[18] S. Kaczmarz und H. Steinhaus, *Theorie der Orthogonalreihen*, Warszawa-Lwów 1935.

[19] S. Kakutani, *Concrete representation of abstract* (M)*-spaces*, Ann. of Math. 2 (42) (1941), p. 994-1024.

[20] J. L. Kelley, *Banach spaces with the extension property*, Trans. Amer. Math. Soc. 72 (1952), p. 323-326.

[21] H. Komatuzaki, *Sur les projections dans certaines espaces du type* (B), Proc. Imp. Acad. Tokyo 16 (1940), p. 274-279.

[22] — *Une remarque sur les projections des certains espaces du type* (B), ibidem 17 (1941), p. 238-240.

[23] C. Kuratowski, *Topologie I*, Warszawa 1958.

[24] J. Marcinkiewicz, *Quelques théorèmes sur les séries orthogonales*, Ann. Soc. Pol. Math. 17 (1938), p. 51-56.

[25] F. J. Murray, *On complementary manifolds and projections in spaces* L_p *and* l_p, Trans. Amer. Math. Soc. 41 (1937), p. 138-152.

[26] L. Nachbin, *A theorem of Hahn-Banach type for linear transformations*, ibidem 68 (1950), p. 28-46.

[27] В. Н. Никольский, *Наилучшая аппроксимация и базис в пространстве Фреше*, Докл. Акад. Наук СССР (Н. С.) 59 (1948), p. 639-642.

[28] A. Pełczyński, *On the isomorphism of the spaces m and M*, M. Bull. Acad. Pol. Sci., Série math. astr. et phys. 6 (1958), p. 695-696.

[29] R. S. Philips, *On linear transformations*, Trans. Amer. Math. Soc. 48 (1940), p. 516-541.

[30] A. Sobczyk, *Projections in Minkowski and Banach spaces*, Duke Math. J. 8 (1941), p. 78-106.

[31] — *Projections of the space* (m) *on its subspace* (c_0), Bull. Amer. Math. Soc. 47 (1941), p. 938-947.

[32] В. Ф. Гапошкин, *Об одном свойстве безусловных базисов в пространстве* L^p, Усп. мат. наук, вып. 4 (88), т. XIV (1959), p. 143-148.

[33] B. Grünbaum, *Projections onto some function spaces*, Technical Report 23, University of Kansas 1959, p. 1-14.

Reçu par la Rédaction le 5. 10. 1959

STUDIA MATHEMATICA, T. XIX. (1960)

On the theory of non-linear operator equations on conjugately similar spaces

by

S. YAMAMURO (Sapporo)

1. Introduction. It is the purpose of this paper to consider an eigenvalue problem for some operators F which map a Banach space R into the conjugate space $\bar{R}$. For this purpose, we take, as the Banach space R, a special kind of vector lattice, a *conjugately similar space* which has been introduced by Nakano [9]. Roughly speaking, this is a Banach space R such that a one-to-one correspondence T exists between R and $\bar{R}$. This correspondence T enables us to define a proper value λ and a proper element $a \in R$ of the operator F from R into $\bar{R}$ by the following equation:

$$Fa = \lambda Ta.$$

In the case of L_p-spaces $(p > 1)$, this definition agrees with that of E. S. Citlanadze [4].

The definitions and elementary properties of the conjugately similar spaces will be given in § 2. In the next section we will prove a theorem of L. A. Ljusternik in its special form. The simple proof may be interesting. In § 4 we will consider the eigenvalue problem of a non-linear operator. The last section contains an application.

We express here our hearty thanks to Dr. Musielak for his valuable remarks on various points in this paper.

2. Conjugately similar spaces. Let R be a vector lattice which satisfies the following condition: for any system of positive elements x_λ $(\lambda \in \Lambda)$ there exists an "infimum" element $\bigcap_{\lambda\in\Lambda} x_\lambda$. The conjugate space $\bar{R}$ of R is the totality of all linear (additive and homogeneous) functionals $\bar{x}$ on R which satisfy the following condition: if $x_\lambda \downarrow_{\lambda\in\Lambda} 0$ (1), then

$$\inf_{\lambda\in\Lambda} |\bar{x}(x_\lambda)| = 0.$$

(1) We write $x_\lambda \downarrow_{\lambda\in\Lambda} 0$ when $\{x_\lambda\ (\lambda \in \Lambda)\}$ is a non-increasing directed system and $\bigcap_{\lambda\in\Lambda} x_\lambda = 0$.

By this notion of conjugate space we define the reflexivity of $\bar{R}$ ([9], p. 93).

Definition ([9], p. 258). R is said to be *conjugately similar* if R is reflexive and there exists a one-to-one correspondence T between R and $\bar{R}$: $R \ni x \leftrightarrow Tx \in \bar{R}$ such that

(1) $T(-x) = -Tx$;
(2) $Tx \geqslant Ty$ if and only if $x \geqslant y$;
(3) $(Tx, x) = 0$ [2], $x \geqslant 0$ implies $x = 0$.

The correspondence T is called the *conjugately similar correspondence.* By this correspondence T we define a functional $m(x)$ on R by

$$m(x) = \int_0^1 (T\xi x, x)\,d\xi.$$

We have $m(x) < +\infty$ for any $x \in R$, because $m(x) \leqslant (Tx, x)$. The functional $m(x)$ satisfies the following conditions:

1) $0 \leqslant m(x) < +\infty$;
2) $m(x) = 0$ implies $x = 0$;
3) $|x| \geqslant |y|$ implies $m(x) \geqslant m(y)$;
4) $x \frown y = 0$ [3] implies $m(x+y) = m(x)+m(y)$;
5) $m(\alpha x+\beta y) \leqslant \alpha m(x)+\beta m(y)$ if $\alpha+\beta = 1$, $\alpha, \beta \geqslant 0$;
6) $0 \leqslant x_{\lambda} \uparrow_{\lambda \in \Lambda} x$ implies $\sup_{\lambda \in \Lambda} m(x_\lambda) = m(x)$ (see [9], p. 261).

Therefore, the space R is a modulared semi-ordered linear space in the sense of (9) [4].

