Dieser Wert b ist für $a > 1$ kleiner als a. Darüber hinaus genügt er der Ungleichung (28) mit $m = 3$, wenn

$$a \geqslant 4 \tag{30}$$

ist. Folglich treten in dem Fall (29), (30) auf der rechten Seite von (26) nicht 4 sondern nur 3 Potenzen von s mit negativen Exponenten auf. Eine entsprechende Diskussion kann man auch für größere m durchführen, man braucht lediglich weitere Koeffizienten d_m zu berechnen.

Die Richtigkeit der Formeln (16) und (17) findet man bestätigt, wenn man die Gleichung

$$\int_0^s e^{-g(s,t)} g_1(s,t)\,dt = 0$$

wegen (7) in der Form

$$\alpha a \int_0^s e^{-g(s,t)} t^{-a-1}\,dt = \beta b \int_0^s e^{-g(s,t)} (s-t)^{-b-1}\,dt$$

schreibt und auf beide Seiten dieser Gleichung die Formel (17) bzw. (16) anwendet. Um dies einzusehen, braucht man nur die Formeln (14) bzw. (13) heranzuziehen und die Fälle $\mu = -a-1$, $\nu = 0$ und $\mu = 0$, $\nu = -b-1$ zu betrachten. Eine weitere Kontrollmöglichkeit erhält man, wenn man den oben angedeuteten Weg über die Laplace-Transformation formal durchrechnet.

Zitatennachweis

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Reçu par la Rédaction le 6. 11. 1959

STUDIA MATHEMATICA, T. XIX. (1960)

Spaces of continuous functions (V)

(On linear isotonical embedding of $C(\Omega_1)$ into $C(\Omega_2)$)

by

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1. Introduction. In the sequel Ω will denote a compact Hausdorff space, and $C(\Omega)$ will denote the Banach lattice of all real-valued continuous functions defined on Ω. By a well-known theorem (established, in various forms, by Banach, M. H. Stone, I. Gelfand and A. Kolmogoroff, S. Eilenberg, I. Kaplansky and others; see [2], p. 170, [23], p. 469, [9], [7], [14]), the space $C(\Omega)$ determines Ω topologically. Thus, the topological properties of Ω determine the linear, metric and lattice properties of $C(\Omega)$, and conversely.

From the topological point of view, the relation *a space Ω_1 is smaller than Ω_2* may be defined variously (e. g. it might mean that $\Omega_1 \underset{\text{top}}{\subset} \Omega_2$, or that Ω_1 is a continuous image of Ω_2). On the other hand, functional analysis gives also many definitions of the relation *a space $C(\Omega_1)$ is smaller than $C(\Omega_2)$* (such a definition may be based on the linear dimension, on isometrical or isotonical embedding, and so on).

These notions suggest the problem whether the statement *Ω_1 is smaller than Ω_2* implies the statement *$C(\Omega_1)$ is smaller than $C(\Omega_2)$* and whether the converse implication is true, both notions *smaller* being suitably defined.

The methods and results of both parts of this general problem — the part concerning topological embedding and that concerning continuous images — are mutually different; moreover, the first part is more difficult and the issues are not complete.

The problem of necessary and sufficient conditions for Ω_0 to be a continuous image of a compact Hausdorff space Ω is completely solved by the following theorem of M. H. Stone (1): *Ω_0 is a continuous image of a com-*

(1) This theorem may be formulated in various ways (e. g. in ring terms or in lattice terms). It has been proved and discussed by M. H. Stone ([23], p. 475), G. Šilov [21], H. Yoshizawa [24], S. B. Myers ([17], p. 240), Ky Fan [8], K. Gęba and Z. Semadeni [11]. It is closely related to the Stone-Weierstrass approximation theorem and to the theory of semicontinuous decompositions of a compact set Ω.

pact space Ω if and only if there exists a one-to-one multiplicative linear transformation T of $C(\Omega_0)$ into $C(\Omega)$ such that $T(e_0) = e$ (where e_0 and e denote the units of $C(\Omega_0)$ and $C(\Omega)$, respectively). The condition of multiplicativity $T(x_1 \cdot x_2) = T(x_1) \cdot T(x_2)$ may be replaced by $T(x_1 \vee x_2) = T(x_1) \vee T(x_2)$, which means that T is a lattice isomorphism (²).

Now, we introduce the following definitions. Given a space $X = C(\Omega)$, a subset X_0 of X will be termed a B_+*-subspace* of X if X_0 is linear and closed and if X_0 is a vector lattice with respect to the order induced by X. In other words, a subspace X_0 of X is a B_+-subspace if, for every pair x, y of elements of X_0, there exists their relative l. u. b. $x \vee_0 y$ (i. e. an element $z \in X_0$ such that $z \geqslant x$, $z \geqslant y$ and such that $z' \in X_0$, $z' \geqslant x$, $z' \geqslant y$ imply $z' \geqslant z$). Obviously, $x \vee_0 y \geqslant x \vee y$ for all $x \in X_0$, $y \in X_0$, and $x \vee_0 y = x \vee y$ if and only if $x \vee y \in X_0$. The relative g. l. b. is defined by $x \wedge_0 y = -[(-x) \vee_0 (-y)]$.

Example 1. Let $\Omega = \langle 0, 1\rangle$ and let X_0 be the set of all functions $x(t)$ of $C\langle 0, 1\rangle$ which are linear on the interval $\langle \frac{1}{3}, \frac{2}{3}\rangle$. X_0 is a B_+-subspace of $C\langle 0, 1\rangle$ and is equivalent to the Cartesian square of $C\langle 0, 1\rangle$; the unit of X_0 coincides with the unit e of $C\langle 0, 1\rangle$ and the condition $x \vee_0 y = x \vee y$ is fulfilled if and only if $x(t) \geqslant y(t)$ on $\langle \frac{1}{3}, \frac{2}{3}\rangle$ or $x(t) \leqslant y(t)$ on $\langle \frac{1}{3}, \frac{2}{3}\rangle$.

Next, a subset X_0 of the space $X = C(\Omega)$ will be termed an *MI-subspace* of X if it is a B_+-subspace, if the unit of $C(\Omega)$ belongs to X_0 and if $x \vee y = x \vee_0 y$ for $x \in X_0$, $y \in X_0$ (the last condition is equivalent to the following one: *$x \in X_0$, $y \in X_0$ imply $x \vee y \in X_0$*).

We shall write $Y \underset{B_+}{\subset} X$, or $Y \underset{MI}{\subset} X$, if there exists a one-to-one, linear, isometrical and isotonical map of Y onto a B_+ -subspace of X, or onto an MI-subspace of X, respectively. As we have mentioned, the condition $C(\Omega_0) \underset{MI}{\subset} C(\Omega)$ is equivalent to existence of a continuous mapping of Ω onto Ω_0.

Every MI-subspace X_0 of $X = C(\Omega)$ is an MI-space (³) and, by a representation theorem of S. Kakutani and of M. Krein and S. Krein (see [13] and [16]), X_0 is equivalent in the linear, metric and lattice sense to a space $C(\Omega_0)$.

(²) The l. u. b. $x \vee y$ is defined (in $X = C(\Omega)$) by $(x \vee y)(t) = \max(x(t), y(t))$; we shall also write $\alpha \vee \beta$ instead of $\max(\alpha, \beta)$, α, β being real numbers.

The terms *lattice isomorphism*, *lattice homomorphism*, *Banach lattice* etc. have the same meaning as in Birkhoff's monography [3].

(³) A Banach lattice Y is called an *M-space* if $x \geqslant 0$, $y \geqslant 0$ imply $\|x \vee y\| = \|x\| \vee \|y\|$. The unit of Y is an element $e \in Y$ such that $\|x\| \leqslant 1$ is equivalent to $-e \leqslant x \leqslant e$.

