Hinweis *(30. 12. 1959)*. Von O. Szász [3] wurde die Poissonsche Verteilung für die Approximation nutzbar gemacht; die approximierenden Ausdrücke sind dabei Potenzreihen multipliziert mit einer Exponentialfunktion. Als neuere Arbeit, in der ein Approximationssatz auf wahrscheinlichkeitstheoretische Weise bewiesen wird, sei die Arbeit [4] von Aratò und Rényi genannt.

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Generalized bases in topological linear spaces

by

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1. Introduction. Let $\mathcal{U}$ be a topological linear space over a scalar field a. By a *basis* in $\mathcal{U}$ we mean a sequence $\{x_n\}$ of points of $\mathcal{U}$ such that to every x in $\mathcal{U}$ there corresponds a unique sequence $\{a_n\}$ of scalars for which

$$x = \sum_{n=1}^{\infty} a_n x_n, \tag{1.1}$$

convergence of the series being that of the topology on $\mathcal{U}$. Each coefficient here determines a corresponding linear functional φ_n on $\mathcal{U}$, and in this notation (1.1) can be written as

$$x = \sum_{n=1}^{\infty} \varphi_n(x) x_n.$$

The above definition of a basis was introduced by Schauder [15] with $\mathcal{U}$ taken as a Banach space, and in this special case the coefficient functionals are automatically continuous. In general, whenever all the coefficient functionals φ_n are continuous, we shall refer to $\{x_n\}$ as a *Schauder basis*.

Although the utility of these basis concepts has been amply demonstrated in various branches of mathematics, certain generalizations appear to be at least as important. Examples in this direction have, in fact, long been implicit in summability theory. To be specific, we need only focus attention on the Banach space of all continuous functions x of period 2π on the real line, normed by $\|x\| = \max|x|$. The functions

$$x_1(t) = 1, \quad x_{2k}(t) = \sin kt, \quad x_{2k+1}(t) = \cos kt \quad (k = 1, 2, \ldots)$$

then do not constitute a basis is $\mathcal{U}$, since there do not generally exist expansions of the form (1.1). However, it is known that for appropriately chosen coefficients a_n the series in (1.1) is always summable C_1 to x in the topology of $\mathcal{U}$, so that $\{x_n\}$ behaves very much like a basis.

There remained for Markushevich [12, 13], with this as motivation, to make the crucial step — that of generalizing the notion of a Schauder basis in such a way as to avoid all mention of series expansions. Thus, suppose that $\{x_n\}$ is any sequence of points total in $\mathcal{U}$ (i. e. having finite linear combinations dense in $\mathcal{U}$). If there exists a sequence $\{\varphi_n\}$ of continuous linear functionals biorthogonal to $\{x_n\}$ and such that

$$\varphi_n(x) = 0 \quad (n = 1, 2, \ldots) \Rightarrow x = 0 \quad (\text{all } x \in \mathcal{U}), \tag{1.2}$$

then Markushevich has called $\{x_n\}$ a "basis in the wide sense" for $\mathcal{U}$. We shall employ the terminology *Markushevich basis* for such a sequence $\{x_n\}$. Every Schauder basis is a Markushevich basis, but not conversely (the sequence $\{x_n\}$ in the example just cited is easily seen to be a Markushevich basis).

A trivial extension of the basis concept involves the replacement of $\{x_n\}$ by a family $\{x_i\}$ which is not assumed countable. With this modification in the definition of a basis, Schauder basis, of Markushevich basis we shall refer to $\{x_i\}$ as an *extended basis*, *extended Schauder basis*, or *extended Markushevich basis* (1).

In the present paper we carry the generalization one step farther by discarding the requirement of totalness, along with that of countability. Thus pared down to its topological essentials, the resulting concept of a generalized basis gives rise to certain paradoxes. For example, the family of coefficient functionals is no longer unique, and, moreover, there exists a non-separable Banach space admitting a countable generalized basis. On the other hand, the concept of a generalized basis is just strong enough to allow an extension of the isomorphism theorem (given in [1] for Schauder bases in Fréchet spaces) to the case of generalized bases in complete metric linear spaces.

A number of examples are exhibited to indicate the relative scope of the various basis concepts. Also, as might be expected, the total generalized bases (or extended Markushevich bases) turn out to be of particular interest, and we are led to examine them in greater detail. Translational bases on discrete abelian groups, previously studied from a somewhat different viewpoint in [9], are then reconsidered in the present context. Our final section deals with continuity of the coefficient functionals and includes a proof of the fact that strict inductive limits of Fréchet spaces do not tolerate non-Schauder bases (this generalizes a theorem of Newns [14], pp. 431-432, which makes the same assertion for Fréchet spaces).

(1) Extended Markushevich bases have already been considered by Markushevich [12]. For extended Schauder bases it is agreed that the basis expansions are to be carried out according to a fixed linear ordering of the indexing set. (In practice, it is usually assumed that the expansions converge unconditionally).

The authors wish to express their gratitude to Dr. Czesław Bessaga for reading the manuscript and calling attention to the weak basis theorem for Fréchet spaces (currently unpublished, but to appear shortly in a paper by Bessaga and Pełczyński [4]) (2). In addition, the study of continuity of the coefficient functionals for the case of inductive limits of Fréchet spaces (§ 6) has its origin in a question raised by Dr. Bessaga.

2. Definition and examples of generalized bases. Let I be any non-empty set, and let us denote by $\mathcal{A}$ the linear space of all indexed families $\{a_i\}$ $(i \in I)$ of scalars, the operations in $\mathcal{A}$ being pointwise operations relative to the scalar field a. For brevity we shall henceforth use the terminology "the family $\{a_i\}$" to mean "the indexed family $\{a_i\}$" (i. e. the function on I whose value at any $i \in I$ is a_i).