If we define $\overline{m}(\bar{x})$ for $\bar{x} \in \bar{R}$ by the relation

$$\overline{m}(\bar{x}) = \sup_{x \in R}\{\bar{x}(x) - m(x)\},$$

then $\bar{R}$ is also a modulared semi-ordered linear space. Moreover, we have

$$\overline{m}(\bar{x}) = \int_0^1 (\bar{x}, T^{-1}\xi\bar{x})\,d\xi$$

and

$$(Tx, x) = m(x) + \overline{m}(Tx).$$

[2] For $x \in R$ and $\bar{x} \in \bar{R}$, $(\bar{x}, x)$ means the value of $\bar{x}$ at x, namely, $\bar{x}(x)$.

[3] Elements x and y are said to be *mutually orthogonal* if $|x| \frown |y| = 0$.

[4] In [9] the modulared semi-ordered linear space has been defined by more general conditions. Namely, the conjugately similar space is a special class of modulared semi-ordered linear spaces.

By the modular, we define two kinds of norms:

$$\|x\| = \inf_{\xi > 0} \frac{1+m(\xi x)}{\xi}; \qquad |||x||| = \inf_{m(\xi x) \leqslant 1} \frac{1}{|\xi|},$$

which satisfy the following relation:

$$|||x||| \leqslant \|x\| \leqslant 2|||x|||,$$

and $m(x) = 1$ if and only if $|||x||| = 1$ (see [9], p. 179 and [12]). By these norms, R is monotone complete [5], and, therefore, R is a Banach space and the Banach dual of R is $\bar{R}$ (see [1] and [13]).

Finally, we must illustrate the notion of projection operators, which are indispensable in this paper. A subset N of R is said to be *normal* if every element $x \in R$ can be decomposed into two orthogonal elements y and z such that

$$x = y + z, \quad y \in N \quad \text{and} \quad z \in N^{\perp}.$$

In this case, we define the projection operator $[N]$ by

$$y = [N]x \quad (x \in R).$$

This operator $[N]$ is linear and idempotent ([9], § 5). The set

$$\{p\}^{\perp} = \{x \in R : |p| \frown |x| = 0\}$$

is normal for any element $p \in R$. We denote $[\{p\}^{\perp\perp}]$ by $[p]$ ([9], § 6).

Example 1. L_p is the totality of all measurable functions $x(t)$ $(0 \leqslant t \leqslant 1)$ such that $\int_0^1 |x(t)|^p dt < +\infty$. If $1 < p < +\infty$, the conjugate space of L_p is L_q, where $q = p/(p-1)$. If we put

$$Tx(t) = |x(t)|^{p-1}\operatorname{sign} x(t),$$

T is a conjugately similar correspondence between L_p and L_q, and

$$m(x) = \frac{1}{p}\int_0^1 |x(t)|^p dt,$$

$$|||x||| = \left(\frac{1}{p}\int_0^1 |x(t)|^p dt\right)^{1/p}, \qquad \|x\| = p^{1/p}q^{1/q}|||x|||.$$

[5] R is said to be *monotone complete* if $0 \leqslant x_\lambda \uparrow_{\Lambda}$ and $\sup_{\lambda \in \Lambda} m(x_\lambda) < +\infty$ imply that $\{x_\lambda\,(\lambda \in \Lambda)\}$ is order-bounded.

Example 2. Let L_Φ be an Orlicz space. We assume that the function $\varphi(\xi)$ which appears in the integral

$$\Phi(\xi) = \int_0^\xi \varphi(\eta)\,d\eta$$

is strictly increasing and its strictly increasing inverse function $\psi(\xi)$ generates the conjugate function $\Psi(\xi)$ by the following relation:

$$\Psi(\xi) = \int_0^\xi \psi(\eta)\,d\eta.$$

Then, L_Ψ is the conjugate space of L_Φ and, if both spaces satisfy condition (Δ_2) for large arguments (6), then

$$Tx(t) = \varphi\big(x^+(t)\big) - \varphi\big(x^-(t)\big)$$

is a conjugately similar correspondence between L_Φ and L_Ψ. In this case

$$m(x) = \int_0^1 \Phi(|x(t)|)\,dt.$$

Example 3. For a measurable function $p(t) \geqslant 1$ $(0 \leqslant t \leqslant 1)$, we define a function space $L_{p(t)}$ as the totality of all measurable functions $x(t)$ such that

$$\int_0^1 \frac{1}{p(t)} |\xi x(t)|^{p(t)}\,dt < +\infty \quad \text{for some} \quad \xi > 0.$$

When

$$1 < \inf_{0\leqslant t\leqslant 1} p(t) \leqslant \sup_{0\leqslant t\leqslant 1} p(t) < +\infty,$$

the conjugate space of $L_{p(t)}$ is $L_{q(t)}$ where $q(t) = p(t)/\big(p(t)-1\big)$, and

$$Tx(t) = |x(t)|^{p(t)-1}\operatorname{sign} x(t)$$

is a conjugately similar correspondence between $L_{p(t)}$ and $L_{q(t)}$. We have

$$m(x) = \int_0^1 \frac{1}{p(t)} |x(t)|^{p(t)}\,dt.$$

(6) Namely, we assume $m(x) < +\infty$ for every $x(t) \in L_\Phi$. For the condition (Δ_2), see [5].

In this paper R is assumed to be conjugately similar by the correspondence T, and the modular generated by T is denoted by $m(x)$.

3. Fréchet-derivatives and the Eigenvalue Problem. We will begin by the following

LEMMA 1. *The functional of* ξ:

$$m(x+\xi y),$$

is differentiable and

$$\frac{d}{d\xi} m(x+\xi y) = \big(T(x+\xi y), y\big).$$

Proof. Although there are shorter proofs, we adopt here the one which uses a simple inequality between the modular and the conjugate modular. Putting $z = x+\xi y$ for any number $\varepsilon > 0$, we have

$$\begin{aligned} m\big(x+(\xi+\varepsilon)y\big) - m(x+\xi y) - \big(T(x+\xi y), \varepsilon y\big) &= m(z+\varepsilon y) - m(z) - (Tz, \varepsilon y) \\ &= m(z+\varepsilon y) - \{(Tz, z) - \overline{m}(Tz)\} - (Tz, \varepsilon y) \\ &= m(z+\varepsilon y) + \overline{m}(Tz) - (Tz, z+\varepsilon y) \geqslant 0, \end{aligned}$$

and hence it follows that

$$\varliminf_{\varepsilon\to 0} \frac{m(z+\varepsilon y) - m(z)}{\varepsilon} \geqslant (Tz, y).$$

On the other hand,

$$\begin{aligned} m(z+\varepsilon y) - m(z) &= \big(T(z+\varepsilon y), z+\varepsilon y\big) - \overline{m}\big(T(z+\varepsilon y)\big) - m(z) \\ &= \big(T(z+\varepsilon y), z\big) - m(z) + \big(T(z+\varepsilon y), \varepsilon y\big) - \overline{m}\big(T(z+\varepsilon y)\big) \\ &\leqslant \big(T(z+\varepsilon y), \varepsilon y\big), \end{aligned}$$

which implies that

$$\varlimsup_{\varepsilon\to 0} \frac{m(z+\varepsilon y) - m(z)}{\varepsilon} \leqslant \varlimsup_{\varepsilon\to 0} \big(T(z+\varepsilon y), y\big) = (Tz, y).$$

Therefore, the right-hand derivative of $m(x+\xi y)$ is $\big(T(x+\xi y), y\big)$. Analogously, we can prove that the left-hand derivative is also $\big(T(x+\xi y), y\big)$.

