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Spaces of continuous functions (IV)

(On isomorphical classification of spaces of continuous functions)

by

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In this paper (Theorem 1) we give a complete isomorphical and dimensional [1] classifications of the spaces of (all) continuous functions defined on countable intervals of ordinal numbers.

Applying Theorem 1 we obtain: a) the complete isomorphical classification of the spaces $C(Q)$ (i. e. of the spaces of all continuous real functions defined on Q), Q being zero-dimensional metrisable compact spaces (Theorem 3) [2] and b) the complete dimensional classification of all the spaces $C(Q)$ for arbitrary metrisable compact spaces Q (Collorary 1).

In the last part of this paper we formulate several problems concerning the spaces of continuous functions.

1. Preliminaries. Two Banach spaces X and Y are called *isomorphic* (written $X \sim Y$), if and only if there exists a linear homeomorphic mapping of X onto Y.

It is known that $X \sim Y$ if and only if there are a linear mapping U of X onto Y and a constant K such that

$$\|x\| \leqslant \|U(x)\| \leqslant K\|x\|. \tag{1}$$

If condition (1) is satisfied for some U we shall write $X \overset{K}{\sim} Y$. In particular, $X \overset{1}{\sim} Y$ means that X and Y are isometric.

The spaces X and Y are said to *have an equal linear dimension* (written $X \underset{\dim}{=} Y$) if each of the spaces X and Y is isomorphic to some subspace of the other. We say that X *has a smaller linear dimension than* Y (written $X \underset{\dim}{<} Y$) if there is a subspace of Y isomorphic to X and no subspace of X is isomorphic to Y.

(1) i. e. classification with respect to linear dimension.

(2) Hence, in particular we obtain a solution of the problem 48 in the Scottish Book posed by Banach and Mazur.

In the sequel we shall denote by $t, s, \alpha, \beta, \gamma, \ldots$ — arbitrary ordinal numbers [3], by $m, n, N, \ldots$ — finite ordinal numbers; the symbols ω and ω_1 will denote the first infinite ordinal number and the first uncountable ordinal number, respectively. If $\alpha < \beta$, then $\langle\alpha, \beta\rangle = \{t\colon \alpha \leqslant t \leqslant \beta\}$, and $(\alpha, \beta\rangle = \{t\colon \alpha < t \leqslant \beta\}$, $\langle\alpha, \beta) = \{t\colon \alpha \leqslant t < \beta\}$.

Sets of ordinal numbers will be always assumed to be topological spaces with the order topology.

The symbols $Q, Q_1, \ldots$ will denote metrisable compact topological spaces.

The symbols X, Y will be used for denoting arbitrary Banach spaces.

C will denote the one-dimensional Banach space (we do not use the standard notation, in which C denotes the space of continuous functions on the unit interval).

$C(Q)$ will denote the Banach space of all continuous real functions $x = x(q)$ defined on Q [4] with the norm $\|x\| = \sup_q |x(q)|$; in particular, by $C(\mathcal{I})$ and $C(\mathcal{C})$ we shall denote the spaces of all continuous real functions defined on the unit interval $\mathcal{I}$ of real numbers and on the Cantor discontinuum $\mathcal{C}$ respectively.

X^α will denote the space of all continuous functions $x = x(t)$ defined on $\langle 1, \alpha\rangle$ having values in the Banach space X with the norm $\|x\| = \sup_t \|x(t)\|$; we set

$$X^\alpha = \{x \in X^\alpha\colon x(\alpha) = 0\}.$$

(Observe that the symbols $C(\langle 1, \alpha\rangle)$ and C^α denote the same.)

The symbol $X \times Y$ will denote the *Cartesian product* of the spaces X and Y, i. e the space of all pairs (x, y), $x \in X$, $y \in Y$ — with the norm $\|(x, y)\| = \max(\|x\|, \|y\|)$.