An M-space with a unit will be called an *MI-space*.

On the other hand, a B_+-subspace need not be equivalent to any space of continuous functions (⁴) and a B_+-subspace need not be an MI-subspace even if it is equivalent (in a linear, isometrical and isotonical sense, with respect to its relative order) to a space $C(\Omega_0)$.

Let $C(\Omega_0) \underset{B_+}{\subset} X = C(\Omega)$, and let X_0 be a B_+-subspace of X corresponding to $C(\Omega_0)$. The question arises whether there exist, for every point $t \in \Omega$, functions $x \in X_0$, $y \in X_0$ such that $(x \vee y)(t) \neq (x \vee_0 y)(t)$.

The answer is negative: the set Ω_l of all points $t \in \Omega$ such that

$$(x \vee y)(t) = (x \vee_0 y)(t) \quad \text{for all} \quad x \in X_0,\ y \in X_0$$

is non-empty; moreover Ω_l separates X_0 (⁵).

Next, X_0 being an M-space (with respect to the relative order induced by X), it may possess a relative unit e_0 (i. e. an element $e_0 \in X_0$ such that $\|e_0\| = 1$ and such that $x \in X_0$, $\|x\| \leqslant 1$ imply $x \leqslant e_0$). Evidently, $e_0 \leqslant e$, but the equality $e_0 = e$ is not true in general.

Example 2. Let $\Omega = \langle 0, 1\rangle$ and let X_0 be the set of all functions $x(t)$ belonging to $C\langle 0, 1\rangle$, which are linear in each interval $\langle \frac{1}{3}, \frac{1}{2}\rangle$ and $\langle \frac{1}{2}, \frac{2}{3}\rangle$, and equal to 0 at $t = \frac{1}{2}$. As in the preceding example, X_0 is a B_+-subspace of $C\langle 0, 1\rangle$, equivalent to the Cartesian square of $C\langle 0, 1\rangle$; $x \vee_0 y = x \vee y$ is valid for all $x \in X_0, y \in X_0$, but the unit e_0 of X_0, equal to 0 at $t = \frac{1}{2}$, differs from e in the interval $(\frac{1}{3}, \frac{2}{3})$.

Let us write $\Omega_e = \{t \in \Omega \colon e_0(t) = 1\}$. In general, neither inclusion $\Omega_e \subset \Omega_l$ nor $\Omega_l \subset \Omega_e$ is true. Indeed, in Example 1 we have $\Omega_e = \Omega = \langle 0, 1\rangle$ (since the unit e of $C(\Omega)$ belongs to X_0) and $\Omega_l = \langle 0, \frac{1}{3}\rangle \cup \langle \frac{2}{3}, 1\rangle$; however, in Example 2, $\Omega_l = \langle 0, 1\rangle$ and $\Omega_e = \langle 0, \frac{1}{3}\rangle \cup \langle \frac{2}{3}, 1\rangle$.

The set $\Omega_s = \Omega_e \cap \Omega_l$ will be termed the *support* of X_0. The main theorem of this paper (Theorem 1) states that *Ω_s is a closed, non-empty subset of Ω separating X_0*, and that

$$\|x\| = \sup\{|x(t)| \colon t \in \Omega_s\}$$

for every $x \in X_0$. Moreover, *Ω_0 is a continuous image of Ω_s* (⁶).

(⁴) Moreover, M. Krein [15] proves that, for every Banach lattice $\langle Y, \|\ \|\rangle$, there exists an equivalent norm $\|\ \|_1$ in Y, defined by the formula $\|y\|_1 = \sup\{|\eta(y)| \colon \eta \in S_+^*\}$ where $S_+^* = \{\eta \in Y^* \colon \|\eta\| \leqslant 1, \eta \geqslant 0\}$, such that $\langle Y, \|\ \|_1\rangle \underset{B_+}{\subset} C(S_+^*)$, S_+^* being compact in the *-weak topology $\sigma(Y^*, Y)$. If Y is separable, then also $\langle Y, \|\ \|_1\rangle \underset{B_+}{\subset} C\langle 0, 1\rangle$.

(⁵) We shall say that a subset F of Ω *separates* a subspace X_0 of X if $\sup\{|x(t)| \colon t \in F\} \neq 0$ for every $x \in X_0$, $x \neq 0$.

(⁶) This theorem was published (without proof) in [10].

Thus, if $C(\Omega_0) \underset{B_+}{\subset} C(\Omega)$, then Ω_0 is a continuous image of a suitable closed subset Ω_s of Ω. This last condition is thus necessary (but not sufficient, even if $\Omega_s = \Omega$) in order that $C(\Omega_0) \underset{B_+}{\subset} C(\Omega)$. In the special case of $C(\Omega)$ being separable, this consequence of Theorem 1 is quite trivial, since every uncountable metric space contains a subset homeomorphic to the Cantor discontinuum $\mathcal{C}$, every compact metric space is a continuous image of $\mathcal{C}$, and every countable compact space is homeomorphic to a closed countable segment of ordinals. Thus, $C(\Omega_1) \underset{B_+}{\subset} C(\Omega_2)$ and $C(\Omega_2) \underset{B_+}{\subset} C(\Omega_1)$ hold for every pair of uncountable compact metric spaces Ω_1 and Ω_2.

Obviously, the condition $\Omega_s = \Omega$ means that X_0 is an MI-subspace of $C(\Omega)$; thus, the continuous image theorem (quoted above as a variant of a theorem of Stone) is a special case of Theorem 1.

K. Borsuk [4] proved the following very important extension theorem: *if E_0 is a closed and separable subset of a metric space E* (in particular, if E_0 is a closed subset of a compact metrisable space E), *then every bounded continuous function $x(t)$ defined on E_0 may be extended to a function $x^*(t)$ continuous on E, so that*

(1) $$\sup_{t \in E_0} |x(t)| = \sup_{t \in E} |x^*(t)|,$$

(2) *the correspondence $x \to x^*$ is linear, i. e. if $x \to x^*$ and $y \to y^*$, then $x+y \to x^*+y^*$,*

(3) *if $x(t) \geqslant y(t)$ for all $t \in E_0$, then also $x^*(t) \geqslant y^*(t)$ for $t \in E$,*

(4) *if $x(t) = 1$ for $t \in E_0$, then $x^*(t) = 1$ for $t \in E$.*

Thus, $x \to x^*$ is a linear, isometrical and isotonical map transforming the space $C_b(E_0)$ of bounded continuous functions on E_0 into the space $C_b(E)$. Any extension $x \to x^*$ satisfying conditions (1), (2) and (3) is called a *simultaneous extension*. Every simultaneous extension determines a non-negative (7) projection P, of norm 1, transforming the space $C_b(E)$ onto the subspace X_0 of all functions $x^*(t)$ with $x \in C_b(E_0)$; namely $P(x)$ is the extension of the restricted function $x|E_0$.

Obviously, X_0 is a B_+-subspace of $C_b(E)$ and P is a lattice homomorphism of $C_b(E)$ onto X_0, i. e.

$$P(x \vee y) = P(x) \vee_0 P(y) \quad \text{for all} \quad x \in X,\ y \in X.$$

(7) A linear operation P is termed *non-negative* if $P(x) \geqslant 0$ whenever $x \geqslant 0$. A map P from X onto a subspace X_0 will be called a *projection* if P is linear and if $P^2 = P$, i. e. if $P(x) = x$ for $x \in X_0$.

Condition (3) being satisfied, neither condition $x \vee y \to x^* \vee y^*$ nor $x \cdot y \to x^* \cdot y^*$ need to be satisfied. Yoshizawa [24] has shown that, in the case of compact E, the last condition is satisfied if and only if E_0 is a retract (8) of E.

Borsuk's proof gives an effective integral formula for the extended functions, founded on the existence of a continuous map transforming a set of positive linear Lebesgue measure onto E_0.