Remark. In this proof, we have used the following fact:

$$\text{if} \quad \lim_{\nu\to\infty} \|x_\nu - x\| = 0, \quad \text{then} \quad \lim_{\nu\to\infty} \|Tx_\nu - Tx\| = 0.$$

This is true, because, since the norm is complete and continuous, the norm convergence is equivalent to the star-order-convergence (see [9], Theorems 33.4, 33.5), and T is order-continuous.

Definition. A functional $f(x)$ on R is said to be *Fréchet-differentiable at x* if there exists an operator $F\epsilon(R\to\bar{R})$ such that

$$f(x+y)-f(x) = (Fx, y)+\gamma(x, y)$$

and

$$\lim_{\|y\|\to 0}\frac{|\gamma(x, y)|}{\|y\|} = 0.$$

The operator F is called the *gradient mapping* of f, and it is, in general, non-linear. Sometimes we write Fx as $f'(x)$. For various properties of the gradient mapping (see [5], [6], [10] and [11]).

THEOREM 1. *$m(x)$ is Fréchet-differentiable and*

$$m(x+y)-m(x) = (Tx, y)+\nu(x, y)$$

for the conjugately similar correspondence T.

Proof. For any $\varepsilon>0$ there exists $\delta=\delta(\varepsilon, x)>0$ such that $\|x-y\|<\delta$ implies $\|Tx-Ty\|<\varepsilon$. Therefore for those ε, δ we have

$$|m(x+y)-m(x)-(Tx, y)| = \Big|\int_0^1\{(T(x+\xi y), y)-(Tx, y)\}d\xi\Big|$$

$$\leqslant\int_0^1\|T(x+\xi y)-Tx\|\cdot|||y|||\,d\xi\leqslant\varepsilon|||y|||$$

if $\|y\|\leqslant\delta$, because

$$\|x+\xi y-x\|\leqslant\xi\cdot\|y\|<\delta\qquad(0\leqslant\xi\leqslant 1).$$

Therefore we have

$$\overline{\lim_{\|y\|\to 0}}\frac{|\gamma(x, y)|}{\|y\|}\leqslant\varepsilon,$$

which is to be proved.

Definition. Let F be an operator from R into $\bar{R}$. A proper value λ and a proper element $a\epsilon R$ of F are defined by

$$Fa = \lambda Ta.$$

In the case where $R=L_p$ $(1<p<+\infty)$, this definition agrees with that of Citlanadze [4]. In particular, for $p=2$, we get the usual definition of the proper element in Hilbert spaces.

The following modification of a theorem of Ljusternik ([6], p. 239) is fundamental in our theory.

THEOREM 2. *Let $S=\{x\epsilon R: m(x)=1\}$ and let N be a normal manifold of R such that $S\frown N$ is not empty. Let $f(x)$ be a functional on R, which takes an extremum at $a\epsilon S\frown N$, i. e.*

$$f(a)\leqslant f(x)\quad(\text{or } f(a)\geqslant f(x))\quad\text{for every } x\epsilon S\frown N.$$

If $f(x)$ is Fréchet-differentiable at a, then

$$f'(a)[N] = \lambda Ta,$$

where $\lambda=(f'(a), a)/(Ta, a)$. Therefore, if $N=R$, then the element a is a proper element of the gradient mapping $f'(x)$ and the proper value, to which a belongs, is $f'((a), a)/(Ta, a)$.

Proof. Let us assume that $f(a)\leqslant f(x)$ $(x\epsilon S\frown N)$. Putting

$$g(x) = f(x)-\lambda\cdot m(x),$$

where $\lambda=(f'(a), a)/(Ta, a)$, we have

$$g(a) = f(a)-\lambda m(a)\leqslant f(x)-\lambda m(x) = g(x)\qquad(x\epsilon S\frown N),$$

which means that the function $g(x)$ also takes its minimum at a. Since $f(x)$ and $m(x)$ are Fréchet-differentiable, $g(x)$ is also Fréchet-differentiable and

$$(g'(a), a) = (f'(a), a)-\lambda(Ta, a) = 0.$$

Now, for any $y\epsilon R$ such that $a+\xi[N]y\neq 0$ for any ξ, the elements

$$a(\xi) = a+\xi[N]y/|||a+\xi[N]y|||\qquad(-\infty<\xi<+\infty)$$

belong to $S\frown N$, and

$$g(a(\xi)) = g(a)+(g'(a), a(\xi)-a)+\gamma(a, a(\xi)-a)$$
$$= g(a)+(g'(a), a(\xi))+\gamma(a, a(\xi)-a).$$

Since $g(a)\leqslant g(a(\xi))$ $(-\infty<\xi<+\infty)$, if $g(a(\xi))$ is right-hand differentiable at $\xi=0$, then the right-hand derivative must be non--negative.