A family $\{\varphi_i\}$ of continuous linear functionals is said to be *biorthogonal* to a family $\{x_i\}$ of points of $\mathcal{U}$ provided that $\varphi_i(x_j) = \delta_{ij}$ for all i, j in I, where δ_{ij} is the Kronecker delta.

DEFINITION 1. Let $\{x_i\}$ be a family of points of $\mathcal{U}$. If there exists a family $\{\varphi_i\}$ of continuous linear functionals on $\mathcal{U}$ biorthogonal to $\{x_i\}$ and such that

$$\varphi_i(x) = 0 \quad (i \in I) \Rightarrow x = 0 \tag{2.1}$$

for all $x \in \mathcal{U}$, then $\{x_i\}$ will be called *a generalized basis* in $\mathcal{U}$. We shall also refer to $\{\varphi_i\}$ as a *family of coefficient functionals* corresponding to $\{x_i\}$.

Condition (2.1) is obviously equivalent to the condition that the mapping Φ of $\mathcal{U}$ into $\mathcal{A}$ defined by $\Phi(x) = \{\varphi_i(x)\}$ be one-to-one. We shall call Φ the *coefficient mapping* determined by $\{\varphi_i\}$.

A given family $\{\varphi_i\}$ of continuous linear functionals can be the family of coefficient functionals for at most one generalized basis $\{x_i\}$. Furthermore, every total generalized basis $\{x_i\}$ uniquely determines the family $\{\varphi_i\}$ of coefficient functionals.

As we now show, the converse also holds provided we assume $\mathcal{U}$ to be locally convex and a to be the real or complex field.

THEOREM 1. *Let $\mathcal{U}$ be a locally convex topological linear space over the real or complex field. For the family $\{\varphi_i\}$ of coefficient functionals to be uniquely determined by the generalized basis $\{x_i\}$ in $\mathcal{U}$ it is necessary and sufficient that $\{x_i\}$ be total in $\mathcal{U}$.*

Proof. It suffices to show that if $\{x_i\}$ is not total, there exist distinct families $\{\varphi_i\}$ and $\{\varphi_i'\}$ of coefficient functionals. For this we invoke the

(2) A special case of the weak basis theorem was given in the original manuscript of the present paper.

Hahn-Banach theorem to infer the existence of a continuous linear functional λ on $\mathcal{U}$ vanishing at each x_i but not vanishing identically. Then if $\{\varphi_i\}$ is a family of coefficient functionals corresponding to $\{x_i\}$, we can take for $\{\varphi_i'\}$ any family obtained from $\{\varphi_i\}$ by adding λ to finitely many of the functionals φ_i.

More generally, let Σ be the subspace of $\mathcal{U}$ spanned finitely by $\{x_i\}$ (i. e. comprised of all finite linear combinations of the points x_i), let Φ be the coefficient mapping determined by $\{\varphi_i\}$, and let $\{\lambda_i\}$ be any family of continuous linear functionals on $\mathcal{U}$ annihilating $\{x_i\}$: $\lambda_i(x_j) = 0$ for all i, j in I. If $\{\lambda_i(x)\} \in \Phi[\overline{\Sigma}]$ for all x in $\mathcal{U}$ and

$$\varphi_i' = \varphi_i + \lambda_i \quad (i \in I), \tag{2.2}$$

then $\{\varphi_i'\}$ is a family of coefficient functionals corresponding to $\{x_i\}$. In fact, $\{\varphi_i'\}$ is obviously biorthogonal to $\{x_i\}$, and we have

$$\begin{aligned} \varphi_i'(x) = 0 \quad (i \in I) \Rightarrow x \in \overline{\Sigma} \Rightarrow \lambda_i(x) = 0 \quad (i \in I) \\ \Rightarrow \varphi_i(x) = 0 \quad (i \in I) \Rightarrow x = 0 \end{aligned}$$

for all x in $\mathcal{U}$.

The condition just given in (2.2) is a sufficient one only. If $\{\varphi_i\}$ and $\{\varphi_i'\}$ are families of coefficient functionals corresponding to the same generalized basis $\{x_i\}$, then the functionals $\varphi_i' - \varphi_i$ must of course annihilate $\{x_i\}$. However, as the following example shows, the family $\{\varphi_i'(x) - \varphi_i(x)\}$ may fail to belong to $\Phi[\overline{\Sigma}]$ for some x in $\mathcal{U}$.

Example 1. Let I be an infinite set and $\mathcal{U}$ the linear space of all bounded real-valued functions on I. This is plainly a Banach space under the sup norm. For each $i \in I$ let us define x_i as the function assuming the value 1 at i and 0 elsewhere. The point functionals $\varphi_i(x) = x(i)$ are all continuous, and $\{\varphi_i\}$ is biorthogonal to $\{x_i\}$. Since

$$\varphi_i(x) = 0 \quad (i \in I) \Rightarrow x = 0$$

for all x in $\mathcal{U}$, it follows that $\{x_i\}$ is a generalized basis in $\mathcal{U}$. However, $\mathcal{U}$ is easily seen to be non-separable, and $\{x_i\}$ is not total in $\mathcal{U}$. Thus, there exists a non-separable Banach space admitting a countable generalized basis (such a space admits neither a Schauder nor a Markushevich basis).

Fixing I now as the set of positive integers, we observe that the Mazur generalized limit ([3], p. 34) yields a continuous linear functional λ on $\mathcal{U}$ annihilating $\{x_i\}$. Moreover, the linear subspace Σ spanned finitely by $\{x_i\}$ has as closure the space of all sequences converging to 0. It is then readily verified that $\{\varphi_i + \lambda\}$ is a family of coefficient functionals corresponding to $\{x_i\}$, but the family all of whose terms are $\lambda(x)$ does not belong to $\Phi[\overline{\Sigma}]$ unless $x = 0$.