The following properties are obvious:

I. *If $X \overset{k}{\sim} Y$, and $l \geqslant k$, then $Y \overset{l}{\sim} X$; if $X \overset{k}{\sim} Y$, $Y \overset{l}{\sim} Z$, then $X \overset{k\cdot l}{\sim} Z$.*

II. $X^\alpha \times X^\beta \overset{1}{\sim} X^\beta \times X^\alpha \overset{1}{\sim} X^{\alpha+\beta} \overset{1}{\sim} X^{\beta+\alpha}$.

III. $X^1 \overset{1}{\sim} X$; $(X^\alpha)^\beta \overset{1}{\sim} X^{\alpha\cdot\beta}$.

IV. *If $X \sim Y$, $X_1 \sim Y_1$, then $X^\alpha \sim Y^\alpha$ and $X \times X_1 \sim Y \times Y_1$ (moreover if $X \overset{k}{\sim} Y$, $X_1 \overset{k}{\sim} Y_1$, then $X^\alpha \overset{k}{\sim} Y^\alpha$ and $X \times X_1 \overset{k}{\sim} Y \times Y_1$).*

2. LEMMA 1. *If $\omega \leqslant \alpha < \omega_1$ and $\alpha \leqslant \beta < \alpha^\omega$, then for arbitrary X we have $X^\alpha \sim X^\beta$.*

[3] For terminology, notation and basic arithmetical properties concerning ordinal numbers see [11], Chap. XIV.

[4] With the usual definitions of addition and multiplication by scalars.

The proof will be given in several stages:

2.1. *If $\alpha \geqslant \omega$, then $X^\alpha \overset{2}{\sim} X_0^\alpha$.*

Indeed, the required isomomorphism is realized by the mapping $U(x) = x'$ $(x \in X^\alpha,\ x' \in X_0^\alpha)$, where $x'(1) = x(\alpha)$, $x'(1+t) = x(t) - x(\alpha)$.

2.2. *Let $\omega \leqslant \alpha < \omega_1$. Then for arbitrary n and X we have:* 1° $X^{\alpha n} \sim X^\alpha$ (2° $X^{\alpha\omega} \sim X^\alpha$).

For any ordinal number α let the symbol α' denote the greatest *prime component* $\leqslant \alpha$ [5]. Proposition 2.2 is an immediate consequence of the properties $(\alpha n)' = \alpha'$, $(\alpha\omega)' = \alpha'\cdot\omega$ (for $\alpha \geqslant \omega$) and of the following two propositions:

2.21. *If α with $\omega \leqslant \alpha < \omega_1$ is a prime component, then for arbitrary n and X we have* 1° $X^{\alpha n} \sim X^\alpha$ (2° $X^{\alpha\omega} \sim X^\alpha$).

2.22. *If $\omega \leqslant \alpha \leqslant \omega_1$, then $X^\alpha \sim X^{\alpha'}$.*

Proof of 2.21. The set $\langle 1, \alpha)$ can be decomposed into n disjoint parts $\Delta_1, \ldots, \Delta_n$ ($\aleph_0$ parts $\Delta_1, \Delta_2, \ldots$) in such a way that each of the sets Δ_i is ordered according to the type α and that each two of these sets have unique common limit-point α (and, moreover, in case 2° if $t_n \in \Delta_n$ $(n < \omega)$, then $t_n \to \alpha$) [6]. Thus we obtain $X_0^\alpha \sim (X_0^\alpha)^n$ $(X_0^\alpha \sim (X_0^\alpha)_0^\omega)$. This, according to 2.1 and IV, gives our assertion.

Proof of 2.22. The number α is of the form $\alpha = \alpha' n + \gamma$, where $\gamma < \alpha'$; therefore by 2.21 and by II, $X^\alpha \sim X^{\gamma+\alpha' n} \sim X^{\alpha' n} \sim X^{\alpha'}$ (because $\gamma + \alpha' n = \alpha' n$).

2.3. *If $\omega \leqslant \alpha < \omega_1$ and $\beta < \alpha$, then $X^{\alpha+\beta} \sim X^{\beta+\alpha} \sim X^\alpha$.*

This follows from 2.22 and from the property: $(\alpha+\beta)' = (\beta+\alpha)' = \alpha'$.