We will prove that $g(a(\xi))$ is right-hand differentiable at $\xi=0$. For any $\varepsilon>0$, we have

$$g(a(\varepsilon))-g(a) = (g'(a), a(\varepsilon))+\gamma(a, a(\varepsilon)-a),$$

and

$$(g'(a), a(\varepsilon)) = \left(g'(a), \frac{a+\varepsilon[N]y}{|||a+\varepsilon[N]y|||}\right) = \frac{\varepsilon}{|||a+\varepsilon[N]y|||}(g'(a), [N]y).$$

Therefore,

$$\frac{g(a(\varepsilon))-g(a)}{\varepsilon}=\frac{1}{|||a+\varepsilon[N]y|||}\big(g'(a),[N]y\big)+\frac{\gamma\big(a,a(\varepsilon)-a\big)}{\varepsilon}.$$

On the other hand, since

$$\frac{|||a(\varepsilon)-a|||}{\varepsilon}=\frac{1}{\varepsilon}\left|\left|\left|\left\{\left(\frac{1}{|||a+\varepsilon[N]y|||}-1\right)a+\frac{\varepsilon[N]y}{|||a+\varepsilon[N]y|||}\right\}\right|\right|\right|$$

$$\leqslant\frac{1}{\varepsilon}\left|\frac{|||a+\varepsilon[N]y|||-1}{|||a+\varepsilon[N]y|||}\right|+\frac{|||[N]y|||}{|||a+\varepsilon[N]y|||},$$

we have

$$\overline{\lim_{\varepsilon\to 0}}\frac{\big|\gamma\big(a,a(\varepsilon)-a\big)\big|}{\varepsilon}=\overline{\lim_{\varepsilon\to 0}}\frac{\gamma\big|\big(a,a(\varepsilon)-a\big)\big|}{|||a(\varepsilon)-a|||}\cdot\frac{|||a(\varepsilon)-a|||}{\varepsilon}$$

$$\leqslant\overline{\lim_{\varepsilon\to 0}}\frac{\big|\gamma\big(a,a(\varepsilon)-a\big)\big|}{|||a(\varepsilon)-a|||}\cdot\left|\lim_{\varepsilon\to 0}\frac{|||a+\varepsilon[N]y|||-|||a|||}{\varepsilon}+|||[N]y|||\right|=0,$$

because, by a result of S. Mazur [7],

$$\lim_{\varepsilon\to 0}\frac{|||a+\varepsilon[N]y|||-|||a|||}{\varepsilon}<+\infty,$$

and, by the definition of Fréchet-differentiation,

$$\lim_{\varepsilon\to 0}\frac{\big|\gamma\big(a,a(\varepsilon)-a\big)\big|}{|||a(\varepsilon)-a|||}=0.$$

Therefore we have

$$\lim_{\varepsilon\to 0}\frac{g\big(a(\varepsilon)\big)-g(a)}{\varepsilon}=\big(g'(a),[N]y\big)=\big(g'(a)[N],y\big).$$

Hence it follows that

$$\big(g'(a)[N],y\big)\geqslant 0\qquad(y\in R),$$

which means that

$$g'(a)[N]=0.$$

Thus we obtain

$$f'(a)[N]=\lambda Ta[N]=\lambda Ta.$$

Remark. In the definition of S the condition that $m(x)=1$ is not essential. Namely, for

$$S_\lambda=\{x\in R: m(x)=\lambda\},$$

the modular

$$m_\lambda(x)=\frac{1}{\lambda}m(x)$$

enables us to get the same conclusion. By that m_λ we can define a conjugately similar correspondence T_λ which satisfies

$$T_\lambda x=\frac{1}{\lambda}Tx.$$

Therefore

$$f'(a)[N]=\frac{\big(f'(a),a\big)}{(T_\lambda a,a)}T_\lambda a=\frac{\big(f'(a),a\big)}{(Ta,a)}Ta.$$

For the properties of m_λ, we refer to [12].

Lemma 2. *Let us suppose that $f(x)$ $(x\in R)$ satisfies the following two conditions:*

i) *$f(x)$ is weakly continuous;*

ii) *there exists a number $\alpha\geqslant 0$ such that $f(\xi x)\geqslant\xi^\alpha f(x)$ for any number $\xi\geqslant 1$ and element $x\in R$.*

Then, there exists an element $a\in R$ such that

$$m(a)=1\quad\text{and}\quad\sup_{m(a)=1}|f(x)|=|f(a)|.$$

Proof. The existence of such an element $a\in R$ that

$$\sup_{m(x)=1}|f(x)|=|f(a)|$$

follows from the facts that the unit sphere is weakly compact (7) and that $f(x)$ is weakly continuous. The norm of the element a is not greater than one. If $m(a)<1$, then

$$|f(a)|\geqslant\left|f\left(\frac{a}{|||a|||}\right)\right|\geqslant\left(\frac{1}{|||a|||}\right)^\alpha|f(a)|,$$

which shows that the element $a/|||a|||$ is our solution.

Theorem 3. *Let us assume that:*

i) *$f(x)$ is Fréchet-differentiable;*

ii) *there exists a number $\alpha\geqslant 0$ such that $f(\xi x)\geqslant\xi^\alpha f(x)$ for any number $\xi\geqslant 1$ and element $x\in R$;*

iii) *$f(x)$ is positive: $f(x)\geqslant 0$ $(x\in R)$;*

iv) *the gradient mapping F of $f(x)$ is completely continuous.*

(7) R is reflexive as a Banach space.

Then, there exists an element $a \in R$ such that $m(a) = 1$ and

$$Fa = \frac{(Fa, a)}{(Ta, a)} Ta,$$

namely, a is a proper element of F and the proper value to which a belongs is $(Fa, a)/(Ta, a)$.

Proof. By the assumptions ii), iii) and Lemma 2, we can find $a \in R$ such that

$$m(a) = 1 \quad \text{and} \quad \sup_{m(x)=1} f(x) = f(a),$$

because the complete continuity of F implies the weak continuity of $f(x)$ (see [10], Theorem 3.2). Therefore, by Theorem 2, we can complete this proof.

Before proceeding to the next theorem, we need the following

Definition. *A subset A of R* is said to be *complete* if $x \perp A$ implies $x = 0$. *An element* is said to be *complete* if it is complete as a one-point set.

Theorem 4. *We assume the conditions* i)-iv) *of Theorem* 3. *Moreover, we suppose that*

v) *there exists a complete element;*

vi) $f(x) \leqslant f(|x|)$ *for every* $x \in R$.