An alternative example of the same kind is obtained by taking $\mathcal{U}$ as the space of all bounded sequences $\{x(i)\}_{i=1}^{\infty}$ for which the quantity

$$\lambda(x) = \lim_{n\to\infty} \frac{x(1)+x(2)+\ldots+x(n)}{n}$$

exists finitely. This in again a Banach space under the sup norm, and λ is a continuous linear functional annihilating the generalized basis $\{x_i\}$ defined as before. Here, however, there is the advantage of having an analytical expression for λ.

That a Markushevich basis need not also be a Schauder basis has been mentioned in connection with the Fourier series example of § 1. Perhaps even simpler is the following example, due to Markushevich [13].

Example 2. Let $\mathcal{U}$ be the Fréchet space of all functions analytic on the open unit disc ($|z| < 1$), topologized by the metric of uniform convergence on compact sets. Choosing a as any complex number such that $0 < |a| < 1$, we set

$$f_n(z) = (z-a)^n \quad (n = 0, 1, \ldots).$$

These functions form a Markushevich basis in $\mathcal{U}$, the corresponding biorthogonal sequence of continuous linear functionals being given by

$$\varphi_n(f) = \frac{f^{(n)}(a)}{n!} \quad (n = 0, 1, \ldots).$$

On the other hand, $\{f_n\}$ is not a Schauder basis, since the series

$$\sum_{n=0}^{\infty} \frac{(z-a)^n}{(1-a)^{n+1}}$$

corresponding to the function $f(z) = (1-z)^{-1}$ diverges for z outside the circle $|z-a| = |1-a|$.

We observe further that if $\{x_i\}$ is a generalized basis in $\mathcal{U}$ and $\{\varphi_i\}$ is any corresponding family of coefficient functionals, then $\{x_i\}$ and $\{\varphi_i\}$ form a maximal biorthogonal system relative to $\mathcal{U}$. In fact, if this biorthogonal system were not maximal, then there would exist a point x' of $\mathcal{U}$ and a continuous linear functional φ' on $\mathcal{U}$ such that $\varphi'(x') = 1$ and $\varphi_i(x') = 0$ for all $i \in I$; but these conditions are clearly incompatible with (2.1).

To show that a maximal biorthogonal system may fail to yield a generalized basis, we make use of an example cited in a different context by Markushevich [13] ([3]).

([3]) Similar examples have been published by Dieudonné [8].

EXAMPLE 3. Let us take $\mathcal{U}$ as the space employed in Example 2 but this time put

$$\begin{cases} f_0(z) = 1, \\ f_1(z) = 1+z, \\ \dots\dots\dots \\ f_n(z) = 1+z+\dots+z^n, \\ \dots\dots\dots\dots \end{cases}$$

Defining φ_n on $\mathcal{U}$ by

$$\varphi_n(f) = \frac{f^{(n)}(0)}{n!} - \frac{f^{(n+1)}(0)}{(n+1)!} \quad (n = 0, 1, \dots),$$

we see that $\{f_n\}$ and $\{\varphi_n\}$ form a biorthogonal system over $\mathcal{U}$. However $\{f_n\}$ is not a generalized basis in $\mathcal{U}$, since, in the first place, the only possible candidate for a sequence of coefficient functionals is $\{\varphi_n\}$ (in view of the fact that $\{f_n\}$ is total), and, in the second place, the condition $\varphi_n(f) = 0$ $(n = 0, 1, \dots)$ does not force $f = 0$ (4).

To verify that the biorthogonal system $\{f_n\}$, $\{\varphi_n\}$ must be maximal, let us assume the contrary. There then exists a continuous linear functional annihilating $\{f_n\}$ but not vanishing identically. This, however, is impossible, since every continuous linear functional φ on $\mathcal{U}$ has the representation

$$\varphi(f) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} h_n$$

for $\{h_n\}$ a suitably chosen sequence of complex numbers.

Summarizing our conclusions, we have the following strict ordering in terms of increasing generality:

Schauder basis $\prec$ *Markushevich basis*

$\prec$ *generalized basis* $\prec$ *maximal biorthogonal system.*

3. Similar generalized bases and the isomorphism theorem. Let us consider now two topological linear spaces $\mathcal{U}$ and $\mathcal{V}$ having scalar fields α and β, respectively. We shall assume further that $\mathcal{U}$ admits a generalized basis $\{x_i\}$ $(i \in I)$ and that $\mathcal{V}$ admits a generalized basis $\{y_j\}$ $(j \in J)$.

DEFINITION 2. The generalized bases $\{x_i\}$ in $\mathcal{U}$ and $\{y_j\}$ in $\mathcal{V}$ will be called *similar* provided there exist corresponding families $\{\varphi_i\}$ and $\{\psi_j\}$ of coefficient functionals such that

$$\Phi[\mathcal{U}] = \Psi[\mathcal{V}],$$

where Φ and Ψ are the coefficient mappings determined by $\{\varphi_i\}$ and $\{\psi_j\}$, respectively.