S. Kakutani [12], Dugundji [6a], R. Arens [1] gave new proofs of this theorem of Borsuk; their proofs are simpler and use methods of functional analysis. At the same time, they generalize Borsuk's assumptions concerning the topological properties of E and E_0. The most general formulations of the simultaneous extension theorem are discussed by E. Michael [19]; in particular, the theorem holds if E_0 is metrisable and compact and E is paracompact, or if E is metric and E_0 is any closed subset of E. Although Tietze's extension theorem is valid for normal spaces, an extension satisfying conditions (1), (2) and (4) does not need to satisfy (3). Arens, Michael and, finally, M. M. Day [6] investigate counter-examples to show that *simultaneous extension is not possible in the domain of all normal spaces, even if both spaces E and E_0 are compact.*

E. g., the Stone-Čech compactification $\beta(N_1)$ of an uncountable isolate set N_1 is topologically contained in a Tychonoff cube $\mathcal{I}^{\aleph_\tau}$ and, on the other hand, the space $C(\beta(N_1))$ is isomorphic to no subspace of $C(\mathcal{I}^{\aleph_\tau})$, since $C(\mathcal{I}^{\aleph_\tau})$ is isomorphic to a strictly convex space and $C(\beta(N_1))$ does not possess this property. Next, N being the set of integers, there exists no simultaneous extension of the continuous functions on $\beta(N) \setminus N$ onto the whole of $\beta(N)$, since there exists no projection of the space m of all bounded sequences onto its subspace c_0 of null-convergent sequences (Phillips [20], Sobczyk [22]), and the space

$$X_0 = \bigcap_{t \in \beta(N) \setminus N} \{x \in C(\beta(N)) : x(t) = 0\}$$

is equivalent to c_0 (9).

(8) E_0 is termed a *retract* of E if there exists a continuous map, called *retraction*, transforming E onto E_0 so that $\sigma(u) = u$ for $u \in E_0$. Any retract σ induces a simultaneous extension given by $x^*(t) = x(\sigma(t))$.

(9) A. Sobczyk [22] proves that if Y_0 is a subspace of a separable Banach space Y and if Y_0 is isomorphic to the space c_0, then there exists a projection of Y onto Y_0. An analysis of the proof of this theorem of Sobczyk (given by A. Pełczyński [19]) leads to the following conclusion: if Y_0 is a subspace of $C(\Omega)$ isometric to c_0 and if there exists no projection of $C(\Omega)$ onto Y_0, then there exists a closed subset Ω_0 of Ω such that the continuous functions on Ω_0 do not admit a simultaneous extension onto Ω.

In particular, the above statement is valid if $C(\Omega)$ contains a subspace isometric to the space m.

The negative solution of the general problem of simultaneous extensions in compact spaces leads to many unsolved particular problems.

The inverse principal question (whether the isometric and lattice-isomorphic embedding $C(\Omega_1) \subset C(\Omega_2)$ implies $\Omega_1 \underset{\text{top}}{\subset} \Omega_2$) has a negative solution, as shown by simple examples. In the second part of this paper we shall give the following conversion of Borsuk's theorem: if $C(\Omega_0)$ is isometric to a subspace X_0 of $X = C(\Omega)$ and if there exists a lattice-homomorphism projection P of norm 1, transforming X onto X_0, then Ω_0 may be embedded topologically into Ω in a way admitting a simultaneous extension corresponding naturally to the initial embedding of $C(\Omega_0)$ into $C(\Omega)$. Moreover, in this case X_0 is a B_+-subspace of $C(\Omega)$ and the homeomorphic image Ω_P of Ω_0 in Ω is a retract of the support Ω_s *of* X_0. Hence, if $\Omega_s = \Omega$ (i. e. if X_0 is an MI-subspace of $C(\Omega)$), then Ω_P is a retract of Ω (this is a generalization of the theorem of Yoshizawa which we have quoted).

If $P \geqslant 0$ and $P^2 = P$, then the assumption $\|P\| = 1$ is equivalent to $P(e) = e_0$; however, it implies the relation $P(x \vee y) = x \vee_0 y$ only for $x \in X_0$ and $y \in X_0$. Thus, the hypothesis of Theorem 3 and Theorem 4 of [10] that P is a non-negative projection of norm 1 is essentially too weak, and the theorems need the additional hypothesis that $P(x \vee y) = x \vee_0 y$ for all $x \in X_0$, $y \in X_0$, to be correct.

We shall also consider the non-negative projections of norm $\|P\| \geqslant 1$. If such a projection of $X = C(\Omega)$ onto its subspace X_0 exists, then X_0 is a B_+-subspace of X, and X_0 is an MI-space with respect to the order induced by X and with respect to the norm $\|x\|^* = \inf\{\lambda : |x| \leqslant \lambda P(e)\}$, which is equivalent to the initial norm $\| \ \|$ in X_0.

2. Support of a B_+-subspace. In the sequel we make the following assumptions:

1° *Ω is a compact Hausdorff space and X is the space $C(\Omega)$,*

2° *X_0 is a B_+-subspace of X,*

3° *Ω_0 is a compact Hausdorff space such that $C(\Omega_0)$ is equivalent to X_0, and this equivalence is established by a one-to-one, linear, isometrical and isotonical transformation T of $C(\Omega_0)$ onto X_0.*

The elements $T^{-1}(x), T^{-1}(x'), T^{-1}(y), T^{-1}(y_n), \ldots$ will be denoted by $\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{y}, \boldsymbol{y}_n, \ldots$, respectively. Since T is a lattice isomorphism,

$$x \vee_0 y = T(\boldsymbol{x} \vee \boldsymbol{y}) \quad \text{and} \quad x \wedge_0 y = T(\boldsymbol{x} \wedge \boldsymbol{y})$$

are the relative l. u. b. and g. l. b., respectively. Moreover

$$T(\boldsymbol{x} \wedge \boldsymbol{y}) \leqslant T(\boldsymbol{x}) \wedge T(\boldsymbol{y}) \leqslant T(\boldsymbol{x}) \vee T(\boldsymbol{y}) \leqslant T(\boldsymbol{x} \vee \boldsymbol{y}),$$

which means that

$$(x \vee_0 y)(t) \geqslant (x \vee y)(t) \quad \text{and} \quad (x \wedge_0 y)(t) \leqslant (x \wedge y)(t)$$

for all $t \in \Omega$.

By $\boldsymbol{e}_0$ we shall denote the unit of $C(\Omega_0)$ (i. e. the function $\boldsymbol{e}_0(u) = 1$ for $u \in \Omega_0$) and, in turn, by e_0 we shall denote the element $T(\boldsymbol{e}_0)$. Obviously, e_0 is the relative unit of X_0, whence $e_0 \leqslant e$. By $|x|_0$ (for $x \in X_0$) we shall denote the relative absolute value of x, defined as

$$|x|_0 = x \vee_0 (-x) = T[\boldsymbol{x} \vee (-\boldsymbol{x})] = T(|\boldsymbol{x}|).$$

Next, we shall write successively:

$$\Phi(x) = \{t \in \Omega : |x|_0(t) = |x(t)|\} \quad \text{for} \quad x \in X_0,$$

$$\Omega_l = \bigcap_{x \in X_0} \Phi(x), \qquad S^+ = \{x \in X_0 : x \geqslant 0,\ \|x\| \leqslant 1\},$$

$$Z(x) = \{t \in \Omega : x(t) = 0\} \quad \text{for} \quad x \in X,$$

$$Z_0(x) = \{u \in \Omega_0 : \boldsymbol{x}(u) = 0\} \quad \text{for} \quad x \in X_0,$$

$$A(u) = \{x \in S^+ : \boldsymbol{x}(u) = 0\} \quad \text{for} \quad u \in \Omega_0,$$

$$\Omega(u) = \bigcap_{x \in A(u)} Z(x) \quad \text{for} \quad u \in \Omega_0.$$

LEMMA 1. *Let F be any closed subset of Ω separating X_0 and such that $\|x\| = \sup\{|x(t)| : t \in F\}$ for every $x \in X_0$. Then the operation U defined as the restriction $y = U(x)$, where*

$$y(t) = x(t) \text{ for } t \in F, \ x \in C(\Omega) \text{ and } y \in C(F),$$

is a linear lattice-homomorphism of norm 1 transforming $C(\Omega)$ onto $C(F)$, and the restricted operation $U|X_0$ (considered only on X_0) is a one-to-one isometrical and isotonical map of X_0 onto a B_+-subspace X_1 of $C(F)$.