Then there exists a sequence of projection operators P_ν in $\bar{R}$ such that

1) $P_\nu \downarrow_{\nu=1}^{\infty} 0$ *and* $P_1 = \bar{I}$ *(the identity in* $\bar{R}$*)*;

2) *there exist* $a_\nu \geqslant 0$ *and* $\lambda_\nu \geqslant 0$ *such that*

$$m(a_\nu) = 1, \quad P_\nu F a_\nu = \lambda_\nu T a_\nu \quad \text{and} \quad P_\nu^R a_\nu = a_\nu \text{ [8]};$$

3) *the sequence* a_ν *(*$\nu = 1, 2, \ldots$*) is complete in* R;

4) *if* $\lambda_\nu \neq 0$ *(*$\nu = 1, 2, \ldots$*), then*

$$\inf_{\nu \geqslant 1} \lambda_\nu \|Ta_\nu\| = 0.$$

Proof. We start by the proper element a of the preceeding theorem. By vi), we can take positive a. We denote it by a_1. When a_1 is not a complete element, then the non-empty set

$$N_{a_1} = \{x \in R : (Ta_1, |x|) = 0\}$$

is normal and

$$[N_{a_1}] + [a_1] = I \quad \text{(the identity of } R).$$

[8] For the definition of P_ν^R, see [9], p. 82.

By the same method as in the proof of Lemma 2, we can find $a_2 \in R$ such that $m(a_2) = 1$, $a_2 \in [Na_1]$ and

$$\sup_{m(x)=1, x \in Na_1} f(x) = f(a_2).$$

Then, by Theorem 3, we have

$$Fa_2[Na_1] = \lambda_2 Ta_2,$$

where

$$\lambda_2 = (Fa_2, a_2)/(Ta_2, a_2) = (Fa_2, a_2)/\|Ta_2\|.$$

Suppose that we could find an orthogonal $a_\nu > 0$ ($\nu = 1, 2, \ldots, \mu$) such that

$$m(a_\nu) = 1, \quad a_\nu \in Na_1 \cap Na_2 \cap \ldots \cap Na_{\nu-1}$$

and

$$Fa_\nu\, [Na_1][Na_2]\ldots[N_{\nu-1}] = \lambda_\nu Ta_\nu \quad \text{where} \quad \lambda_\nu = (Fa_\nu, a_\nu)/\|Ta_\nu\|.$$

If $\{a_\nu\ (\nu = 1, 2, \ldots, \mu)\}$ is not complete, we can continue this process. Since R is superuniversally continuous, the set of such elements must be at most countable. Therefore, 3) is true.

Now, since

$$[Na_\nu] = I - [a_\nu] = I - [Ta_\nu]^R,$$

we have

$$\begin{aligned} Fa_\nu[Na_1][Na_2]\ldots[Na_{\nu-1}] &= Fa_\nu\big(I - [Ta_1]^R\big)\ldots\big(I - [Ta_{\nu-1}]^R\big) \\ &= \big(\bar{I} - [Ta_1]\big)\ldots\big(\bar{I} - [Ta_{\nu-1}]\big)Fa_\nu. \end{aligned}$$

Putting $P_1 = \bar{I}$ and

$$P_\nu = \big(\bar{I} - [Ta_1]\big)\ldots\big(\bar{I} - [Ta_{\nu-1}]\big) \quad (\nu = 2, 3, \ldots),$$

we have

$$P_\nu F a_\nu = \lambda_\nu T a_\nu \quad (\nu = 1, 2, \ldots)$$

and

$$P_\nu^R a_\nu = [Na_1]\ldots[Na_{\nu-1}]a_\nu = a_\nu \quad (\nu = 2, 3, \ldots).$$

Thus, we could prove 2).

It is evident that $P_1 \geqslant P_2 \geqslant \ldots$ We will prove that $\bigcap_{\nu=1}^{\infty} P_\nu = 0$ (the projection operator defined by $0 \cdot \bar{x} = 0$). Since

$$P_\nu^R = [Na_1]\ldots[Na_{\nu-1}],$$

we have only to prove that

$$\bigcap_{\nu=2}^{\infty}[Na_1]\ldots[Na_{\nu-1}] = 0.$$

Suppose, on the contrary, that

$$\bigcap_{\nu=2}^{\infty}[Na_1]\ldots[Na_{\nu-1}] \neq 0.$$

Then there exists a projector $[p] \neq 0$ such that

$$[p] \leqslant [Na_1]\ldots[Na_{\nu-1}] \qquad (\nu = 2, 3, \ldots),$$

which shows that

$$[p][a_\nu] = 0 \qquad (\nu = 1, 2, \ldots).$$

This contradicts the fact that $\{a_\nu\ (\nu = 1, 2, \ldots)\}$ is complete.

Finally, we will prove that 4) is true. If

$$\inf_{\nu\geqslant 1}\lambda_\nu\|Ta_\nu\| \neq 0,$$

we can take a subsequence (we denote it also by $\lambda_\nu\|Ta_\nu\|$) such that $\lambda_\nu\|Ta_\nu\| > \varepsilon$ $(\nu = 1, 2, \ldots)$ for some $\varepsilon > 0$. Then the elements $a_\nu/\lambda_\nu\|Ta_\nu\|$ $(\nu = 1, 2, \ldots)$ are norm-bounded. Therefore, by iv), the set

$$\left\{\frac{Fa_\nu}{\lambda_\nu\|Ta_\nu\|} \qquad (\nu = 1, 2, \ldots)\right\}$$

is compact, and hence it follows that

$$\left\{\frac{P_\nu Fa_\nu}{\lambda_\nu\|Ta_\nu\|} = \frac{Ta_\nu}{\|Ta_\nu\|} \qquad (\nu = 1, 2, \ldots)\right\}$$

is compact. But this is impossible, because, as a_ν $(\nu = 1, 2, \ldots)$ are mutually orthogonal, we have

$$\overline{m}\left(\frac{Ta_\nu}{\|Ta_\nu\|} - \frac{Ta_\mu}{\|Ta_\mu\|}\right) = \overline{m}\left(\frac{Ta_\nu}{\|Ta_\mu\|} + \frac{Ta_\mu}{\|Ta_\mu\|}\right)$$

$$\geqslant \overline{m}\left(\frac{Ta_\nu}{2|||Ta_\nu|||}\right) + \overline{m}\left(\frac{Ta_\nu}{2|||Ta_\mu|||}\right) \geqslant 2\inf_{\overline{m}(\bar{x})=1}\overline{m}(\tfrac{1}{2}\bar{x}) > 0,$$

since $\overline{R}$ is uniformly simple ([13], Lemma 2.1). Thus 4) is established.

Remark. It is possible that, for an a, there exist infinite $x_\lambda \geqslant 0$ $(\lambda \in \Lambda)$ such that

$$F_{x_\lambda} = aTx_\lambda \quad \text{and} \quad m(x_\lambda) = 1.$$

But the set $\{x_\lambda\ (\lambda \in \Lambda)\}$ contains only a finite number of mutually orthogonal elements because of the complete continuity of F. It is evident that mutually orthogonal elements are linearly independent. But the inverse is not always true.

