mellin transform of the variable

$$a\cdot X_1^{k_1}\cdot X_2^{k_2}\cdot\ldots\cdot X_n^{k_n}$$

is

$$|a|^z\cdot h_1(k_1 z)\cdot h_2(k_2 z)\cdot\ldots\cdot h_n(k_n z). \tag{9}$$

The correspondence between the frequency function $f(x)$ of a symmetrical random variable X and its Mellin transform $h(z)$ ($c_1 < \mathrm{Re}\,z < c_2$, $c_1 < 0$, $c_2 > 0$), will be written as follows:

$$f(z);\ h(z). \tag{10}$$

It is easily inferred that

$$\frac{|q|}{2\Gamma(b)}|x|^{bq-1}e^{-|x|^q};\ \frac{\Gamma\left(b+\frac{z}{q}\right)}{\Gamma(b)} \quad \left(b > 0,\ q \neq 0,\ \frac{\mathrm{Re}\,z}{q} > -b\right). \tag{11}$$

Hence

$$\frac{1}{\sqrt{\pi}}e^{-x^2};\ \frac{1}{\sqrt{\pi}}\Gamma\left(\frac{1+z}{2}\right) \quad (\mathrm{Re}\,z > -1), \tag{12}$$

$$\frac{1}{\sqrt{\pi}}\cdot\frac{1}{x^2}e^{-1/x^2};\ \frac{1}{\sqrt{\pi}}\Gamma\left(\frac{1-z}{2}\right) \quad (\mathrm{Re}\,z < 1). \tag{13}$$

Further we have

$$\frac{|q|\cdot 2^p\cdot\cos^p b\pi}{8\pi^2\Gamma(p)}|x|^{bq-1}\Gamma\left(\frac{p}{2}+i\frac{q}{2\pi}\ln|x|\right)\Gamma\left(\frac{p}{2}-i\frac{q}{2\pi}\ln|x|\right);\ \frac{\cos^p b\pi}{\cos^p\left(b+\frac{z}{q}\right)\pi} \tag{14}$$

$$(q \neq 0,\ p > 0,\ |b| < \tfrac{1}{2};\ -b-\tfrac{1}{2} < (\mathrm{Re}\,z)/q < -b+\tfrac{1}{2}).$$

Hence

$$\frac{1}{\pi}\cdot\frac{1}{1+x^2};\ \frac{1}{\cos\frac{z}{2}\pi} \quad (-1 < \mathrm{Re}\,z < 1), \tag{15}$$

$$\frac{1}{(2\pi)^{3/2}\pi|x|}\Gamma\left(\frac{1}{4}+i\frac{\ln|x|}{\pi}\right)\Gamma\left(\frac{1}{4}-i\frac{\ln|x|}{\pi}\right);\ \frac{1}{\sqrt{\cos\frac{z}{2}\pi}} \quad (-1 < \mathrm{Re}\,z < 11), \tag{16}$$

$$\frac{\sqrt{2}}{\pi}\cdot\frac{1}{1+x^4};\ \frac{1}{\sqrt{2}\cos\frac{1-z}{4}\pi} \quad (-1 < \mathrm{Re}\,z < 3), \tag{17}$$

$$\frac{\sqrt{2}}{\pi}\cdot\frac{x^2}{1+x^4};\ \frac{1}{\sqrt{2}\cos\frac{1+z}{4}\pi} \quad (-3 < \mathrm{Re}\,z < 1). \tag{18}$$

Expanding the function $1/\Gamma(z)$ into an infinite product by the formula

$$\frac{1}{\Gamma(z)} = \lim_{n\to\infty} e^{\left(\sum_{k=1}^{n}\frac{1}{k}-\ln n\right)z}\cdot z\cdot\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)c^{-z/k} \quad (\mathrm{Re}\,z > 0) \tag{19}$$