Proof. The first part of the lemma is obvious. We shall prove that if $x \in X_0$ and $U(x) \geqslant 0$, then $x \geqslant 0$ (i. e. that the statements $x(t) \geqslant 0$ for $t \in F$ and $x(t) \geqslant 0$ for $t \in \Omega$ are equivalent for $x \in X_0$); we may assume that $\|x\| = 1$.

Let $x \in X_0$, $U(x) \geqslant 0$ and $\|x\| = 1$. Then $x \leqslant e_0$, whence $0 \leqslant x(t) \leqslant e_0(t) \leqslant 1$ for $t \in F$, Consequently,

$$\|e_0 - x\| = \sup\{|e_0(t) - x(t)| : t \in F\} \leqslant 1,$$

which means that $e_0 - x \leqslant e_0$ and $x \geqslant 0$. $U|X_0$ being isotonical, the set X_1 of all restricted functions $y = x|F$ with $x \in X_0$ is a B_+-subspace of $C(F)$.

LEMMA 2. *The set* $\Omega_e = \{t \in \Omega : e_0(t) = 1\}$ *is identical with the set*

$$H = \bigcup_{\substack{\|x\|=1 \\ x \in X_0}} \{t \in \Omega : x(t) = 1\} = \bigcup_{\substack{\|x\|=1 \\ x \in X_0}} \{t \in \Omega : |x(t)| = 1\}.$$

Proof. Inclusion $\Omega_e \subset H$ is trivial. Let us assume $t_0 \in H$ and $x(t_0) = 1$, $\|x\| = 1$ for a certain function $x \in X_0$. Then, by the definition of e_0, we have $e \geqslant e_0 \geqslant x$, whence $1 \geqslant e_0(t_0) \geqslant x(t_0) = 1$. We infer $e_0(t_0) = 1$, whence $t_0 \in \Omega_e$.

LEMMA 3. *The set* Ω_e *separates* X_0; *moreover*

$$\|x\| = \sup\{|x(t)| : t \in \Omega_e\} \quad \text{for} \quad x \in X_0.$$

This is a consequence of Lemma 2.

LEMMA 4. *The conditions* $x \in X_0$, $y \in X_0$ *and*

$$x(t) \geqslant y(t) \quad \text{for} \quad t \in \Omega_e$$

imply $x \geqslant y$.

This is a consequence of Lemmas 1 and 3.

LEMMA 5. *The set* $\Omega_e \cap \Omega(u)$ *is non-empty for every* $u \in \Omega_0$.

Proof. Let $u_0 \in \Omega_0$. By the compactness of Ω, it suffices to prove that the family $\{\Omega_e \cap Z(x)\}_{x \in A(u_0)}$ has the finite intersection property. Let us suppose, a contrario, that there exist functions $x_1, \ldots, x_n$ belonging to $A(u_0)$ and such that $\Omega_e \cap \bigcap_{i=1}^{n} Z(x_i) = 0$. This means that $x_1(t) + \ldots + x_n(t) > 0$ for all $t \in \Omega_e$ (since $x_i \geqslant 0$ for $i = 1, \ldots, n$). By the compactness of Ω_e, there exists $\delta > 0$ such that

$$x_1(t) + \ldots + x_n(t) \geqslant \delta = \delta \cdot e_0(t) \quad \text{for} \quad t \in \Omega_e,$$

whence, by Lemma 4, we infer $x_1 + \ldots + x_n \geqslant \delta e_0$ and $\boldsymbol{x}_1 + \ldots + \boldsymbol{x}_n \geqslant \delta \boldsymbol{e}_0$, which contradicts $\boldsymbol{x}_1(u_0) + \ldots + \boldsymbol{x}_n(u_0) = 0$.

LEMMA 6. *Let* $t \in \Omega(u)$ *and let* $x \in S^+$. *If* $\boldsymbol{x}(u) = 0$, *then* $x(t) = 0$; *if* $\boldsymbol{x}(u) = 1$ *and* $t \in \Omega_e$, *then* $x(t) = 1$.

Proof. The first implication follows immediately from the definition of $\Omega(u)$; in order to prove the second, we observe that if $x \in S^+$, then also $e_0 - x \in S^+$, and $(e_0 - x)(t) = 0$ implies $x(t) = 1$.

LEMMA 7. *Let* $u_1 \in \Omega_0$, $u_2 \in \Omega_0$ *and* $u_1 \neq u_2$. *Then*

$$\Omega_e \cap [\Omega(u_1) \cap \Omega(u_2)] = 0.$$

Proof. There exists a function $\boldsymbol{x} \in C(\Omega_0)$ such that $\|\boldsymbol{x}\| = 1$, $\boldsymbol{x}(u) \geqslant 0$ for all $u \in \Omega_0$, $\boldsymbol{x}(u_1) = 0$ and $\boldsymbol{x}(u_2) = 1$. Then $x = T(\boldsymbol{x}) \in S^+$ and, by Lemma 6, we have $x(t) = 0$ for all $t \in \Omega(u_1)$ and $x(t) = 1$ for all $t \in \Omega_e \cap \Omega(u_2)$, which implies $\Omega_e \cap \Omega(u_1) \cap \Omega(u_2) = 0$.

By the above lemmas, the map φ defined by the relation

$$u = \varphi(t) \quad \text{if and only if} \quad t \in \Omega_e \cap \Omega(u)$$

is uniquely defined on the set

$$\Omega_1 = \Omega_e \cap \bigcup_{u \in \Omega_0} \Omega(u)$$

and $\Omega(u) = \varphi^{-1}(u)$ for all $u \in \Omega_0$.

Thus, the map φ is defined by the following property: *if* $e_0(t) = 1$ *and if* $x \geqslant 0$ *and* $\boldsymbol{x}(u) = 0$ *imply* $x(t) = 0$, *then* $u = \varphi(t)$.

Let $e_0(t) = 1$ and $u = \varphi(t)$. Then $\boldsymbol{x}(u) = 0$ implies $\boldsymbol{x}_+(u) = 0$ and $\boldsymbol{x}_-(u) = 0$, whence $x(t) = 0$. Consequently, the map φ is determined by the null-sets of the functions x and $\boldsymbol{x}$ with $x \in X_0$.

LEMMA 8. φ *is a continuous map of* Ω_1 *onto* Ω_0.

Proof. The identity $\varphi(\Omega_1) = \Omega_0$ has been established; we shall prove that $\varphi^{-1}(A)$ is closed for every closed subset A of Ω_0. Let $t_0 \in \Omega_1 \setminus \varphi^{-1}(A)$. Then there exists a function $x \in S^+$ such that $\boldsymbol{x}(u) = 0$ for $u \in A$ and $\boldsymbol{x}[\varphi(t_0)] = 1$. Hence, by Lemma 6, we conclude that $x(t) = 0$ for $t \in \varphi^{-1}(A)$ and $x(t_0) = 1$. Thus, the set $G = \{t \in \Omega : x(t) > \frac{1}{2}\}$ is an open neighbourhood of t_0 disjoint with $\varphi^{-1}(A)$.