2.4. *If $\omega \leqslant \alpha < \omega_1$, $0 < \beta \leqslant \alpha$, then for arbitrary X we have $X^{\alpha\beta} \sim X^\alpha$.*

Proof. It can easily be verified that the space $X_0^{\alpha\beta}$ is a direct sum

$$X_0^{\alpha\beta} = Y \oplus Z, \tag{2}$$

where Y consists of those functions $y(t)$ $(1 \leqslant t \leqslant \alpha\beta)$ belonging to $X_0^{\alpha\beta}$ which are constant on each of the intervals

$$T_\xi = (\alpha\xi, \alpha(\xi+1)\rangle \quad \text{for} \quad 0 \leqslant \xi < \beta;$$

[5] α is called a *prime component* if the condition $\alpha = \gamma + \delta$ implies $\delta = \alpha$.

[6] Let us prove this fact, for instance, for $n = 2$. The condition that α is a prime component makes it possible to choose a sequence of ordinal numbers (γ_n) in such a way that $\gamma_n \to \alpha$ and the ordinal types of the segments $(\gamma_n, \gamma_{n+1}\rangle$ are also convergent to α. Now the required decomposition is

$$\Delta_1 = \bigcup_{k=1}^{\infty} (\gamma_{2k-1}, \gamma_{2k}\rangle, \qquad \Delta_2 = \bigcup_{k=1}^{\infty} (\alpha_{2k}, \alpha_{2k+1}\rangle \qquad (\gamma_0 = 0).$$

Z is composed of the functions $z(t)$ $(1 \leqslant t \leqslant \alpha\beta)$ vanishing at all the points $\alpha\xi$ $(1 \leqslant \xi \leqslant \beta)$.

It is easily seen that $Y \overset{1}{\sim} X_0^\beta$, whence, by 2.1,

(3) $$Y \sim X^\beta.$$

One can easily establish that if $z \in Z$ then for every $\varepsilon > 0$ the set $\{\xi: \sup_{t \in T_\xi}\|z(t)\| \geqslant \varepsilon\}$ is finite. It follows that $Z \overset{1}{\sim} (X_0^\alpha)_0^\omega$. Hence, by 2.1, 2.2, III and IV, we obtain

(4) $$Z \sim X^\alpha.$$

Thus, using in turn: 2.1, (2), (3), (4), IV, II, and 2.3, we obtain [7] $X^{\alpha\beta} \sim X_0^{\alpha\beta} \sim Y \times Z \sim X^\beta \times X^\alpha \sim X^{\beta+\alpha} \sim X^\alpha$, q. e. d.

2.5. *If $\omega \leqslant \alpha < \omega_1$, then for every n and X we have $X^\alpha \sim X^{\alpha n}$.*

For $n = 2$ this proposition follows from 2.4; for arbitrary n one can give a simply inductive proof by the use of property IV.

2.6. Now the proof of Lemma 1 can be completed. If $\omega \leqslant \alpha < \omega_1$ and $\alpha \leqslant \beta < \alpha^\omega$, then β is of the form $\beta = \alpha^n\gamma + \delta$, where $\gamma < \alpha \leqslant \alpha^n$, $\delta < \alpha^n$. By 2.3 we have $X^{\alpha^n\gamma+\delta} \sim X^{\alpha^n\gamma}$, and, by 2.5 and 2.4, $X^{\alpha^n\gamma} \sim X^\alpha$. Hence $X^\beta \sim X^\alpha$, q. e. d.