4. Symmetric operators. In this section we will introduce the notion of symmetric operators on Banach spaces and apply the results of the preceeding section.

Definition. An operator $F \in (R \to \overline{R})$ (9) is said to be *symmetric* if $(Fx, y) = (Fy, x)$ $(x, y \in R)$.

Obviously the symmetric operator is additive and homogeneous. The conjugately similar correspondence T is symmetric if and only if the space R is an *abstract L_2-space* (10).

For a symmetric operator F, we put

$$(*) \qquad f_F(x) = \tfrac{1}{2}(Fx, x).$$

Then $f_F(x)$ is homogeneous of order 2, that is, a quadratic form. Moreover, we have

Lemma 3. *If F is continuous and symmetric, $f_F(x)$ is Fréchet-differentiable and $f'_F(x) = Fx$.*

Proof. By $(*)$, we have

$$f_F(x+y) - f_F(x) = \tfrac{1}{2}\{(F(x+y), x+y) - (Fx, x)\}$$

$$= (Fx, y) + \tfrac{1}{2}(Fy, y),$$

and $|(Fy, y)| \leqslant \|Fy\|\cdot|||y|||$. Therefore

$$\lim_{|||y|||\to 0}\frac{|(Fy, y)|}{|||y|||} = 0,$$

because F is continuous. Thus we have

$$(f'(x), y) = (Fx, y) \qquad (y \in R).$$

(9) We will denote by $(R \to \overline{R})$ the set of all operators from R into $\overline{R}$.

(10) If F is symmetric, it is homogeneous. Therefore

$$m(\xi x) = \int_0^\xi (T\eta x, x)\,d\eta = \frac{\xi^2}{2}(Tx, x),$$

which means that $m(\xi x) = \xi^2 m(x)$. Hence it follows that $\|x+y\|^2 = \|x\|^2 + \|y\|^2$ if $x \frown y = 0$. See [3].

Remark. If a function $f(x)$ is Fréchet-differentiable and its gradient mapping F is symmetric, then

$$f(x) = \tfrac{1}{2}(Fx, x) + f(0).$$

In fact,

$$f(x) - f(0) = \int_0^1 \big(F(\xi x), x\big)\, d\xi$$

$$= \int_0^1 \xi (Fx, x)\, d\xi = \tfrac{1}{2}(Fx, x).$$

Definition. An operator $F \epsilon (R \to \bar{R})$ is said to be *positive definite*, if $(Fx, x) \geqslant 0$ $(x \epsilon R)$. F is said to be *positive* if $x \geqslant 0$ implies $Fx \geqslant 0$. Obviously, these two notions are independent.

Lemma 3. *If F is positive, then $f_F(x) \leqslant f_F(|x|)$.*

Proof.

$$2f_F(x) = (Fx, x)$$

$$= \big(F(x^+ - x^-), x^+ - x^-\big)$$

$$= (Fx^+, x^+) - 2(Fx^+, x^-) + (Fx^-, x^-)$$

$$\leqslant (Fx^+, x^+) + 2(Fx^+, x^-) + (Fx^-, x^-)$$

$$= (F|x|, |x|) = 2f_F(|x|)\ (^{11}).$$

Thus, we can easily see that the following theorem is true:

Theorem 5. *If F is symmetric, completely continuous, positive and positive definite, then the functional $f_F(x)$ satisfies the conditions* i)-iv) *and* vi) *of Theorem* 3 *and Theorem* 4. *Therefore we can apply Theorem* 4. *In this case we can replace* 4) *of Theorem* 4 *by*

4′) *if $\lambda_\nu > 0$, then $\lambda_1 \|Ta_1\| \geqslant \lambda_2 \|Ta_2\| \geqslant \ldots$ and*

$$\lim_{\nu \to \infty} \lambda_\nu \|Ta_\nu\| = 0.$$

In fact, $\lambda_\nu \|Ta_\nu\| = (Fa_\nu, a_\nu) = 2f_F(a_\nu)$.

Remark. If a symmetric operator F is positive definite, we can easily prove the following equality:

$$\|F\| = \sup_{\||x|\|=1} \|Fx\| = \sup_{m(x)=1} (Fx, x).$$

[11] $|x| = x^+ + x^-$, $x^+ = x \smile 0$ and $x^- = (-x)^+$.

5. An integral operator on modulared function spaces. For $\xi > 0$ and $0 \leqslant t \leqslant 1$, we consider a finite function $\varphi(\xi, t)$, which is assumed to be strictly increasing with respect to $\xi > 0$, measurable with respect to t and $\varphi(0, t) = 0$ for almost all t. Let the inverse function of $\varphi(\xi, t)$ as a function of ξ be $\psi(\xi, t)$, which is also a finite, strictly increasing function of $\xi > 0$ and measurable with respect to t.

We define Young's functions as follows:

$$\Phi(\xi, t) = \int_0^\xi \varphi(\eta, t)\, d\eta \quad \text{and} \quad \Psi(\xi, t) = \int_0^\xi \psi(\eta, t)\, d\eta.$$

Then it is easy to see that

$$\xi\eta \leqslant \Phi(\xi, t) + \Psi(\eta, t),$$

and

$$\alpha\beta = \Phi(\alpha, t) + \Psi(\beta, t) \quad \text{if} \quad \beta = \varphi(\alpha, t) \text{ or } \alpha = \psi(\beta, t).$$

Let L_Φ be the totality of all measurable functions $x(t)$ $(0 \leqslant t \leqslant 1)$ such that

$$m(\xi x) = \int_0^1 \Phi(\xi |x(t)|, t)\, dt < +\infty \quad \text{for some} \quad \xi > 0.$$

Then the functional $m(x)$ satisfies the "modular conditions" of [9], p. 153. This space L_Φ is reflexive and its conjugate space is L_Ψ. Every element $\bar{x} \epsilon \bar{L}_\Phi$ takes the form of

$$\bar{x}(x) = \int_0^1 x(t)\bar{x}(t)\, dt \quad (x(t) \epsilon L_\Phi)$$

for some $\bar{x}(t) \epsilon L_\Psi$.

Throughout this section we assume that $m(x) < +\infty$ for every $x(t) \epsilon L_\Phi$.

Lemma 4. *The transformation*

$$Tx(t) = \varphi\big(x^+(t), t\big) - \varphi\big(x^-(t), t\big)$$

is a conjugately similar correspondence between L_Φ and L_Ψ.