LEMMA 9. *Let* $x \in X_0$, $x \neq 0$, *and* $x \geqslant 0$. *Then*

$$\varphi^{-1}[Z_0(x)] \subset Z(x).$$

Proof. $t \in \varphi^{-1}[Z_0(x)]$ means that $\boldsymbol{x}[\varphi(t)] = 0$, whence, by Lemma 6 and by the formula $Z(x) = Z(x/\|x\|)$, we obtain $x(t) = 0$.

LEMMA 10. *Let* $x \in X_0$, $y \in X_0$; *if* $x \wedge_0 y = 0$, *then* $x(t) \cdot y(t) = 0$ *for all* $t \in \Omega_1$.

Proof. Let $x \wedge_0 y = 0$, then $\boldsymbol{x} \wedge \boldsymbol{y} = 0$ in $C(\Omega_0)$, which means that $Z_0(x) \cup Z_0(y) = \Omega_0$. We apply Lemma 9:

$$\Omega_1 = \varphi^{-1}(\Omega_0) = \varphi^{-1}[Z_0(x)] \cup \varphi^{-1}[Z_0(y)] \subset Z(x) \cup Z(y),$$

whence, for every $t \in \Omega_1$, either $x(t) = 0$ or $y(t) = 0$.

LEMMA 11. *Given a point* $t_0 \in \Omega$, *the following statements are equivalent*:

(a) $(x \vee_0 y)(t_0) = (x \vee y)(t_0)$ *for all* $x \in X_0$, $y \in X_0$,

(b) $(x \wedge_0 y)(t_0) = (x \wedge y)(t_0)$ *for all* $x \in X_0$, $y \in X_0$,

(c) $|x|_0(t_0) = |x|(t_0)$ *for all* $x \in X_0$,

(d) *if* $(x \wedge_0 y)(t_0) = 0$, *then* $x(t_0) \cdot y(t_0) = 0$, *for all* $x \in X_0$, $y \in X_0$.

Proof. The equivalence between (a), (b) and (c) follows by the identities

$$x \vee_0 y = -[(-x) \wedge_0 (-y)] = \tfrac{1}{2}[x + y + |x - y|_0].$$

The implication (b) $\Rightarrow$ (d) is trivial. Thus, let us assume (d) and let $x\in X_0$, $y\in X_0$. Then, by the identity

$$[x-(x\wedge_0 y)]\wedge_0[y-(x\wedge_0 y)]=0,$$

we infer either $x(t_0)=(x\wedge_0 y)(t_0)$ or $y(t_0)=(x\wedge_0 y)(t_0)$, whence $(x\wedge y)(t_0)=(x\wedge_0 y)(t_0)$.

LEMMA 12. *The set Ω_1 is identical with $\Omega_s=\Omega_e\cap\Omega_l$, whence it is closed.*

Proof. The inclusion $\Omega_1\subset\Omega_e\cap\Omega_l$ being a consequence of Lemmas 10 and 11, we have to prove that $\Omega_e\setminus\Omega_1\subset\Omega_e\setminus\Omega_l$. Let $t_0\in\Omega_e\setminus\Omega_1=\Omega_e\setminus\bigcup_{u\in\Omega_0}\Omega(u)$. Then, for every $u\in\Omega_0$, there exists a function $x\in A(u)$ such that $x(t_0)\neq 0$. Hence, for every $u\in\Omega_0$ we can choose a function $y_u\in X_0$ such that $y_u(t_0)=1$, $y_u(u)=0$ and $y_u\geqslant 0$. Writing $G_u=\{v\in\Omega_0: y_u(v)<\frac{1}{2}\}$ we obtain an open covering of Ω_0, whence, by the compactness of Ω_0, there exist points $u_1,\ldots,u_n\in\Omega_0$ such that $\Omega_0=\bigcup_{i=1}^{n}G_{u_i}$. Writing $y=\bigwedge_{i=1}^{n}y_{u_i}$ we conclude that $y(v)<\frac{1}{2}$ for all $v\in\Omega_0$, whence $\|y\|=\|y\|<\frac{1}{2}$. However, $y_{u_i}(t_0)=1$ for $i=1,\ldots,n$ which means that

$$\Big[\bigwedge_{0\; i=1,\ldots,n} y_{u_i}\Big](t_0)=y(t_0)\neq\min_{i=1,\ldots,n}y_{u_i}(t_0)=\Big(\bigwedge_{i=1}^{n}y_{u_i}\Big)(t_0).$$

Thus, there exist functions x and y in X_0 such that $(x\wedge_0 y)(t_0)\neq\neq(x\wedge y)(t_0)$; this implies $t_0\notin\Omega_l$.

THEOREM 1. *Let Ω, Ω_0, X_0 and T satisfy conditions* 1°, 2°, 3° *(written above, see p. 308). Then the support $\Omega_s=\Omega_e\cap\Omega_l$ of X_0 has the following properties:*

(i) *Ω_s is closed and non-empty,*

(ii) *Ω_s separates X_0 and $\|x\|=\sup\{|x(t)|: t\in\Omega_s\}$ for every $x\in X_0$,*

(iii) *if $x\in X_0$, $y\in X_0$ and $x(t)\geqslant y(t)$ for $t\in\Omega_s$, then $x(t)\geqslant y(t)$ for all $t\in\Omega$,*

(iv) *the relation*

$$u=\varphi(t)\ \textit{if}\ e_0(t)=1\ \textit{and}\ x(t)=0\ \textit{whenever}\ x\geqslant 0\ \textit{and}\ \boldsymbol{x}(u)=0$$

determines a uniquely defined continuous map of Ω_s onto Ω_0,

(v) *$\boldsymbol{x}[\varphi(t)]=x(t)$ for every $x=T(\boldsymbol{x})\in X_0$,*

(vi) *the functions $x(t)$ continuous on Ω_s and constant on every member of the semicontinuous decomposition $\Omega_s=\bigcup_{u\in\Omega_0}\varphi^{-1}(u)$ have a simultaneous extension on Ω (determined by T).*

Proof. Conditions (i) and (iv) have been established in Lemmas 5, 8 and 12.

Condition (v) has been proved in two special cases (see Lemma 6): if $\boldsymbol{x}[\varphi(t)]=0$ or if $\boldsymbol{x}[\varphi(t)]=1$. The general case may easily be reduced (by the substitution $y(s)=x(s)/\boldsymbol{x}[\varphi(t)]$) to the preceding cases. Condition (ii) is an immediate consequence of (v), and condition (iii) follows by (ii) and by Lemma 1.

Finally, let us recall the following known theorem, due to Šilov [21] (see also [11], p. 57): every continuous function $z(t)$ on Ω_s, constant on the members of the decomposition $\Omega_s=\bigcup_{u\in\Omega_0}\varphi^{-1}(u)$, is of the form $z(t)=\boldsymbol{x}[\varphi(t)]$ with $\boldsymbol{x}\in C(\Omega_0)$. By (v), the function $x=T(\boldsymbol{x})$ (defined on Ω) coincides with z on Ω_s, which means that the correspondence $z\to x$ is a simultaneous extension.

COROLLARY. *If (assuming conditions* 1°, 2°, 3°*) the set X_0 separates Ω_s (in particular, if X_0 separates Ω), then $\Omega_0\underset{\text{top}}{\subset}\Omega$.*

Indeed, the assumption of separation is equivalent to the existence, for every pair of different points t_1, t_2 of Ω_s, of a function $x\in X_0$ such that $x(t_1)\neq x(t_2)$ and consequently, by Lemma 6, to the fact that φ is one-to-one. Hence $\Omega_0\underset{\text{top}}{=}\Omega_s\subset\Omega$.