(see [7], p. 267, formula (132)) we may represent the Mellin transform (11) on the imaginary axis as

$$\varphi(t) = h(it) = \frac{\Gamma\left(b+\frac{it}{q}\right)}{\Gamma(b)} = \lim_{n\to\infty} e^{\frac{it}{q}\ln n}\prod_{k=0}^{n}\left[1+\frac{it}{(k+b)q}\right]^{-1}. \tag{20}$$

We see that it is the characteristic function of some infinitely divisible random variable (see [5], p. 78, example 4, p. 80, theorem 3). Hence, the function

$$h(z)=\left[\frac{\Gamma\left(b+\frac{z}{q}\right)}{\Gamma(b)}\right]^p \tag{21}$$

is for every $p\geqslant 0$ the Mellin transform (see [5], p. 80, theorem 4) of some symmetrical random variable on some strip S containing the imaginary axis and parallel to it.

It may easily be verified that the random variable defined by (14) is a quotient of two independent random variables given by (21) for $b=\frac{1}{2}+b_1$ and $b=\frac{1}{2}-b_1$, where $|b_1|<\frac{1}{2}$.

3. Method of determining other random variables in $\mathfrak{X}$. By (9) and (15), instead of solving equation (4) it is sufficient to solve the functional equation

$$h(z)\cdot h(-z)=\frac{1}{\cos\frac{z}{2}\pi}\qquad(-1<\operatorname{Re}z<1) \tag{22}$$

in the class of Mellin transforms of symmetrical random variables.

Let us remark that if equation (22) holds for some function $h_0(z)$, it holds also for $h_0(-z)$, and for the functions $|a|^z h_0(z)$ and $|a|^z h_0(-z)$, provided that $a\neq 0$. Thus, if some random variable X belongs to $\mathfrak{X}$, the variables $1/X$, $a\cdot X$, a/X also belong to $\mathfrak{X}$ for all $a\neq 0$.

It is easy to show that equation (22) holds for Mellin transforms (12), (13), (16), (17), (18). Thus the random variables defined by those expressions multiplied by an arbitrary $a\neq 0$ also belong to $\mathfrak{X}$.

To obtain a method of constructing new random variables belonging to $\mathfrak{X}$ we use the following formula for the function $\Gamma(w)$ (see [7], p. 272, formula (145)):

$$\Gamma(nw)=\frac{n^{nw-\frac{1}{2}}}{(2\pi)^{\frac{n-1}{2}}}\prod_{k=1}^{n}\Gamma\left(w+\frac{k-1}{n}\right). \tag{23}$$

Using this formula we can write

$$\frac{1}{\cos\frac{z}{2}\pi}=\frac{\Gamma\left(\frac{1+z}{2}\right)}{\Gamma(\frac{1}{2})}\cdot\frac{\Gamma\left(\frac{1-z}{2}\right)}{\Gamma(\frac{1}{2})}=\prod_{k=1}^{n}\frac{\Gamma\left(\frac{2k-1+z}{2n}\right)}{\Gamma\left(\frac{2k-1}{2n}\right)}\cdot\prod_{k=1}^{n}\frac{\Gamma\left(\frac{2k-1-z}{2n}\right)}{\Gamma\left(\frac{2k-1}{2n}\right)}. \tag{24}$$

Hence and from (9) and (11) it follows that for every positive integer n and for $\varepsilon_k^{(n)}$ equal to $+1$ or -1, the function

$$h_n(z)=\prod_{k=1}^{n}\frac{\Gamma\left(\frac{2k-1+\varepsilon_k^{(n)}z}{2n}\right)}{\Gamma\left(\frac{2k-1}{2n}\right)} \tag{25}$$

is the Mellin transform of some symmetrical random variable belonging to $\mathfrak{X}$. Taking $n=1$, $\varepsilon_1^{(1)}=1$ we obtain the random variable (12); taking $n=1$, $\varepsilon_1^{(1)}=-1$ we obtain (13). Taking $n=2$, $\varepsilon_1^{(2)}=+1$, $\varepsilon_2^{(2)}=-1$ we obtain (17); taking $n=2$, $\varepsilon_1^{(2)}=-1$, $\varepsilon_2^{(2)}=+1$ we obtain (18) (*).