Whenever $\{x_i\}$ and $\{y_j\}$ are similar, we must evidently have $I = J$ and $\alpha = \beta$. *From this point on, we shall adhere to the convention that the linear spaces which enter the discussion shall all be taken over the same field α of all real or complex numbers.*

Assuming now that $\mathcal{U}$ and $\mathcal{V}$ are complete metric linear spaces, we arrive at the following extension of Theorem 1 of [1] (5):

THEOREM 2. *Let $\mathcal{U}$ and $\mathcal{V}$ be complete metric linear spaces, and let $\{x_i\}$ $(i \in I)$ be a generalized basis in $\mathcal{U}$. If T is an isomorphism of $\mathcal{U}$ onto $\mathcal{V}$ and $y_i = Tx_i$ $(i \in I)$, then $\{y_i\}$ is a generalized basis in $\mathcal{V}$ similar to $\{x_i\}$. Conversely, if $\{y_i\}$ is a generalized basis in $\mathcal{V}$ similar to $\{x_i\}$, then there exists an isomorphism T of $\mathcal{U}$ onto $\mathcal{V}$ such that $y_i = Tx_i$ $(i \in I)$. In either case, $\{y_i\}$ is an extended Markushevich [Schauder] basis if and only if $\{x_i\}$ has the same property.*

Proof of the direct assertion amounts simply to the observation that the properties involved are all invariant under isomorphisms. For the converse let us put

$$\mathcal{A}_0 = \Phi[\mathcal{U}] = \Psi[\mathcal{V}],$$

so that $\mathcal{A}_0$ is a linear subspace of $\mathcal{A}$. In view of the fact that $\mathcal{A}_0$, as a linear space, is isomorphic to both $\mathcal{U}$ and $\mathcal{V}$, we can metrize $\mathcal{A}_0$ either by the metric of $\mathcal{U}$ or the metric of $\mathcal{V}$. In each case $\mathcal{A}_0$ becomes a complete metric linear space over α.

If we can show that both of these metrics define the same topology on $\mathcal{A}_0$, then the mapping $T = \Psi^{-1}\Phi$ will furnish the desired isomorphism of $\mathcal{U}$ onto $\mathcal{V}$ (the biorthogonality yielding $y_i = Tx_i$ for $i \in I$). That these two metrics do, indeed, define the same topology is a consequence of the following general theorem (6):

THEOREM 3. *Let $\mathcal{E}$ be a linear space and $\{F_i\}$ a separating family* (7) *of functions on $\mathcal{E}$. Further, let ϱ' and ϱ'' be two metrics, each making $\mathcal{E}$ into*

(4) It is easily seen that the condition $\varphi_n(f) = 0$ $(n = 0, 1, \dots)$ is necessary and sufficient for f to be of the form $f(z) = c(1+z+z^2+\dots+z^n+\dots)$, where c is an arbitrary constant.

(5) The present proof is simpler than that in [1], but the development in [1] yields additional insight into the properties of similar sequences in Fréchet spaces.

(6) The argument here is essentially that used by Gelfand [11, p. 17] in showing that algebraic isomorphism of normed rings implies topological isomorphism.

(7) That is, if $F_i(x) = F_i(y)$ for all $i \in I$, then x and y must coincide. Note that condition (2.1) is just the requirement that $\{\varphi_i\}$ be a separating family of continuous linear functionals on $\mathcal{U}$, or, equivalently ([6], p. 51), that $\{\varphi_i\}$ be weakly total in the dual space of $\mathcal{U}$.

a complete metric linear space. If every F_i is continuous with respect to each of these metrics, then ϱ' and ϱ'' define the same topology on $\mathcal{E}$.

Proof. It is clear that $\varrho = \varrho' + \varrho''$ is a metric on $\mathcal{E}$, and we proceed to show that $\mathcal{E}$ is complete in this metric. Thus, let $\{z_n\}$ be a Cauchy sequence in the metric ϱ, so that $\{z_n\}$ in automatically a Cauchy sequence with respect to each of the metrics ϱ' and ϱ''. By the assumed completeness, $\{z_n\}$ converges (ϱ') to some point z' of $\mathcal{E}$ and converges (ϱ'') to some point z'' of $\mathcal{E}$. We next invoke the continuity properties of the functions F_i to infer that

$$F_i(z') = \lim_{n\to\infty} F_i(z_n) = F_i(z'') \qquad (i \in I).$$

The hypothesis that $\{F_i\}$ is a separating set then forces $z' = z''$, and it follows that $\{z_n\}$ converges (ϱ) to the point $z = z' = z''$. Hence, $\mathcal{E}$ is complete in the metric ϱ.

Since $\varrho' \leqslant \varrho$ and $\varrho'' \leqslant \varrho$, the open mapping theorem [3, p. 41, Theorem 5] ensures that ϱ' and ϱ'' define on $\mathcal{E}$ the same topology as ϱ. This completes the proof.

In applying Theorem 3 to the proof of Theorem 2, we have but to set $\mathcal{E} = \mathcal{A}_0$ and $F_i(z) = \varphi_i(x) = \psi_i(y)$, where $z = \Phi(x) = \Psi(y)$.

4. Total generalized bases. The assumption that a given generalized basis is total allows certain properties of finite sums to be carried over to arbitrary points of the space. It is natural, therefore, to expect that some of the theorems for Schauder bases will have counterparts for total generalized bases. We present here a few results of this sort.

Let us suppose that $\mathcal{U}$ is a topological linear space admitting a total generalized basis $\{x_i\}$ $(i \in I)$. For I' any non-empty subset of I we shall consider the closed subspace $\mathcal{U}'$ of $\mathcal{U}$ generated by $\{x_i\}$ $(i \in I')$. That is, $\mathcal{U}'$ is the closure of the set of all finite linear combinations of those x^i having indices i in I'. The following theorem on extensions of isomorphisms appears as a direct analogue of Corollary 1.3 of [1]:

Theorem 4. *Let $\mathcal{U}$ and $\mathcal{V}$ be complete metric linear spaces having $\{x_i\}$ and $\{y_i\}$ $(i \in I)$, respectively, as total generalized bases. Further, let I' be a subset of I for which $I' - I$ is finite, and let $\mathcal{U}'$ and $\mathcal{V}'$ be the closed subspaces generated by $\{x_i\}$ $(i \in I')$ and $\{y_i\}$ $(i \in I')$, respectively. If T' is an isomorphism of $\mathcal{U}'$ onto $\mathcal{V}'$ such that $T'x_i = y_i$ $(i \in I')$, then T' can be extended to an isomorphism T of $\mathcal{U}$ onto $\mathcal{V}$ such that $Tx_i = y_i$ $(i \in I)$.*