3. Lemma 2. *Let α be an arbitrary ordinal number. If for every $\gamma < \alpha$ the relation $C^\gamma \underset{\dim}{<} C^\alpha$ holds, then $C^\alpha \underset{\dim}{<} C^{\alpha^\omega}$.*

Proof. According to 2.1 it is enough to prove that the assumptions of Lemma 2 imply that $C_0^\alpha \underset{\dim}{<} C^{\alpha^\omega}$. Suppose that the last condition is false, i. e. that there exists a subspace X of the space C_0^α and a constant $K > 0$ such that $C^{\alpha^\omega} \overset{K}{\sim} X$. Let N be arbitrary fixed positive integer. Since the space C^{α^N} is isometric with a subspace of C^{α^ω}, there exists a subspace X_N of the space X such that $C^{\alpha^N} \overset{K}{\sim} X_N$, i. e. there exists a linear mapping U of the space X_N onto C^{α^N} such that

(6) $$\|x\| \leqslant \|U(x)\| \leqslant K\|x\| \quad \text{for each } x \in X_N.$$

We shall show that this is impossible for $N > 4K$, whence it follows that our supposition that $C_0^\alpha \geqslant C^{\alpha^\omega}$ leads to a contradiction.

Let $y_0 \in C^{\alpha^N}$ be the function identically equal 1; $x_0 = U^{-1}(y_0)$. Let γ_1 with $\gamma_1 < \alpha$ be chosen in such a way that $x_0(t) < 1/(N+1)$ for $t > \gamma_1$ (such a number γ_1 exists because $\lim_{t \to \alpha} x(t) = 0$). Write

$$\Delta_\xi^1 = \langle \alpha^{N-1}\xi, \alpha^{N-1}(\xi+1)\rangle \quad \text{for } 0 \leqslant \xi < \alpha.$$

[7] Here we make use of the obvious fact that if a Banach space X is a direct sum of its two subspaces Y and Z, then that space is isomorphic to the Cartesian product $Y \times Z$.

Let

$$Y_1 = \bigcap_{\xi < \alpha}\{y \in C^{\alpha^N}: y(t) \text{ is constant on } \Delta_\xi^1\}.$$

Obviously Y_1 is a subspace of C^{α^N} and

(7) $$Y_1 \overset{1}{\sim} C^\alpha.$$

To begin with, we shall prove that there exist elements x_1 in X_N and y_1 in Y_1 such that

$$x_1 = U^{-1}(y_1), \quad \|x_1\| \leqslant \|y_1\| = 1,$$

and

$$|x_1(t)| < \frac{1}{N+1} \quad \text{for} \quad t \leqslant \gamma_1.$$

For every $x \in C^\alpha$ let $P_{\gamma_1}(x)$ denote the “restriction” of the function x to the set $\langle 1, \gamma_1\rangle$, more exactly $P_{\gamma_1}(x) = z$, where $z \in C^{\gamma_1}$ and $z(t) = x(t)$ for $t \leqslant \gamma_1$. We consider the operation $P_{\gamma_1}U^{-1}$ of the space Y_1 into the space C^{γ_1}. By (7) and according to the fact that $C^{\gamma_1} \underset{\dim}{<} C^\alpha$, this operation cannot be any isomorphic mapping of Y_1 into C^{γ_1}. Hence, for every $\varepsilon > 0$, there exists an element y in Y_1 such that $\|P_{\gamma_1}U^{-1}(y)\| < \varepsilon\|y\|$. In particular we may choose an element $y_1 \in Y_1$ in such a way that $\|y_1\| = 1$ and $\|P_{\gamma_1}U^{-1}(y_1)\| < 1/(N+1)$. Putting $x_1 = U^{-1}(y_1)$ we obtain

$$\sup_{t < \gamma_1}|x_1(t)| = \|P_{\gamma_1}(x_1)\| = \|P_{\gamma_1}U^{-1}(y_1)\| < \frac{1}{N+1}.$$

By (6) we have

$$\|x_1\| \leqslant \|U(x_1)\| = \|UU^{-1}(y_1)\| = \|y_1\| = 1.$$

Now let ξ_1 be such an ordinal number that $|y_1(t)| \geqslant 1/2$ for $t \in \Delta_{\xi_1}^1$ (such a number must exist because $\|y_1\| = 1$). Consider the new family of intervals

$$\Delta_\xi^2 = \langle \alpha^{N-1}\xi_1 + \alpha^{N-2}\xi, \alpha^{N-1}\xi_1 + \alpha^{N-2}(\xi+1)\rangle \quad \text{for} \quad 0 \leqslant \xi < \alpha.$$

Let

$$Y_2 = \bigcap_{\xi < \alpha}\{y \in C^{\alpha^N}: y(t) \text{ is constant on } \Delta_\xi^2 \text{ and } y(t) = 0 \text{ for } t \notin \Delta_{\xi_1}^1\}.$$

It is easy to establish that $Y_2 \overset{1}{\sim} C^\alpha$.