Evidently the function

$$h(z)=|a|^z\prod_{r=1}^{N}\left[\prod_{k=1}^{n_r}\frac{\Gamma\left(\frac{2k-1+\varepsilon_k^{(n_r)}z}{2n_r}\right)}{\Gamma\left(\frac{2k-1}{2n_r}\right)}\right]^{p_r} \tag{26}$$

where $a\neq 0$, $p_r\geqslant 0$, $\sum_{r=1}^{N}p_r=1$, is the Mellin transform of some random variable from $\mathfrak{X}$. In particular, taking $a=1$, $N=2$, $n_1=n_2=1$, $p_1=p_2=\frac{1}{2}$, $\varepsilon_1^{(n_1)}=+1$, $\varepsilon_1^{(n_2)}=-1$, we obtain the variable (16).

4. The characteristics of the set $\mathfrak{X}$ by Mellin transforms. We shall characterize the set $\mathfrak{X}$ by solving equation (22) in the class of Mellin transforms of symmetrical random variables. To do this, we represent the unknown function $h(z)$ as a product

$$h(z)=h_0(z)\cdot H(z), \tag{27}$$

where $h_0(z)$ is a known Mellin transform which is a special solution of equation (22) (for instance the Mellin transform given on the right side of (16)), and $H(z)$ is a new unknown function, defined in a strip S. Equation (22) is in this way reduced to the equation

$$H(z)\cdot H(-z)=1. \tag{28}$$

Let us write (28) in the form

$$\ln H(-z)=-\ln H(z); \tag{29}$$

we see that the function

$$\gamma(z)=\ln H(z) \tag{30}$$

is an odd one, analytical in a strip S. Then we may represent the unknown

(*) (Added in proof.) I have recently learned that the particular cases given by the formulae (13), (16) and (18) have been considered by Steck [7a].

function $H(z)$ in the form

$$H(z) = e^{\gamma(z)} \tag{31}$$

and the unknown function $h(z)$ according to (27) by the formula

$$h(z) = h_0(z)\cdot e^{\gamma(z)}. \tag{32}$$

Since the frequency function is determined by its Mellin transform on the imaginary axis ($c = 0$ in formula (7)), we shall examine the function

$$\varphi(t) = h(it) = h_0(it)\cdot e^{\gamma(it)}. \tag{33}$$

According to (8) the functions $\varphi(t)$ and $h_0(it)$ are the characteristic functions of some random variables, and that is why they satisfy the conditions

$$\varphi(-t) = \overline{\varphi(t)}, \tag{34}$$

$$h_0(-it) = \overline{h_0(it)} \tag{35}$$

(see [3], formula (4.1.9)).

Splitting the odd function $\gamma(it)$ into the real part $\alpha(t)$ and the imaginary part $\beta(t)$

$$\gamma(it) = \alpha(t)+i\beta(t), \tag{36}$$

we write according to (33)

$$\begin{cases} \varphi(-t) = h_0(-it)\cdot e^{-\alpha(t)-i\beta(t)}, \\ \overline{\varphi(t)} = \overline{h_0(it)}\cdot e^{+\alpha(t)-i\beta(t)}. \end{cases} \tag{37}$$

Comparing equations (37) and taking into account (34) and (35) we see that $\alpha(t) \equiv 0$, and we obtain

$$\varphi(t) = h(it) = h_0(it)\cdot e^{i\beta(t)}. \tag{38}$$

As we see, if the random variable X belongs to $\mathcal{X}$, its Mellin transform has on the imaginary axis the form (38), where $h_0(z)$ is the Mellin-transform of some particular variable belonging to $\mathcal{X}$, and $\beta(t)$ is a real odd function, analytic along the axis of t. It is the necessary condition for a random variable to belong to $\mathcal{X}$. To find the condition that would also be sufficient, we shall give the matter some further consideration.