To begin with, it is evident that $\{x_i\}$ $(i \in I')$ and $\{y_i\}$ $(i \in I')$ are total generalized bases in $\mathcal{U}'$ and $\mathcal{V}'$, respectively. The existence of the isomorphism T' then ensures that these bases are similar. Since this plainly implies similarity of the original bases $\{x_i\}$ and $\{y_i\}$, there exists an isomorphism T of $\mathcal{U}$ onto $\mathcal{V}$ carrying x_i into y_i $(i \in I)$. That T is actually an extension of T' is immediate from the fact that these two linear mappings coincide on a total subset of $\mathcal{U}'$.

We turn now to a generalization of the classical Paley-Wiener theorem ([8]). Here $\mathcal{U}$ is taken as a complete metric linear space having ϱ as a translation-invariant metric, and we use the notation of Banach:

$$\|x\| = \varrho(x, 0).$$

Theorem 5. *Let $\mathcal{U}$ be a complete metric linear space and $\{x_i\}$ $(i \in I)$ a total generalized basis in $\mathcal{U}$. If $\{y_i\}$ $(i \in I)$ is a family of points in $\mathcal{U}$ and λ a real number $(0 < \lambda < 1)$ such that*

$$\Big\|\sum_{n=1}^{m} a_n(y_{i_n} - x_{i_n})\Big\| \leqslant \lambda \Big\|\sum_{n=1}^{m} a_n x_{i_n}\Big\| \tag{4.1}$$

holds for all finite sequences $i_1, i_2, \ldots, i_m$ of indices in I and all finite sequences $a_1, a_2, \ldots, a_m$ of scalars, then

(1) $\{y_i\}$ is a total generalized basis in $\mathcal{U}$,

and

(2) there exists an automorphism A on $\mathcal{U}$ such that $y_i = Ax_i$ $(i \in I)$ and $(1-\lambda)\|x\| \leqslant \|Ax\|$ for all x in $\mathcal{U}$.

Proof. On the subspace Σ spanned finitely by $\{x_i\}$ we define a linear operator T by putting

$$Tx = \sum_i \varphi_i(x) \cdot (y_i - x_i).$$

Since T is bounded on Σ by virtue of (4.1), we can consider T as extended continuously onto $\mathcal{U}$ $(= \bar{\Sigma})$. Condition (4.1) also yields

$$\|T^n x\| \leqslant \lambda^n \|x\| \qquad (n = 0, 1, \ldots)$$

for all x in $\mathcal{U}$.

By comparison with the corresponding geometric series in λ we at once infer convergence of the operator series

$$U = \sum_{n=0}^{\infty} (-T)^n, \tag{4.2}$$

and there follows

$$\|Ux\| \leqslant (1-\lambda)^{-1}\|x\| \qquad (x \in \mathcal{U}).$$

([8]) A somewhat restricted theorem of this sort has been stated by Markushevich [12]. It should be mentioned also that the device of using the operator series (4.2) to prove the Paley-Wiener theorem has been suggested by Buck [7], p. 410. (For a discussion of the classical Paley-Wiener theorem and certain of its variants, see [2]).

Thus, U is a one-to-one continuous linear operator on $\mathcal{U}$. The expansion (4.2) shows, in fact, that $U = (I+T)^{-1}$, where I is the identity operator. Hence $A = I+T (= U^{-1})$ is an automorphism on $\mathcal{U}$ carrying x_i into y_i, and the elementary direct assertion in Theorem 2 allows us to conclude that $\{y_i\}$ is a generalized basis in $\mathcal{U}$. This generalized basis is obviously total, and the proof is complete.

A further result on total generalized bases is concerned with the characterization of similar such bases directly by means of inequalities. The following theorem is an extension of Theorem 4 of [1]:

THEOREM 6. *Let $\mathcal{U}$ and $\mathcal{V}$ be Fréchet spaces with topologies defined by sequences $\{u_n\}$ and $\{v_n\}$, respectively, of continuous semi-norms, and let $\{x_i\}$ and $\{y_i\}$ be total generalized bases in the respective spaces. Then a necessary and sufficient condition for $\{x_i\}$ and $\{y_i\}$ to be similar is that to every positive integer p there correspond a positive integer q and a positive number M such that*

$$u_p\Big(\sum_{n=1}^{m} a_n x_{i_n}\Big) \leqslant M v_q\Big(\sum_{n=1}^{m} a_n y_{i_n}\Big)$$

and

$$(4.3) \qquad v_p\Big(\sum_{n=1}^{m} a_n y_{i_n}\Big) \leqslant M u_q\Big(\sum_{n=1}^{m} a_n x_{i_n}\Big)$$

hold for all finite sequences $i_1, i_2, \ldots, i_m$ of indices in I and all finite sequences $a_1, a_2, \ldots, a_m$ of scalars.

Proof. (Necessity). This is an immediate consequence of Theorem 2, coupled with Proposition 9, p. 100-101, of [5].

(Sufficiency). Denoting by Σ the space spanned finitely by $\{x_i\}$, we define on Σ the linear transformation

$$T\Big(\sum_n a_n x_{i_n}\Big) = \sum_n a_n y_{i_n}.$$

Condition (4.3), in conjunction with Proposition 9 (cited above), ensures that T is continuous on Σ. By linearity, T must, in fact, be uniformly continuous and can therefore be extended to a continuous linear mapping of $\mathcal{U} (= \bar{\Sigma})$ into $\mathcal{V}$.