Let γ_2 with $\gamma_1 < \gamma_2 < \alpha$ be chosen in such a way that

$$|x_1(t)| < \frac{1}{N+1} \quad \text{for} \quad t > \gamma_2.$$

Since no subspace of C^{γ_1} is isomorphic to Y_2, we infer in the same way as above that there exist $y_2 \in Y_2$ and $x_2 \in X_N$ such that

$$x_2 = U^{-1}(y_2), \qquad \|x_2\| \leqslant \|y_2\| = 1$$

and

$$|x_2(t)| < \frac{1}{N+1} \qquad \text{for } t \leqslant \gamma_2.$$

Now we choose ξ_2 such that $|y_2(t)| \geqslant 1/2$ for $t \in \Delta^2_{\xi_2}$ etc.

Repeating this procedure N times we shall find the elements

$$x_0, x_1, \ldots, x_N; \qquad y_0 = U(x_0), \ldots, y_N = U(x_N),$$

the ordinal numbers

$$1 = \gamma_0 < \gamma_1 < \gamma_2 < \ldots < \gamma_N < \alpha,$$

and the sets of ordinal numbers (intervals)

$$\Delta_0 = \langle 1, \alpha^N \rangle \supset \Delta_1 = \Delta_1^{\xi_1} \supset \Delta_2 = \Delta^2_{\xi_2} \supset \ldots \supset \Delta_N$$

such that

(8) $$\|x_k\| \leqslant 1, \qquad |x_k(t)| < \frac{1}{N+1} \qquad \text{for} \quad t \notin \langle \gamma_k, \gamma_{k+1} \rangle,$$

(9) $y_k(t) = a_k = \text{const}$ for $t \in \Delta_k$, where $|a_k| \geqslant 1/2$, $k = 0, 1, \ldots, N$.

Let us put $\varepsilon_k = \operatorname{sgn} a_k$ for $k = 0, 1, \ldots, N$ and $z = \sum_{k=0}^{N} \varepsilon_k x_k$. Since $\Delta_N = \bigcap_{k=1}^{N} \Delta_k$ there is a point t_0 belonging to all Δ_k $(k = 0, 1, \ldots, N)$. We have

$$\|U(z)\| = \Big\|\sum_{k=0}^{N} \varepsilon_k U(x_k)\Big\| \geqslant \sum_{k=0}^{N} \varepsilon_k y_k(t_0) = \sum_{k=0}^{N} |a_k| \geqslant \frac{N+1}{2}.$$

On the other hand, by (8) and by the fact that the intervals $\langle \gamma_k, \gamma_{k+1} \rangle$ are disjoint for $k = 0, 1, \ldots, N-1, N$, we have

$$|x_i(t)| < \frac{1}{N+1}$$

for every $t \leqslant \alpha$ and for all indices i $(i = 0, 1, \ldots, N)$ except at most one. Thus, according to the fact $\|x_k\| \leqslant 1$, we obtain

$$\|z\| \leqslant 1 + N \cdot \frac{1}{N+1} < 2.$$

Hence $\|U\| \geqslant \|U(\|z\|^{-1} z)\| \geqslant N/4$ and for $N > 4K$ we obtain a contradiction with (6), q. e. d.

4. THEOREM 1. *Let $\omega \leqslant \alpha \leqslant \beta < \omega_1$. Then $C^\alpha \sim C^\beta$ if and only if $C^\alpha \underset{\dim}{=} C^\beta$ if and only if $\beta < \alpha^\omega$.*

This theorem is an immediate consequence of Lemmas 1 and 2 ([8]).