PROPOSITION 1. *Let Y_0 be an arbitrary B_+-subspace of $X=C(\Omega)$ and let Y_1 be a dense subset of Y_0 such that $Z(y)\neq 0$ for every $y\in Y_1$. Then there exists a point $t_0\in\Omega$ such that $y(t_0)=0$ for all $y\in Y_0$.*

If, additionally, Y_0 is a B_+-subspace satisfying conditions 1°, 2°, 3° (written above), *then*

$$\bigcap_{x\in Y_0}Z(x)=Z(e_0)\subset\Omega_l.$$

Proof. Let $y\in Y_0$ and let $\|y-y_n\|\to 0$ with $y_n\in Y_1$. Then the compactness of Ω and $Z(y_n)\neq 0$ imply $Z(y)\neq 0$.

Now, the family $\{Z(x)\}_{x\in Y_0, x\geqslant 0}$ has the finite intersection property, since $Z(x_1)\cap\ldots\cap Z(x_n)=Z(x_1+\ldots+x_n)\neq 0$ for $x_1\geqslant 0,\ldots,x_n\geqslant 0$. Hence

$$F=\bigcap_{\substack{x\in Y_0\\ x\geqslant 0}}Z(x)\neq 0.$$

Every element of Y_0 being the difference of non-negative elements of Y_0, we infer $y(t)=0$ for all $y\in Y_0$ and for any point t belonging to F.

The second part of Proposition 1 is obvious.

3. Projections onto B_+-subspaces. P being a projection of a Banach space X onto its subspace X_0, the adjoint transformation P^*, defined by the formula

$$(P^*\xi)(x)=\xi(Px)\quad\text{for}\quad x\in X,\ \xi\in X_0^*,$$

is an isomorphism of X_0^* into X^*, $\|P^*\| = \|P\|$, and $P^*\xi$ is an extension of ξ onto the whole of X, whence $\|(P^*)^{-1}\| \leqslant 1$. Moreover, P^* is a homeomorphism with respect to the *-weak topologies $\sigma(X_0^*, X_0)$ and $\sigma(X^*, X)$.

LEMMA 13. *Let X be a vector lattice, let X_0 be any subspace of X and let P be any non-negative projection of X onto X_0. Then X_0 is a vector lattice with respect to the order induced by X and*

$$x\vee_0 y = P(x\vee y), \quad x\wedge_0 y = P(x\wedge y), \quad |x|_0 = P(|x|) \quad \textit{for } x\in X_0,\ y\in X_0.$$

Proof. Let $x\in X_0$, $y\in X_0$. Then $x\vee y \geqslant y$ and $x\vee y \geqslant x$, whence

$$P(x\vee y) \geqslant P(y) = y \quad \text{and} \quad P(x\vee y) \geqslant P(x) = x.$$

Now, let z be any element of X_0 such that $z \geqslant x$ and $z \geqslant y$. Then $z \geqslant x\vee y$, whence

$$z = P(z) \geqslant P(x\vee y).$$

This means that $P(x\vee y)$ is the relative l. u. b. $x\vee_0 y$ of x and y in X_0; x and y being arbitrary elements of X_0, we have proved that X_0 is a vector lattice. The relation $P(x\wedge y) = x\wedge_0 y$ follows from $P(x\vee y) = x\vee_0 y$; finally, $|x|_0 = x\vee_0(-x) = P[x\vee(-x)] = P(|x|)$.

THEOREM 2. *Let $X = C(\Omega)$ and let P be a non-negative projection of X onto its subspace X_0. Then X_0 is a B_+-subspace of X isomorphic to a space $C(\Omega_0)$. Moreover:*

(i) *X_0 is an MI-space with respect to the norm*

$$\|x\|^* = \inf\{\lambda\colon |x| \leqslant \lambda P(e)\}$$

and unit $e_0 = P(e)$ (i. e. $x\in X_0$, $y\in X_0$, $x\geqslant 0$, $y\geqslant 0$ imply $\|x\vee_0 y\|^* = \max(\|x\|^*, \|y\|^*)$, $\|e_0\|^* = 1$, and $\|x\|^* \leqslant 1$ implies $x \leqslant e_0$);

(ii) *the norms $\|\ \|$ and $\|\ \|^*$ are equivalent on X_0 and*

$$\|x\|^* \leqslant \|x\| \leqslant \|P\|\cdot\|x\|^* \quad \textit{for} \quad x\in X_0;$$

(iii) *we have $\|e_0\| = \|P\|$.*

Proof. X_0 is a B_+-subspace of X (by Lemma 13) and $|x| \leqslant \|x\|^* P(e) = \|x\|^* e_0$ (by the continuity of the order relation). Now, we shall prove that $\|\ \|^*$ is a norm in X_0. Since

$$|x+y| \leqslant |x|+|y| \leqslant (\|x\|^* + \|y\|^*)e_0,$$

we conclude that $\|x+y\|^* \leqslant \|x\|^* + \|y\|^*$. The equality $\|\alpha x\|^* = |\alpha|\cdot\|x\|^*$ being trivial, $\|\ \|^*$ is a pseudonorm in X_0.

Let $x\in X_0$; then $0 \leqslant e_0 \leqslant \|P\|\cdot e$ implies

$$\|x\| = \inf\{\lambda\colon |x| \leqslant \lambda e\} \leqslant \inf\{\lambda\colon \|P\||x| \leqslant \lambda e_0\} = \|P\|\cdot\|x\|^*.$$

Thus, we have proved that $\|\ \|^*$ is a norm in X_0, and

$$\|x\| \leqslant \|P\|\cdot\|x\|^*.$$

In turn, let us write

$$\|x\|^0 = \inf\{\lambda\colon |x|_0 \leqslant \lambda e_0\} \quad \text{for} \quad x\in X_0.$$

We shall prove that $\|x\|^0 = \|x\|^*$ for $x\in X_0$. Let $|x| \leqslant \lambda e_0$. Then $x \leqslant \lambda e_0$ and $-x \leqslant \lambda e_0$, whence $|x|_0 = x\vee_0(-x) \leqslant \lambda e_0$. On the other hand, $|x| \leqslant |x|_0$, whence $|x|_0 \leqslant \lambda e_0$ implies $|x| \leqslant \lambda e_0$. Consequently, $|x| \leqslant \lambda e_0$ is equivalent to $|x|_0 \leqslant \lambda e_0$, and $\|x\|^0 = \|x\|^*$.

Next, $|x| \leqslant \lambda e$ implies $|x|_0 = P(|x|) \leqslant \lambda e_0$ for $x\in X_0$. Hence $\|x\|^* = \|x\|^0 = \inf\{\lambda\colon |x|_0 \leqslant \lambda e_0\} \leqslant \inf\{\lambda\colon |x| \leqslant \lambda e\} = \|x\|$. We have proved that $\|\ \|$ and $\|\ \|^*$ are equivalent on X_0. Finally, easy computations show that $\langle X_0, \|\ \|^*\rangle$ is an MI-space with unit e_0, $\|e_0\|^* = 1$ and $\|e_0\| = \|P\|$.

COROLLARY. *If all assumptions of Theorem 2 are fulfilled and if moreover, $e_0(t) > 0$ for all $t\in\Omega$, then there exists a norm $\|\ \|^*$ in X, equivalent to $\|\ \|$ on X, which is an extension of the norm $\|\ \|^*$ considered above and is such that $\|P\|^* = 1$ ($\|P\|^*$ denotes the norm of the projection P of $\langle X, \|\ \|^*\rangle$ onto its subspace $\langle X_0, \|\ \|^*\rangle$).*

Indeed, if $e_0(t) > 0$ for $t\in\Omega$, then the relation

$$\|x\|^* = \inf\{\lambda\colon |x| \leqslant \lambda e_0\}$$

defines a new norm in X. The equivalence of $\|\ \|$ and $\|\ \|^*$ follows at once from the inequalities $\beta^{-1}\|x\| \leqslant \|x\|^* \leqslant \alpha^{-1}\|x\|$ where $\alpha = \inf_{\Omega} e_0(t)$, $\beta = \sup_{\Omega} e_0(t) = \|P\|$. As in the proof of Theorem 3, we obtain

$$\|Px\|^* = \inf\{\lambda\colon |Px| \leqslant \lambda e_0\}$$
$$\leqslant \inf\{\lambda\colon P(|x|) \leqslant \lambda e_0\} \leqslant \inf\{\lambda\colon |x| \leqslant \lambda e_0\} = \|x\|^*,$$

since $P(|x|) \geqslant |P(x)|$. Thus, $\|P\|^* \leqslant 1$, and $\|P\|^* \geqslant 1$, since P is a projection.