Proof. At first, we prove that $T \epsilon (L_\Phi \to L_\Psi)$. Let $0 \leqslant x(t) \epsilon L_\Phi$. Then

$$\int_0^1 x(t) \cdot \varphi\big(x(t), t\big)\, dt = \lim_{\varepsilon \to 0} \frac{m\big((1+\varepsilon)x\big) - m(x)}{\varepsilon} < +\infty.$$

Therefore we have

$$\int_0^1 \Psi\big(\varphi(x(t),t)\big)\,dt = \int_0^1 x(t)\cdot\varphi(x(t),t)\,dt - \int_0^1 \Phi(x(t),t)\,dt < +\infty,$$

which means that

$$Tx(t)\in L_\Psi \quad \text{if} \quad 0 \leqslant x(t)\in L_\Phi .$$

For arbitrary $x(t)\in L_\Phi$ we have

$$Tx(t) = Tx^+(t) - Tx^-(t)\in L_\Psi .$$

Next we prove that

$$Tx = Ty \quad \text{implies} \quad x = y .$$

We can assume that $x(t), y(t)$ are positive. If $x(t)\neq y(t)$, then, for example,

$$x(t) > y(t)$$

on a set of positive measure. Therefore, we can find $\alpha>\beta>0$ such that the set

$$\{t: x(t)>\alpha>\beta>y(t)\}$$

is of positive measure. Since $\varphi(\xi,t)$ is strictly increasing, we have $\varphi(x(t),t)\neq\varphi(y(t),t)$.

To show that T is one-to-one, we have only to prove that for any $0\leqslant y\in L_\Psi$ there exists $0\leqslant x\in L_\Phi$ such that $Tx=y$. But this is obvious, because, for that $y(t)\in L_\Psi$. we have

$$x(t) = \psi(y(t),t)\in L_\Phi \quad \text{and} \quad \varphi(x(t),t) = y(t).$$

We find no difficulty in proving the following conditions: (1) $T(-x) = -Tx$; (2) $x\leqslant y$ if and only if $Tx\leqslant Ty$; (3) $(Tx, x)=0$, $x\geqslant 0$ implies $x=0$.

Thus the proof is established.

Next we will consider the following integral operator:

$$(**)\qquad Fx = y(s) = \int_0^1 K(s,t)x(t)\,dt.$$

Using a method of A. C. Zaanen [14], we can prove the following

THEOREM 6. *If*

$$\int_0^1 \Psi\Big[\int_0^1 \Psi(|K(s,t)|,t)\,dt, s\Big]\,ds < +\infty,$$

then the operator (**) *is completely continuous as an operator from* L_Φ *into* L_Ψ.

Proof. At first, we prove that $F\in(L_\Phi\to L_\Psi)$. For $Fx=y$, we have

$$\int_0^1 \Psi\big(\tfrac12|y(s)|,s\big)\,ds = \int_0^1 \Psi\Big[\tfrac12\Big|\int_0^1 K(s,t)x(t)\,dt\Big|,s\Big]\,ds$$
$$\leqslant \int_0^1 \Psi\Big[\tfrac12\int_0^1 \Psi(|K(s,t)|,t)\,dt + \tfrac12\int_0^1 \Phi(|x(t)|,t)\,dt, s\Big]\,ds$$
$$\leqslant \tfrac12\int_0^1 \Psi\Big[\int_0^1 \Psi(|K(s,t)|,t)\,dt, s\Big]\,ds$$
$$+\tfrac12\int_0^1 \Psi\Big[\int_0^1 \Phi(|x(t)|,t)\,dt, s\Big]\,ds < +\infty,$$

which shows that $Fx\in L_\Psi$.

To prove that F is completely continuous, we note that in this case the norm convergence and the convergence by modular ([12]) coincide ([13], Lemma 2.1).

Let a set $A\subset L_\Phi$ be bounded, namely,

$$|||x|||\leqslant\gamma \quad (x\in A)$$

for some $\gamma>0$. This is equivalent to

$$m\Big(\frac{1}{\gamma}x\Big)\leqslant 1 \quad (x\in A).$$

Since A is weakly compact, there exist x_ν $(\nu=0,1,2,\ldots)$ such that

$$\lim_{\nu\to\infty} x_\nu = x_0 \quad \text{weakly and} \quad m(2x_\nu)\leqslant 1,$$

where $2\gamma x_\nu\in A$. Therefore, for every $y(t)\in L_\Psi$, we have

$$\lim_{\nu\to\infty}\int_0^1 x_\nu(t)y(t)\,dt = \int_0^1 x_0(t)y(t)\,dt.$$

The assumption on $K(s,t)$ shows that $K(s,t)\in L_\Psi$ as a function of t for almost all s. Hence it follows that for

$$y_\nu(s) = \int_0^1 K(s,t)x_\nu(t)\,dt$$

([12]) A sequence x_ν is said to be *convergent by modular to* x_0 if $\lim_{\nu\to\infty} m(x_\nu - x_0) = 0$.

we have

$$\lim_{\nu\to\infty} |y_\nu(s) - y_0(s)| = 0 \quad \text{almost everywhere,}$$

which implies

$$\lim_{\nu\to\infty} \Psi\big(|y_\nu(s) - y_0(s)|, s\big) = 0 \quad \text{almost everywhere.}$$

On the other hand, since

$$|y_\nu(s) - y_0(s)| \leqslant \int_0^1 |K(s,t)| \cdot |x_\nu(t) - x_0(t)|\, dt$$

$$\leqslant \int_0^1 \Psi\big(|K(s,t)|, t\big)\, dt + \int_0^1 \Phi\big(|x\ (t) - x_0(t)|, t\big)\, dt$$

$$\leqslant \int_0^1 \Psi\big(|K(s,t)|, t\big)\, dt + 1,$$

we have

$$\Psi\big(|y_\nu(s) - y_0(s)|, s\big) \leqslant \Psi\Big[\int_0^1 \Psi\big(|K(s,t)|, t\big)\, dt + 1, s\Big],$$

and the right-hand function of s is integrable. Therefore, by a theorem of Lebesgue, we have

$$\lim_{\nu\to\infty} \int_0^1 \Psi\big(|y_\nu(s) - y_0(s)|, s\big)\, ds = 0.$$

This means that Fx_ν converges to Fx_0 by the modular convergence. Therefore,

$$\lim_{\nu\to\infty} \|Fx_\nu - Fx_0\| = 0$$

by the norms defined by the modular. Since $2\gamma x_\nu \in A$ and F is linear, $F(A)$ is compact.