Now, let $\{\varphi_i\}$ and $\{\psi_i\}$ be the families of coefficient functionals corresponding to $\{x_i\}$ and $\{y_i\}$, respectively. In as much as $Tx_i = y_i$ $(i \in I)$, we have

$$\psi_i(Tx) = \varphi_i(x) \qquad (i \in I)$$

for all x in Σ. Moreover, an evident passage to the limit shows that this actually holds for all x in $\mathcal{U}$. That is,

$$\Phi[\mathcal{U}] \subset \Psi[\mathcal{V}],$$

where Φ and Ψ are the coefficient mappings determined by $\{\varphi_i\}$ and $\{\psi_i\}$, respectively. Symmetry of the given data then yields $\Phi[\mathcal{U}] = \Psi[\mathcal{V}]$, completing the proof.

We make use next of the notion of inductive limit of a sequence of locally convex topological linear spaces (see [5, p. 61-69]).

THEOREM 7. *Let $\mathcal{U}$ be the inductive limit of an increasing sequence* (⁹) *$\{\mathcal{U}_n\}$ of locally convex topological linear spaces, and let $\{x_i\}$ be a family of points in $\mathcal{U}_1$. Then*

(1) *if $\{x_i\}$ is a total generalized basis in each $\mathcal{U}_n$, then $\{x_i\}$ is a total generalized basis in $\mathcal{U}$.*

(2) *if $\{x_i\}$ is an extended Schauder basis in each $\mathcal{U}_n$, then $\{x_i\}$ is an extended Schauder basis in $\mathcal{U}$.*

Proof. To establish (1), let us denote by $\{\varphi_i^{(n)}\}$ the family of coefficient functionals corresponding to $\{x_i\}$ relative to the space $\mathcal{U}_n$. Since $\{\mathcal{U}_n\}$ is an increasing sequence and the coefficient family for each of these spaces is unique, it is evident that each $\varphi_i^{(n)}$ is an extension of the corresponding $\varphi_i^{(m)}$ provided $n \geqslant m$. Hence, for each $i \in I$ there exists a linear functional φ_i on $\mathcal{U}$ coinciding with $\varphi_i^{(n)}$ on $\mathcal{U}_n$ $(n = 1, 2, \ldots)$.

Since the continuity of φ_i on each $\mathcal{U}_n$ forces continuity of φ_i on $\mathcal{U}$, the sequences $\{x_i\}$, $\{\varphi_i\}$ form a biorthogonal system over $\mathcal{U}$. That $\varphi_i(x) = 0$ (all $i \in I$) implies $x = 0$ is then apparent from the corresponding property for any of the subspaces $\mathcal{U}_n$ in which x lies. To prove (2), it suffices to observe that each x in $\mathcal{U}$ is given by a unique series expansion in some subspace $\mathcal{U}_n$ and that convergence in this subspace results in convergence in $\mathcal{U}$.

As an application of Theorem 6 we proceed to exhibit a Schauder basis in a non-metrizable space. Let us take $\{R_n\}$ as any sequence of positive numbers decreasing to some number $R \geqslant 0$ and denote by σ_n the closed disc of radius R_n about the origin in the complex plane. For each n, $\mathcal{U}_n$ will then be defined as the space of all complex functions continuous on σ_n and analytic on its interior, the topology being that given by the sup norm. With the convention that we identify any two functions coinciding on some neighbourhood of the origin, it is clear that the hypotheses of Theorem 7 are satisfied. Thus, the functions $f_n(z) = z^n$ $(n = 0, 1, \ldots)$ comprise a Schauder basis in the inductive limit space $\mathcal{U}$. That $\mathcal{U}$ is non-metrizable follows easily from Exercise 13 a), p. 36, of [6] (with $F_n = \mathcal{U}_n$ and $E = F = \mathcal{U}$).

(⁹) More generally, the theorem remains valid for directed families.

5. Translational bases. A broad class of total generalized bases is provided by translates of certain functions on the L^2 space over a discrete abelian group I.

For x in $L^2(I)$ and i in I, we shall use x_i to denote here the translate of x by i. That is,

$$x_i(t) = x(t+i) \qquad (t \in I),$$

the group operation being written as addition. Also, $\hat{I}$ will signify the dual group of I, $\hat{x}$ the Fourier transform of x, and μ the Haar measure on I.

Questions relating to independence of the translates have been investigated in [9]. Using results developed there, we examine the translates now from the point of view of generalized bases.

THEOREM 8. *The condition $1/\hat{x} \in L^2(\hat{I})$ is necessary and sufficient for $\{x_i\}$ to be a total generalized basis in $L^2(I)$.*

Proof. Theorem 3 of [9] asserts that the above condition is necessary and sufficient for $\{x_i\}$ to be linearly independent. Moreover, as noted in [9, p. 39], this condition makes $\{x_i\}$ total in $L^2(I)$, and there then exists a unique function w in $L^2(I)$ such that

$$\int x_i \overline{w}\, d\mu = \begin{cases} 1 & \text{for } i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Setting

$$\varphi_i(y) = \int y \overline{w}_i\, d\mu \qquad \big(i \in I;\ y \in L^2(I)\big), \tag{5.1}$$

we obtain (by means of the Parseval formula) the inequality

$$|\varphi_i(y)| \leqslant \|y\|/\|\hat{x}\|.$$

Hence, each φ_i is a continuous linear functional on $L^2(I)$, and we observe that $\{\varphi_i\}$ is biorthogonal to $\{x_i\}$. There remains only to establish that

$$\varphi_i(y) = 0 \quad (\text{all } i \in I) \Rightarrow y = 0.$$

This, however, follows from (5.1) and the conditions on p. 38-39 of [9], since the symmetry of these conditions shows that $\{w_i\}$ shares with $\{x_i\}$ the property of being total in $L^2(I)$.