PROPOSITION 2. *Let X_0 be a B_+-subspace of X and an MI-space with respect to the relative order and norm induced by X, and let P be any projection of X onto X_0. Then the following conditions are equivalent:*

(a) $\|P\| = 1$ *and* $P \geqslant 0$,

(b) $P \geqslant 0$ *and* $P(e) = e_0$,

(c) $P(e) = e_0$ *and* $\|P\| = 1$.

Proof. Implications (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c) follow by conditions (ii) and (iii) of Theorem 2.

To prove (c) $\Rightarrow$ (a), let us assume $\|P\| = 1$, $P(e) = e_0$, $\|e_0\| = 1$. Let $x \in X$, $0 \leqslant x \leqslant e$ and let t be any point of Ω_s. Then

$$1-(Px)(t) = e_0(t)-(Px)(t) = (e_0-Px)(t) = [P(e-x)](t) \leqslant \|P\|\cdot\|e-x\| \leqslant 1,$$

whence $(Px)(t) \geqslant 0$. Since t is an arbitrary point of Ω_s, $Px \geqslant 0$ (by condition (iii) of Theorem 1). Consequently, $P \geqslant 0$.

Corollary. *If X_0 is a B_+-subspace of $X = C(\Omega)$ and an MI-space with respect to the relative order, and if $e \in X_0$ (i.e. if $e_0 = e$), then the conditions*

$$\|P\| = 1 \quad \text{and} \quad P \geqslant 0$$

are equivalent, P being any projection of X onto X_0.

Theorem 3. *Let X_0 be a B_+-subspace of the space $X = C(\Omega)$ and let P be a lattice-homomorphism projection of X onto X_0, with $\|P\| = 1$. Then*

(i) *X_0 is an MI-space with respect to the norm and order induced by X, and the unit of X_0 is $e_0 = P(e)$,*

(ii) *there exist successively: a compact Hausdorff space Ω_0, a linear-isometrical-isotonical transformation T from $C(\Omega_0)$ onto X_0, a closed subset Ω_P of Ω and, finally, a homeomorphism ψ from Ω_0 onto Ω_P, such that*

$$(T^{-1}Px)\big(\psi^{-1}(t)\big) = x(t) \quad \text{for} \quad x \in X \text{ and } t \in \Omega_P,$$

(iii) *the continuous functions on Ω_P have a simultaneous extension onto Ω defined by $z \to z^*$ where*

$$z^*(t) = (T\boldsymbol{y})(t) \quad \text{for} \quad t \in \Omega \quad \text{and} \quad \boldsymbol{y}(u) = z[\psi(u)] \quad \text{for} \quad u \in \Omega_0,$$

(iv) *Px is identical with the extension z^* of the restricted function*

$$z(t) = x(t) \quad \text{for} \quad t \in \Omega_P.,$$

(v) *Ω_P is a retract of the support Ω_s of X_0 and a retraction σ of Ω_s onto Ω_P is given by the formula*

$$\sigma(t) = \psi[\varphi(t)],$$

where φ denotes the continuous map of Ω_s onto Ω_0 defined in Theorem 1.

Proof. Condition (i) is a consequence of Theorem 2.

Let Ω_0 denote the set of all functionals ξ over X_0 such that

(5) $$\|\xi\| = 1, \quad \xi \geqslant 0,$$

and such that $x \wedge_0 y = 0$ implies $\xi(x)\cdot\xi(y) = 0$. Ω_0 is a compact Hausdorff space with respect to the *-weak topology induced by the space X_0^* conjugate to X_0, and by the Kakutani-Krein representation theorem,

there exists a one-to-one linear isometrical and isotonical map T from $C(\boldsymbol{\Omega}_0)$ onto X_0, defined by the relation

$$T^{-1}(x) = \boldsymbol{x}(\cdot), \text{ where } \boldsymbol{x}(\xi) = \xi(x) \text{ for } \xi \in \Omega_0.$$

Next, let $\boldsymbol{\Omega}$ denote the set of all functionals ξ over X satisfying (5) and such that $x \wedge y = 0$ implies $\xi(x)\cdot\xi(y) = 0$ for all $x \in X$, $y \in X$. It is well known that the natural map $\varkappa$, defined by

$$\varkappa(t) = \xi_t(\cdot), \text{ where } \xi_t(x) = x(t) \text{ for } x \in X,$$

is a homeomorphism from Ω onto $\boldsymbol{\Omega}$.

Given $\xi \in \boldsymbol{\Omega}_0$, the functional $\eta = P^*\xi$ is an extension of ξ onto X, and $\eta \in \boldsymbol{\Omega}$. Indeed, $\|\eta\| = 1$, and if $x \geqslant 0$, then $\eta(x) = (P^*\xi)(x) = \xi(Px) \geqslant 0$, for $P \geqslant 0$; if $x \wedge y = 0$, then

$$\eta(x)\wedge\eta(y) = \xi(Px)\wedge\xi(Py) = \xi(Px\wedge_0 Py) = \xi[P(x\wedge y)] = 0.$$

Thus, P^* maps Ω_0 into $\boldsymbol{\Omega}$ homeomorphically (with respect to the *-weak topologies). Accordingly, the map

$$\psi(\xi) = \varkappa^{-1}[P^*(\xi)]$$

is a homeomorphism from $\boldsymbol{\Omega}_0$ onto a subset Ω_P of Ω.

Next, we shall prove that $(T^{-1}Px)\big(\psi^{-1}(t)\big) = x(t)$ for all $x \in X$, $t \in \Omega_P$. Let $t \in \Omega_P$. Then $t = \psi(\xi) = \varkappa^{-1}(P^*\xi)$ and

$$x(t) = (P^*\xi)(x) = \xi(Px) = [T^{-1}(Px)](\xi)$$

(by the definition of T). The proofs of (iii) and (iv) are similar.

Now, the proof of (v) consists of two steps. Firstly we shall prove that $\Omega_P \subset \Omega_s$; in other words, given $t = \psi(\xi) = \varkappa^{-1}P^*\xi$, we shall deduce that $(x\vee_0 y)(t) = x(t)\vee y(t)$ and $e_0(t) = 1$:

$$\begin{aligned}(x\vee_0 y)(t) &= (P^*\xi)(x\vee_0 y) = \xi[P(x\vee_0 y)] = \xi(x\vee_0 y) = \xi(x)\vee\xi(y)\\ &= \xi(Px)\vee\xi(Py) = (P^*\xi)(x)\vee(P^*\xi)(y) = x(t)\vee y(t),\\ e_0(t) &= (P^*\xi)(e_0) = \xi[P(e_0)] = \xi(e_0) = \|\xi\| = 1.\end{aligned}$$

Finally, let φ be the map of Ω_s onto $\boldsymbol{\Omega}_0$ defined in Theorem 1; it is determined by the following relation:

$$\xi = \varphi(t) \text{ if and only if } \boldsymbol{x}(\xi) = x(t), \text{ whenever } x \in X_0, \ \boldsymbol{x} = T^{-1}(x).$$

We shall prove that $\sigma = \psi\varphi$ is a retraction of Ω_s onto Ω_P. Obviously, $\sigma(\Omega_s) = \Omega_P$. Let $t \in \Omega_P$ and let $\xi = \psi^{-1}(t)$. Then

$$\boldsymbol{x}(\xi) = \xi(x) = \xi(Px) = (P^*\xi)(x) = x[\psi(\xi)] = x(t),$$

whence $\xi = \varphi(t)$ and, consequently, $t = \psi[\varphi(t)]$.