Remark 1. For two isomorphic Banach spaces X and Y let us define the function

$$[X, Y] = \inf\{K : X \overset{K}{\sim} Y\}$$

(cf. Banach [1], Remarques, p. 242). Investigating the proofs of Lemmas 1 and 2 we see that if $\omega \leqslant \alpha \leqslant \alpha^N \leqslant \beta < \alpha^{N+1} < \omega_1$, then

$$N \leqslant [C^\beta, C^\alpha] \leqslant 4^{N+3}.$$

It would be interesting to obtain an estimation of the form

$$G(N) \leqslant [C^\alpha, C^\beta] \leqslant H(N),$$

where $\sup(H(N)/G(N)) < +\infty$, or to compute the exact values of $[C^\alpha, C^\beta]$.

Let Q be countable. Denote by $\varkappa(Q)$ the smallest ordinal number γ such that the γ-th derivative $Q^{(\gamma)}$ is empty, and set

$$\chi(Q) = [\varkappa(Q)]^\omega.$$

It is not difficult to verify that

$$\max(\alpha, \beta) \leqslant [\min(\alpha, \beta)]^\omega \textit{ if and only if } \chi(\langle 1, \alpha \rangle) = \chi(\langle 1, \beta \rangle).$$

Using this fact we may give a new formulation of Theorem 1:

(*) *Let α and β be countable infinite ordinal numbers. $C^\alpha \sim C^\beta$ if and only if $C^\alpha \underset{\dim}{=} C^\beta$ if and only if $\chi(\langle 1, \alpha \rangle) = \chi(\langle 1, \beta \rangle)$. $C^\alpha \underset{\dim}{<} C^\beta$ if and only if $\chi(\langle 1, \alpha \rangle) < \chi(\langle 1, \beta \rangle)$.*

According to a well-known theorem of Mazurkiewicz and Sierpiński [6] and to the fact that $\chi(Q)$ is a topological invariant of the space Q (because $\varkappa(Q)$ is invariant), the proposition (*) gives

THEOREM 2. *Let Q and Q_1 be countable compact metric spaces. Then $C(Q) \sim C(Q_1)$ if and only if $C(Q) \underset{\dim}{=} C(Q_1)$ if and only if $\chi(Q) = \chi(Q_1)$. $C(Q) \underset{\dim}{<} C(Q_1)$ if and only if $\chi(Q) < \chi(Q_1)$.*

Also the following is true:

([8]) To prove the necessity of this condition we apply Lemma 2 for the ordinal number α_1 = the smalest γ for which $C^\gamma \underset{\dim}{=} C^\alpha$.

THEOREM 3. *Let Q and Q_1 be zero-dimensional metrisable compact spaces. Then $C(Q) \sim C(Q_1)$ if and only if $C(Q) \underset{\dim}{=} C(Q_1)$ if and only if one of the following conditions:*

(i) *Q and Q_1 are finite and have the same number of elements,*

(ii) *Q and Q_1 are countable and $\chi(Q) = \chi(Q_1)$,*

(iii) *Q and Q_1 are uncountable*

holds.

Proof. In the case where Q and Q_1 are finite, this theorem is obvious; for countable Q and Q_1 it follows from Theorem 2. Now let us suppose that Q is zero-dimensional and uncountable. Then, according to the Cantor-Bendixon Theorem ([5], Chap. II, p. 141) and according to the fact that every zero-dimensional perfect compact metric space is homeomorphic to the Cantor discontinuum, Q is the sum of a set $\mathcal{C}$ homeomorphic to the Cantor discontinuum and a countable set. To complete the proof it is enough to apply the following

LEMMA 3. *Let Q be an uncountable compact metric space. If Q_1 and A are closed subsets of Q such that $Q = Q_1 \cup A$ and A is countable, then $C(Q) \sim C(Q_1)$.*