It is an open problem whether the condition $1/\hat{x} \in L^2(\hat{I})$ also forces $\{x_i\}$ to be an extended Schauder basis, that is, whether it is strong enough to ensure the L^2 convergence of $\sum \varphi_i(y) x_i$ to y for every y in $L^2(I)$. The problem is a special case of the following one. Let G be a compact abelian group, and let δ be a positive integrable function on G such that δ^{-1} is also integrable. Construct the L^2 Hilbert space relative to the measure with density δ. Is each f in this space the L^2 sum of its Fourier series?

To elaborate on the above remarks, we develop certain convergence results for I taken as the discrete additive group of integers and $1/\hat{x}$ in $L^2(\hat{I})$. Let y be fixed as any element of $L^2(I)$, so that (with the aid of Parseval's formula) we can write

$$c_n = \varphi_n(y) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\hat{y}(t)}{\hat{x}(t)} e^{-\iota nt}\, dt \qquad (\iota = \sqrt{-1})$$

for all n in I (see [9], p. 39-40); c_n is thus the n^{th} Fourier coefficient of the function $\hat{y}/\hat{x}$ in $L^1(-\pi, \pi)$.

In discussing convergence of the series $\sum c_n x_n$ to y, it is natural to deal with the symmetric partial sums

$$s_k = \sum_{|n| \leqslant k} c_n x_n \qquad (k = 1, 2, \ldots).$$

Since the Fourier transform of s_k is

$$\hat{s}_k(t) = \sum_{|n| \leqslant k} c_n e^{\iota nt} \hat{x}(t) = S_k(t)\hat{x}(t),$$

where S_k is the k^{th} symmetric partial sum of the Fourier series for $\hat{y}/\hat{x}$, Parseval's formula yields

$$\|y - s_k\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\frac{\hat{y}}{\hat{x}} - S_k\right|^2 |\hat{x}|^2\, dt. \tag{5.2}$$

Thus, the convergence of $\sum c_n x_n$ to y is governed by the degree of approximation to $\hat{y}/\hat{x}$ (in the L^2 space relative to the measure of density $|\hat{x}|^2$) afforded by the partial sums S_k of the Fourier series for $\hat{y}/\hat{x}$. In particular, we have

THEOREM 9. *In order that $s_k \to y$ in $L^2(I)$ it is*

(1) necessary that $S_k \to \hat{y}/\hat{x}$ in $L^1(-\pi, \pi)$,

and

(2) sufficient that $\{S_k\}$ converge boundedly a. e.

Further, convergence takes place whenever $\hat{x}$ is essentially bounded and $\hat{y}/\hat{x}$ belongs to $L^2(-\pi, \pi)$ (convergence therefore takes place for all y in $L^2(I)$, provided that both $\hat{x}$ and $1/\hat{x}$ are essentially bounded).

Proof. In the case of (1) we neeed only note that, for any measurable function f on $[-\pi, \pi]$,

$$\int_{-\pi}^{\pi} |f|\, dt = \int_{-\pi}^{\pi} |f|\hat{x}\cdot(1/\hat{x})\, dt \leqslant \Big\{\int_{-\pi}^{\pi} |f|^2 |\hat{x}|^2 dt\Big\}^{1/2} \Big\{\int_{-\pi}^{\pi} \frac{dt}{|\hat{x}|^2}\Big\}^{1/2};$$

this is then applied, in conjunction with (5.2), to the function $f = \hat{y}/\hat{x} - S_k$. On the other hand, (2) follows at once from the principle of dominated convergence.

If $\hat{y}/\hat{x}$ belongs to $L^2(-\pi,\pi)$, then $S_k \to \hat{y}/\hat{x}$ in this space. If also $\hat{x}$ is essentially bounded, then the right-hand member of (5.2) tends to 0 as $k \to \infty$. Finally, if $1/\hat{x}$ is essentially bounded, then $\hat{y}/\hat{x}$ belongs to $L^2(-\pi,\pi)$ for each y in $L^2(I)$. This completes the proof. (Note that in these latter cases the argument holds when I is replaced by an arbitrary discrete abelian group; the series development is then unconditionally convergent, i. e. convergent according to the increasing directed family of finite subsets of the discrete group in question).

6. Continuity of the coefficient functionals. So far, the relationship of ordinary bases with Schauder bases has received only brief mention. It is known that every basis in a Banach space must be a Schauder basis, and the same result has been established by Newns [14, p. 431-432] for the more general case of a Fréchet space.

On the other hand, it is not true that bases in all topological linear spaces are Schauder bases. For example, if $\mathcal{U}$ is taken as the space of all real functions expandable as power series on the interval $(-1,1)$, the topology being that of uniform convergence on compact sets, then the functions $x_n(t) = t^n$ $(n = 0, 1, \ldots)$ are easily seen to form a basis which is not a Schauder basis. The only coefficient functional continuous on $\mathcal{U}$ is, in fact, φ_0 (10).

The space $\mathcal{U}$ in the above example is certainly not complete (if it were complete, it would be a Fréchet space and the basis a Schauder basis). Moreover, the completion of $\mathcal{U}$ is the space of all real functions continuous on $(-1,1)$, and $\{x_n\}$ is not a basis in the latter space. An example can, however, be given of a Schauder basis for which the space spanned (in the infinite-series sense) by the basis elements fails to be complete. For this we have but to take $\mathcal{U}$ as the space introduced in Example 3 and consider the subspace consisting of all functions on the open unit disc representable as infinite series in the functions f_n. The resulting subspace is not complete, since $\{f_n\}$ itself converges to a function in $\mathcal{U}$ but not in the subspace.

