Decomposition of the line in isometric three-point sets

by

A. D. **Sands** and S. **Świerczkowski** (Dundee, Scotland)

In the present note we have simplified the proof first obtained by K. Koutský and M. Sekanina [1] of the following result.

THEOREM. *Given any three distinct real numbers x, y, z the real line L can be decomposed in sets isometric to $\{x, y, z\}$.*

Proof. By a set isometric to $\{x, y, z\}$ we mean a set which can be obtained from $\{x, y, z\}$ by translation or by reflection and translation. If the result holds for the set $\{x, y, z\}$ it clearly holds for any translate $\{x+a, y+a, z+a\}$ of it. Thus, without loss of generality, we may suppose that $z = 0$. Also if it holds for $\{x, y, z\}$ it must clearly hold for $\{ux, uy, uz\}$ where u is any non-zero real number. If x and y are commensurable we may suppose that they are relatively prime integers. If not, we may suppose that x is irrational and that $y = 1$. For any sets $A, B \subset L$ we define $A+B = \{a+b\colon\ a \in A,\ b \in B\}$.

We consider first the case where x and y are relatively prime integers. Since L can be decomposed in translates of the set N of all integers, it is sufficient to show that N can be decomposed in sets isometric to $\{0, x, y\}$. If 0, x and y form a complete set of residues modulo 3, then this result is a consequence of the more general well-known fact:

(*) *If k integers $x_1, \ldots, x_k$ are different modulo k, then $N = \{x_1, \ldots, x_k\} + \{0, \pm k, \pm 2k, \ldots\}$, i. e. N is decomposed in translates of $\{x_1, \ldots, x_k\}$.*

If $0, x, y$ are not different mod 3, then, since x and y are relatively prime, exactly one of the possibilities

$$3|x, \quad 3|y \quad \text{or} \quad 3|x-y$$

holds. The first two cases are clearly similar and only $3|x$ need be considered. In the third case we may consider instead of the set $\{0, x, y\}$, its translate by $-y$, $\{0, x-y, -y\}$, and so this case also reduces to the first one above. Thus we may suppose that $3|x$.

[1] See Časopis pro Pěstování Matematiky 83 (1958), No. 3, p. 317-326 (in Czech, English summary).

We denote by $l \geqslant 0$ the largest integer l for which $2^l | x$ and we define

$$S = \{0, y, 2y, \dots, (3 \cdot 2^l - 1)y\}.$$

Then all elements of $S \cup (S+x)$ are different mod $3 \cdot 2^{l+1}$. For if s and t are in S, then $s \equiv t \pmod{3 \cdot 2^l}$ is not possible, unless $s = t$, since as x and y are relatively prime and $3 \cdot 2^l | x$ it follows that $3 \cdot 2^l$ and y are relatively prime. Now if $s \equiv t + x \pmod{3 \cdot 2^{l+1}}$, then, since $3 \cdot 2^l | x$, we have $s \equiv t \pmod{3 \cdot 2^l}$ and so $s = t$. But this implies $2^{l+1} | x$ which is not so. Hence, by $(*)$, N can be decomposed in sets isometric to $S \cup (S+x)$. The theorem now follows, in this case, since $S \cup (S+x)$ can be decomposed into sets isometric to $\{0, x, y\}$, namely $\{3ny, (3n+1)y, 3ny+x\}$ and $\{(3n+2)y+x, (3n+1)y+x, (3n+2)y\}$, where $n = 0, \dots, 2^l - 1$.

If now $y = 1$ and x is irrational, then there exists a maximal set of rationally independent real numbers (i. e. a Hamel basis) containing 1 and $x+1$, say $\{1, x+1, \delta_1, \delta_2, \dots\}$. Let A be the set of all numbers $r_{i_0}(x+1) + r_{i_1}\delta_{i_1} + \dots + r_{i_k}\delta_{i_k}$, where k is an integer and r_{i_l} are rational. Then, if R is the set of all rationals, we have

$$L = A + R \tag{1}$$

and moreover each number $l \in L$ has a unique representation as $l = a + r$ with $a \in A$, $r \in R$. We define $B = \{r \in R\colon 3m \leqslant r < 3m+1;\ m = 0, \pm 1, \pm 2, \dots\}$ and $C = A + B$. We shall show that

$$L = C \cup (C+1) \cup (C+x) \tag{2}$$

and that the sets $C, C+1, C+x$ are disjoint. Then clearly L is decomposed in sets $\{c, c+1, c+x\}$, where $c \in C$, which are isometric to $\{0, 1, x\}$.

We first prove (2). We observe that $C+1 = A + (B+1)$ and $C+x = (A+x+1)+B-1 = A+B-1$ since $A+x+1 = A$. Now $R = B \cup (B+1) \cup (B-1)$ and so (2) follows by (1). The sets $C, C+1, C+x$ are disjoint since the sets $B, B+1, B-1$ are disjoint. This completes the proof.

QUEEN'S COLLEGE, DUNDEE

Reçu par la Rédaction le 21. 6. 1959

Sur les limites approximatives

par

A. Matysiak (Łódź)

Examinons l'ensemble $E = \underset{x}{E}\{f(x) > b,\ x > x_0\}$. Nous nommons la *limite approximative droite supérieure* de la fonction $f(x)$ au point x_0 la borne inférieure de tels nombres b, dont la densité de l'ensemble E au point x_0 égale zéro. Nous désignons cette limite par le symbole $\overline{\lim\limits_{x\to x_0+}}\operatorname{apr} f(x)$.

On peut écrire

$$\overline{\lim_{x\to x_0+}}\operatorname{apr} f(x) = \inf_b \left[\lim_{h\to 0+} \frac{|(x_0, x_0+h) \cap \{f(x) > b\}|}{h} = 0\right],$$

où le symbole $|\ |$ signifie la mesure de l'ensemble, lorsque la fonction $f(x)$ est mesurable et la mesure extérieure lorsque $f(x)$ est non-mesurable. Il est évident qu'il faut dans ce cas remplacer dans la désignation de la limite approximative la définition „densité" par la définition „densité extérieure".

D'une façon analogue nous déterminons la *limite approximative gauche supérieure*

$$\overline{\lim_{x\to x_0-}}\operatorname{apr} f(x) = \inf_b \left[\lim_{h\to 0+} \frac{|(x_0-h, x_0) \cap \{f(x) > b\}|}{h} = 0\right].$$

Dans son travail „Résolution d'un problème de M. Z. Zahorski sur les limites approximatives", Fund. Math. 48 (1960), p. 277-286, L. Belowska a indiqué la construction de la fonction $f(x)$, pour laquelle l'ensemble des points x_0, dont la limite approximative supérieure gauche n'est pas égale à la limite approximative supérieure droite, est de la puissance du continu. Ci-dessous nous démontrerons qu'un tel ensemble est de la I-ère catégorie.

THÉORÈME. *L'ensemble E des points x_0, pour lesquels*

$$\overline{\lim_{x\to x_0+}}\operatorname{apr} f(x) \neq \overline{\lim_{x\to x_0-}}\operatorname{apr} f(x),$$

est un ensemble de la I-ère catégorie.