Proof. Let us denote by $C(Q/A)$ the subspace of $C(Q)$ consisting of all functions which vanish on the set A. According to Borsuk's theorem on simultanous extensions [2], one can easily establish that

$$C(Q) \sim C(Q_1) \times C(A/Q_1 \cap A).$$

It may easily be shown that there exists a countable compact B such that $C(B) \sim C(A/Q_1 \cap A)$. Let B' be a subset of Q_1 homeomorphic to B (such subsets exist because Q_1 is an uncountable metric compact space and B is a countable one). According to Borsuk's theorem quoted above and to the fact that $C(B') \times C(B') \sim C(B')$ (this fact follows from proposition 2.2 and the Mazurkiewicz-Sierpiński theorem already cited) we have

$$C(Q_1) \times C(A/Q_1 \cap A) \sim C(Q_1) \times C(B) \sim C(Q_1/B') \times C(B') \times C(B)$$
$$\sim C(Q_1/B') \times \big(C(B') \times C(B')\big) \sim C(Q_1/B') \times C(B') \overset{\cdot}{\sim} C(Q_1).$$

Hence $C(Q) \sim C(Q_1)$, q. e. d.

Banach and Mazur have proved (see [1], p. 186) that for every separable Banach space X there is a subspace X' of the space $C(\mathcal{C})$ such that $X \overset{1}{\sim} X'$.

On the other hand, since every uncountable compact metric space Q contains a subset $\mathcal{C}'$ homeomorphic to $\mathcal{C}$, according to Borsuk's theorem we have $C(Q) \sim C(Q/\mathcal{C}') \times C(\mathcal{C}')$. Thus $C(Q)$ contains a subspace isomorphic to $C(\mathcal{C})$.

From these two facts it follows that $C(Q) \underset{\dim}{=} C(\mathcal{C})$ for arbitrary uncountable Q; and, according to Theorem 2, we obtain

COROLLARY 1. *Let Q and Q_1 be metrisable compact spaces. Then the spaces $C(Q)$ and $C(Q_1)$ have an equal linear dimension if and only if one of the conditions* (i), (ii), (iii) (*formulated in Theorem* 3) *is satisfied.*

Since all the intervals $\langle 1, \alpha \rangle$ are dispersed topological compact spaces, it follows from a result of [8] (see also [7]) and Lemma 2 that all the spaces conjugate to C^{α} with $\bar{\alpha} = \aleph_\tau$ are isometric. Hence

COROLLARY 2. *There are $\aleph_{\tau+1}$ isomorphicallly different (and having different linear dimensions) spaces C^{α} with $\bar{\alpha} = \aleph_\tau$, whose first conjugate spaces are all isometric.*

This implies, in particular,

COROLLARY 3. *There are at least $\aleph_1$ separable Banach spaces, different with regard to linear dimension, whose first conjugate spaces are isometric to the space l (composed of all absolutely convergent real series).*

5. Remarks and unsolved problems

5.1. Give an isomorphic classification of the spaces $C(Q)$ for arbitrary metrisable compact topological spaces Q. In particular establish whether the spaces $C(\mathcal{D})$ and $C(\mathcal{C})$ are isomorphic.

5.2. Let α and β be arbitrary ordinal numbers. Give a necessary and sufficient condition (concerning α and β) for the spaces C^{α} and C^{β} to be isomorphic.

Z. Semadeni [10] has proved that $C^{\omega_1} \underset{\dim}{<} C^{\omega_1 \cdot 2}$; hence Lemma 1 cannot be generalized to the case of uncountable α.

5.3. We say that the space X has *a smaller linear dimension in the sense of Kolmogoroff* [4] *than the space* Y (briefly $X \underset{\delta}{<} Y$) if X is a linear image of a subspace of Y and no subspace of X can be linearly mapped onto Y.

Does Lemma 2 hold true if we replace the symbol "$\underset{\dim}{<}$" by "$\underset{\delta}{<}$"?

We know that such a modification of Lemma 2 is true for $\alpha < \omega^{\omega^3}$ (9).