We have seen that a function $h(z)$ defined in a strip S is the Mellin transform of some random variable if and only if $h(it)$ is a characteristic function of some random variable (formula (8)).

Now we shall make use of the following condition (see [1], p. 91):

In order that a given, bounded and continuous function $\varphi(t)$ be the characteristic function of a random variable, it is necessary and sufficient that $\varphi(0) = 1$ and that the function

$$\psi(x, A) = \int_0^A\int_0^A \varphi(t-u)\cdot e^{ix(t-u)}\,dt\,du \tag{39}$$

be real and non-negative for all real x and all $A > 0$.

Applying this condition to function (38), and taking $h_0(z)$ given by (16), we obtain the following condition: For a function $h(z)$, which is analytic in a strip S containing the imaginary axis and parallel to it, to be the Mellin transform of some random variable X belonging to $\mathcal{X}$, it is necessary and sufficient that it be represented on the imaginary axis in the form

$$h(it) = \frac{e^{i\beta(t)}}{\sqrt{\operatorname{ch}\frac{t}{2}\pi}}, \tag{40}$$

where $\beta(t)$ is a real odd function analytic along the axis t and satisfying the conditions

$$\begin{cases} \displaystyle\int_0^A\int_0^A \frac{\sin[\beta(t-u)+x(t-u)]}{\sqrt{\operatorname{ch}\frac{t-u}{2}\pi}}\,dt\,du = 0, \\ \displaystyle\int_0^A\int_0^A \frac{\cos[\beta(t-u)+x(t-u)]}{\sqrt{\operatorname{ch}\frac{t-u}{2}\pi}}\,dt\,du \geqslant 0 \end{cases} \tag{41}$$

for every real x and every $A > 0$.

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ЗАМЕЧАНИЕ ОБ ОДНОЙ ТЕОРЕМЕ ХИНЧИНА ИЗ ТЕОРИИ СЛУЧАЙНЫХ ПОТОКОВ

Ю. ЛУКАШЕВИЧ (ВРОЦЛАВ)

В монографии [1] по теории массового обслуживания А. Я. Хинчин доказывает следующую теорему:

Теорема. *Для потока Пальма (стационарного ординарного потока с ограниченным последействием) законы распределения $F_k(x)$ расстояний $z_k = t_k - t_{k-1}$ $(z_1 = t_1)$ между последовательными вызовами даются формулами*

$$F_1(x) = \lambda \int_0^x \varphi_0(u)\,du, \qquad F_k(x) = 1-\varphi_0(x) \qquad (k = 2, 3, \ldots),$$

где интенсивность λ определяется из условия, что $F_1(\infty) = \lambda \int_0^\infty \varphi_0(u)\,du = 1$, а $\varphi_0(x)$ означает нулевую функцию Пальма, т. е. условную вероятность того, что в промежутке времени длины x не будет вызовов, если вызов имел место в начале этого промежутка.

В доказательстве этой теоремы существенную роль играет следующая лемма:

Лемма. *Для любого потока Пальма и любого $r = 1, 2, \ldots$*

$$\lim_{u\to 0} \frac{\psi_{r+1}(u)}{\psi_r(u)} = 0, \tag{1}$$

где $\psi_r(u)$ вероятность того, что в промежутке времени длины u будет не менее r вызовов.

Но предел (1) не всегда существует. Можно указать такие потоки Пальма, для которых $\psi_r(u) \equiv 0$ в некотором интервале $[0, a]$ $(a > 0)$ и вследствие этого отношение $\psi_{r+1}(u)/\psi_r(u)$ вблизи нуля не имеет смысла. Примером такого потока служит поток, в котором рас-

(1) А. Я. Хинчин, *Математические методы теории массового обслуживания,* Труды Математического Института им. В. А. Стеклова 49 (1955), стр. 43.