Remark. If $\Phi(\xi, t) = \Phi(\xi)$, i. e. if $\Phi(\xi, t)$ is a function of only $\xi > 0$, L_Φ is an Orlicz space. If $\Phi(\xi, t) = \xi^{p(t)}$ for a measurable function $p(t) \geqslant 1$, L_Φ is $L_{p(t)}$. But in those cases we cannot always obtain the exact forms of their norms that make them Banach spaces. It is easy to see that in the case where

$$\Phi(\xi, t) = \xi^p \quad \text{for} \quad p > 1,$$

i. e. in the case of L_p $(p > 1)$, the condition on $K(s,t)$ of the above theorem can be replaced by a better one:

$$\int_0^1 \int_0^1 |K(s,t)|^{p/(p-1)}\, ds\, dt < +\infty.$$

By Theorem 4 and 6, we can ensure the existence of a positive proper element of the operator (**). Namely,

THEOREM 7. *Let us suppose that the operator* (**) *satisfies the following conditions*:

i) $\int_0^1 \Psi\big[\Psi \int_0^1 \big(|K(s,t)|\ t\big)\, dt, s\big]\, ds < +\infty$;

ii) $K(s,t) \geqslant 0$ *for every* s, t;

iii) $K(s,t) = K(t,s)$.

Then, there exist a positive number a *and a function* $a(t) \in L_\Phi$ *such that*

$$\int_0^1 K(s,t)\, a(t)\, dt = \lambda \varphi\big(a(s), s\big).$$

If, moreover, F *is positive definite, then there exists a sequence of measurable sets* E_ν $(\nu = 1, 2, \ldots)$ *such that*:

1) $[0,1] = E_1 \supset E_2 \supset \ldots$ *and* $\lim_{\nu\to\infty} |E_\nu| = 0$;

2) *there exist* $a_\nu(t) \geqslant 0$ *and* $\lambda_\nu \geqslant 0$ *such that*

$$\int_0^1 \Phi\big(a_\nu(t), t\big)\, dt = 1 \quad \text{and} \quad \int_{E_\nu} K(s,t)\, a_\nu(t)\, dt = \lambda_\nu \varphi\big(a_\nu(s), s\big);$$

3) *if* $a_\nu(t) \cdot x(t) = 0$ $(\nu = 1, 2, \ldots)$ *and* $x(t) \geqslant 0$, *then* $x = 0$;

4) *if* $\lambda_\nu > 0$ $(\nu = 1, 2, \ldots)$, *then* $\lim_{\nu\to\infty} \lambda_\nu = 0$.

References

[1] I. Amemiya, *A generalization of Riesz-Fisher's theorem*, Jour. Math. Soc. Japan 5 (1953), p. 353-354.

[2] G. Birkhoff, *Lattice Theory*, American Math. Soc. 1949.

[3] F. Bohnenblust, *An axiomatic characterization of L_p-spaces*, Duke Math. Journal 6 (1940), p. 627-640.

[4] E. S. Citlanadze, *The method of orthogonal trajectories for non-linear operators of variational type in the space L_p*, Acad. Nauk Gruzin. SSR, Trudy Tbiliss. Mat. Inst. Razmadze 20 (1954), p. 245-278; American Math. Soc. Transactions, Series 2, Vol. 5, p. 305-333.

[5] M. A. Krasnoseliskii and J. B. Rutickii, *Convex functions and Orlicz spaces* (Russian), 1958.

[6] L. A. Ljusternik and W. I. Sobolev, *Elemente der Funktionalanalysis*, 1955.

[7] S. Mazur, *Über konvexe Mengen in linearen normierten Räumen*, Studia Math. 4 (1933), p. 70-84.

[8] E. J. McShane, *Integration*, Princeton 1944.

[9] H. Nakano, *Modulared semi-ordered linear spaces*, Tokyo 1950.

[10] E. Rothe, *Gradient mappings and extrema in Banach spaces*, Duke Math. Jour. 15 (1948), p. 421-431.

[11] M. M. Vainberg, *Some questions of differential calculus in linear spaces*, Uspehi Mat. Nauk, N. S. 7 (1952), p. 55-102 (Russian).

[12] S. Yamamuro, *On conjugate spaces of Nakano spaces*, Transactions Amer. Math. Soc. 90 (1959), p. 291-311.

[13] — *Monotone completeness of normed semi-ordered linear spaces*, Pacific Jour. Math. 7 (1957), p. 1715-1725.

[14] A. C. Zaanen, *On a certain class of Banach spaces*, Annals of Math. 47 (1946), p. 654-666.

HOKKAIDO UNIVERSITY
SAPPORO, JAPAN

Reçu par la Rédaction le 18.3.1959

STUDIA MATHEMATICA, T. XIX. (1960)

Remarks to the paper of L. Kubik "The limiting distributions of cumulative sums of independent two-valued random variables"

(Studia Mathematica 18 (1959), p. 295-309)

In the definition of class $\mathcal{R}$ it is necessary to assume additionally that the limit

$$\lim_{n\to\infty} z_n \Big/ \sqrt{\sum_{k=1}^{n} D^2(X_k)}$$

exists and in the definition of class $\mathcal{K}$ (and in the proof of theorem) it is necessary to assume that $A = -B$, $\nu+\mu = 1$. In the proof of theorem, $\mathfrak{P}(n)$ and $\mathfrak{Q}(n)$ should be defined as $\mathfrak{P}(n) = \mathfrak{P} \frown \mathfrak{N}(n)$, $\mathfrak{Q}(n) = \mathfrak{Q} \frown \mathfrak{N}(n)$ where $\mathfrak{P}$ (respectively $\mathfrak{Q}$) denotes the set of positive integers k such that $\min(p_k, q_k) = p_k$ (respectively $\min(p_k, q_k) < p_k$).

The theorem on page 296 should be formulated as follows:

The class $\mathcal{R}$ coincides with the class of all distributions which are of the same type as any element of $\mathcal{K}$.

The elements of $\mathcal{R}$ which do not belong to $\mathcal{K}$ can be obtained by replacing $B_n = \sqrt{\sum_{k=1}^{n} D^2(X_k)}$ by $B_n' = B_n/\sigma$.

Reçu par la Rédaction le 6. 4. 1960