5.4. We introduce the following classification of the separable Banach spaces:

(9) The proof of this fact is based on a certain property of weakly convergent series in the space C^{ω^ω}. Series having this property may be constructed in the same manner as that by Schreier in [9] in the case of $C([0,1])$.

Let $\mathfrak{U}_0$ be the class of all separable Banach spaces having an unconditional basis (see [3], Chapt. IV, § 4). Suppose that we have defined the classes $\mathfrak{U}_\beta$ for all $\beta < \alpha$ ($\alpha < \omega_1$). We define $\mathfrak{U}_\alpha$ as the class of all separable Banach spaces X which have the following properties:

(a) $X \notin \mathfrak{U}_\beta$ for each $\beta < \alpha$;

(b) there exist sequences (X_n) of subspaces of X and (β_n) of ordinal numbers $< \alpha$ such that $X_n \in \mathfrak{U}_{\beta_n}$ $(n = 1, 2, \ldots)$ and every element $x \in X$ may be uniquely represented as a sum of an unconditionally convergent series $x = \sum_{n=1}^{\infty} x_n$, where $x_n \in X_n$ for $n = 1, 2, \ldots$

We say that the separable Banach space *belongs to the class* $\mathfrak{U}_{\omega_1}$ if $X \in \mathfrak{U}_\alpha$ for no $\alpha < \omega_1$.

Questions:

1. Are all classes $\mathfrak{U}_\alpha$ (for $\alpha \leqslant \omega_1$) non-empty?

2. Does there exist for every $0 \leqslant \alpha \leqslant \omega_1$ a compact metric space Q such that $C(Q) \in \mathfrak{U}_\alpha$?

We know only that $C^\omega \in \mathfrak{U}_0$, $C^{\omega^\omega} \in \mathfrak{U}_1$, $C(Q) \in \mathfrak{U}_{\omega_1}$ for uncountable Q.

5.5. Let X be a Banach space with the conjugate space X^* isomorphic to l. Does there exist an ordinal α such that $X \sim C^\alpha$?

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States of operator algebras

by

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§ 0. Introduction & summary. Let $\mathcal{K}$ be a fixed Hilbert space and denote by $B = B(\mathcal{X})$ the algebra of all bounded endomorphisms of $\mathcal{X}$. B is a complete normed algebra with an involution which carries $T \in B$ into its adjoint T^*; this algebra is non-commutative unless $\mathcal{K}$ is one-dimensional. If $\mathcal{A}$ is a self-adjoint (i. e. stable under *) subalgebra of B, we follow Segal [5] in extending the customary language of statistical quantum mechanics by applying the name "state of $\mathcal{A}$" to any positive-definite linear form f on $\mathcal{A}$, i. e. a linear form f on $\mathcal{A}$ such that $f(T^*) = \overline{f(T)}$ and $f(T^*T) \geqslant 0$ for arbitrary $T \in \mathcal{A}$. These correspond to the "mixed states" of a quantum mechanical assemblage and are therefore thought of as being compounded in some way from the "pure states"

$$f_x \colon T \to (Tx, x), \tag{0.1}$$

where x is an arbitrary element of $\mathcal{X}$. The main aim of this paper is to discover more precisely how some at least of these mixed states are obtained from the pure ones.

When $\mathcal{A} = B$, von Neumann gives one answer to this problem, at least for those states which are weakly continuous. On the other hand, Segal [5] discusses a fairly general type of algebra $\mathcal{A}$ and shows that there exist always sufficiently many pure or minimal states to make plausible the possibility of expressing a wide class of states in terms of these. However, Segal does not concern himself with any explicit representation of this kind. von Neumann's approach ([6], Chapter IV) for $\mathcal{A} = B$ is very direct and leads to a representation in terms of the trace. Unfortunately his approach is not adaptable in any obvious way to states initially defined only on some subalgebra $\mathcal{A}$ of B. This is one reason for seeking an apparently different representation.

The proposed alternative is a representation in terms of positive integral combinations of pure states:

$$f(T) = \int_{\Sigma} (Tx, x)\, dm(x), \tag{A}$$