Corollary. *If X_0 is an MI-subspace of $X = C(\Omega)$ (i. e. if $\Omega_s = \Omega$) and if P is a lattice-homomorphism projection of X onto X_0, with $\|P\| = 1$, then $\mathfrak{Q}_0$ may be embedded topologically into Ω in such a way that this homeomorphic image Ω_P in Ω is a retract of Ω.*

Example 3. Let $\Omega = \langle 0, 1\rangle \times \langle 0, 1\rangle$. Then $X = C(\Omega)$ consists of all real-valued continuous functions $x(t, s)$, defined for $0 \leqslant t \leqslant 1$, $0 \leqslant s \leqslant 1$. Let X_0 consist of all functions constant with respect to the second variable, i. e.

$$X_0 = \bigcap_{t, s_1, s_2} \{x \in X: x(t, s_1) = x(t, s_2)\}.$$

The transformation P defined by the formulas

$$y(t, s) = \int_0^1 x(t, \sigma)\, d\sigma, \qquad Px = y(\cdot, \cdot)$$

is a non-negative projection of norm 1, and $P(X) = X_0$. However, the condition $P(x \vee y) = P(x) \vee_0 P(y)$ is not satisfied for all $x \in X$, $y \in X$, and the functionals ξ_t, defined for $x \in X_0$ by the formula $\xi_t(x) = x(t, 0)$, belong to $\mathfrak{Q}_0$, although the functionals

$$\eta_t(x) = (P^* \xi_t)(x) = \int_0^1 x(t, \sigma)\, d\sigma \quad \text{for} \quad x \in X$$

do not belong to $\mathfrak{Q}$. Thus, we have obtained

Proposition 3. *A non-negative projection P transforming $X = C(\Omega)$ onto an MI-subspace X_0, with $\|P\| = 1$, is not necessarily a lattice homomorphism.*

Example 4. Let $\Omega = \langle 0, 1\rangle$, $X = C(\Omega)$ and let X_0 consist of all functions $x = a_1 z_1 + a_2 z_2$ where

$$z_1(t) = \begin{cases} 0 & \text{on } \langle 0, \frac{1}{5}\rangle, \\ \text{linear} & \text{on } \langle \frac{1}{5}, \frac{2}{5}\rangle, \\ 1 & \text{on } \langle \frac{2}{5}, 1\rangle, \end{cases} \qquad z_2(t) = \begin{cases} 1 & \text{on } \langle 0, \frac{3}{5}\rangle, \\ \text{linear} & \text{on } \langle \frac{3}{5}, \frac{4}{5}\rangle, \\ 0 & \text{on } \langle \frac{4}{5}, 1\rangle, \end{cases}$$

a_1, a_2 being constant.

X_0 is a two-dimensional B_+-subspace of X, and the operation P, defined by

$$Px = y, \text{ where } y(t) = x(1)\cdot z_1(t) + x(0)\cdot z_2(t) \quad \text{for} \quad t \in \langle 0, 1\rangle,$$

establishes a non-negative projection of X onto X_0, with $\|P\| = 2$. Since X_0 is not isometrically and isotonically isomorphic to the space C_2 of pairs $s = (s', s'')$ with $\|s\| = |s'| \vee |s''|$, there exists no non-negative projection of norm 1 transforming X onto X_0.

Example 5. Let $\Omega = \langle 0, 1\rangle \cup \langle 2, 3\rangle$, $X = C(\Omega)$ and let X_0 consist of all functions $x \in X$ such that

$$x(t+2) = 3tx(t) \quad \text{for} \quad 0 \leqslant t \leqslant 1.$$

A non-negative projection P of X onto X_0 may be established as follows: $Px = y$, where $y(t) = x(t)$ for $0 \leqslant t \leqslant 1$ and $y(t) = 3(t-2) \times x(t-2)$ for $2 \leqslant t \leqslant 3$. Evidently, $\|P\| = 3$.

X_0 is a B_+-subspace of X and is an MI-space with respect to the relative order and with respect to the norm $\|x\|^* = \inf\{\lambda: |x| \leqslant \lambda e_0\}$, where $e_0(t) = 1$ for $0 \leqslant t \leqslant 1$ and $e_0(t) = 3(t-2)$ for $2 \leqslant t \leqslant 3$.

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Reçu par la Rédaction le 17. 12. 1959

STUDIA MATHEMATICA, T. XIX. (1960)

Espaces d'Orlicz de champs de vecteurs (IV)

(Opérations linéaires)

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Introduction. Soient Z un espace localement compact et ν une mesure de Radon ([1]) positive sur Z; $\mathcal{E} = (E(z))_{z \in Z}$ une famille d'espaces de Banach et $\mathcal{E}' = (E'(z))_{z \in Z}$ la famille des duals des espaces $E(z)$.

Désignons par $\mathcal{C}(\mathcal{E})$ (respectivement $\mathcal{C}(\mathcal{E}')$) l'ensemble des champs de vecteurs ([2]) $\boldsymbol{x}$ (resp. de fonctionnelles $\boldsymbol{x}'$) définis sur Z tels que $\boldsymbol{x}(z) \in E(z)$ (resp. $\boldsymbol{x}'(z) \in E'(z)$) quel que soit $z \in Z$.

Supposons qu'il existe une famille fondamentale $\mathcal{A} \subset \mathcal{C}(\mathcal{E})$ de champs de vecteurs continus et une famille fondamentale $\mathcal{A}' \subset \mathcal{C}(\mathcal{E}')$ de champs de fonctionnelles continus vérifiant la condition suivante:

La fonction scalaire $z \to \langle \boldsymbol{x}(z), \boldsymbol{x}'(z) \rangle$ est continue quels que soient $\boldsymbol{x} \in \mathcal{A}$ et $\boldsymbol{x}' \in \mathcal{A}'$.

Soit X un espace de Banach et pour tout $z \in Z$ désignons par $G(z)$ l'espace $\mathcal{L}(E(z), X)$ (des applications linéaires et continues de $E(z)$ dans X), par $\mathcal{G}$ la famille $(G(z))_{z \in Z}$ et par $\mathcal{C}(\mathcal{G})$ l'ensemble des champs d'opérations U définis sur Z tels que $U(z) \in G(z)$ quel que soit $z \in Z$.

Soient φ une fonction positive définie sur $[0, +\infty]$, croissante, continue à gauche, telle que $\varphi(0) = 0$ et $0 < \varphi(t) < \infty$ pour $0 < t < +\infty$, ψ la fonction inverse de φ, Φ et Ψ les fonctions définies sur $[0, +\infty]$ par les égalités

$$\Phi(u) = \int_0^u \varphi(t)\,dt, \qquad \Psi(v) = \int_0^v \psi(s)\,ds.$$

Considérons l'espace d'Orlicz ([3]) $L^{\Phi}_{\mathcal{A}}(\nu)$.

Dans [4] nous avons montré que si: 1) $\lim\limits_{t\to\infty} \varphi(t) = 1$, 2) *il existe $M > 0$*

([1]) Pour ce qui concerne l'intégration voir [1].

([2]) Pour ce qui concerne les champs de vecteurs voir [8].

([3]) Pour la définition et les propriétés des espaces $L^{\Phi}_{\mathcal{A}}$ voir un des ouvrages [2], [3], [4], [5].