Two recent generalizations of Newns' theorem deserve mention. Theorem 2 of [2] states that *every basis in a complete metric linear space over the real or complex field is a Schauder basis*. That is, the local convexity hypothesis in Newns' theorem can be omitted. The second generalization retains local convexity but relaxes the basis requirement. This

(10) To show that φ_n $(n \geqslant 1)$ is not everywhere continuous, take δ as a continuous function on $(-1,1)$ vanishing except for a narrow tapered unit step centered at the origin. Then δ can be uniformly approximated by polynomials, and the n-fold integrals of these polynomials are functions approximating 0 in $\mathcal{U}$ but having n^{th} derivatives converging to 1 at the origin.

result, due to Bessaga and Pełczyński [4], asserts that *every weak basis in a Fréchet space is a Schauder basis* (it thus represents an extension to Fréchet spaces of the classical weak basis theorem of Banach).

In general, the qualification "weak", applied to any of the types of bases previously considered, means simply that the topology on $\mathcal{U}$ is to be taken as the weak topology in the basis definition.

We give two further results related to the weak basis theorem.

THEOREM 10. *If $\mathcal{U}$ is a locally convex topological linear T_1-space, then every weak extended Markushevich basis in $\mathcal{U}$ is an extended Markushevich basis for the initial topology.*

THEOREM 11. *Let $\mathcal{U}$ be a locally convex topological linear T_1-space. If $\mathcal{U}$ is a t-space* (11), *then every weak extended Schauder basis in $\mathcal{U}$ is an extended Schauder basis for the initial topology.*

Theorem 10 is based on the important observation that *the concept of an extended Markushevich basis is the same for all locally convex Hausdorff topologies (on a given linear space) which yield the same dual.* This is clear since continuity of the coefficient functionals is obviously unchanged and (by [6], p. 67, Proposition 4) the total subsets are the same for all such topologies. In applying the above observation to the proof of Theorem 10, we use the following facts: (i) the T_1 requirement ensures that $\mathcal{U}$ is a Hausdorff space ([16], p. 126), and (ii) the dual under the weak topology coincides with that under the initial topology ([6], p. 63).

To prove Theorem 11, we take $\{x_n\}$ as a weak Schauder basis in $\mathcal{U}$. There then exists a biorthogonal sequence of continuous (= weakly continuous) linear functionals φ_n on $\mathcal{U}$ such that

$$x = \sum_{n=1}^{\infty} \varphi_n(x) x_n \qquad (x \in \mathcal{U})$$

in the sense of weak convergence. Hence, the partial sums

$$s_N(x) = \sum_{n=1}^{N} \varphi_n(x) x_n$$

are continuous endomorphisms on $\mathcal{U}$ having the property that $s_N(x) \to x$ weakly for each x in $\mathcal{U}$. Pointwise convergence under the weak topology results in pointwise boundedness for the weak topology and thereby for the initial topology, in view of the corollary of Mackey's theorem ([6], p. 70). By the Banach-Steinhaus theorem ([6], p. 27, Theorem 2) coupled with the assumption that $\mathcal{U}$ is a t-space, it follows that the family of functions $s_N(x)$ is equicontinuous under the initial topology.

(11) An *espace tonnelé* in the terminology of Bourbaki [6].

Now, taking cognizance of the fact that $\{x_n\}$ is a weak Markushevich basis in $\mathcal{U}$, we infer from Theorem 10 that $\{x_n\}$ is a Markushevich basis for the initial topology. The finite sums $s_N(x)$ are therefore dense in $\mathcal{U}$. Since they have been shown to be equicontinuous, there results

$$x = \sum_{n=1}^{\infty} \varphi_n(x) x_n$$

(relative to the initial topology) for all x in $\mathcal{U}$. Thus completes the proof for the Schauder case, and the extended Schauder case is dealt with similarly.

Our next result serves to extend Newns' theorem to certain inductive limits of Fréchet spaces.

THEOREM 12. *Let $\mathcal{U}$ be the inductive limit of an increasing sequence $\{U_n\}$ of Fréchet spaces with the following properties:*

(i) *every bounded subset of $\mathcal{U}$ is a bounded subset of some $\mathcal{U}_n$;*

(ii) *for each n, every bounded sequence in $\mathcal{U}_n$ which converges in $\mathcal{U}$ also converges in $\mathcal{U}_n$.*

Then every basis in $\mathcal{U}$ is a Schauder basis.

Proof. Let $\{x_i\}$ be a basis in $\mathcal{U}$, and denote by φ_i the coefficient functional corresponding to x_i $(i = 1, 2, \ldots)$. Our object is to show that each φ_i is continuous on $\mathcal{U}$, and it suffices to show that $\varphi_i|\mathcal{U}_n$ is continuous with respect to the topology t_n on $\mathcal{U}_n$ $(n = 1, 2, \ldots)$.

We put

$$s_N(x) = \sum_{i=1}^{N} \varphi_i(x) x_i \qquad (N = 1, 2, \ldots)$$

and define $\mathcal{E}_n$ $(n = 1, 2, \ldots)$ as the linear subspace of $\mathcal{U}$ formed of those elements x for which the points $s_N(x)$ $(N = 1, 2, \ldots)$ comprise a t_n-bounded subset of $\mathcal{U}_n$. Clearly, $\mathcal{E}_n \subset \mathcal{E}_{n+1}$ $(n = 1, 2, \ldots)$. Also, by (i) and the basis properties we have

$$\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}_n,$$

and by (ii)

$$\mathcal{E}_n \subset \mathcal{U}_n \qquad (n = 1, 2, \ldots).$$

The next step is to introduce a topology t_n' on $\mathcal{E}_n$ as follows. Let $\{p_n^\lambda\}_{\lambda=1}^{\infty}$ be a sequence of semi-norms defining the topology t_n, and put

$$q_n^\lambda(x) = \sup_{N \geqslant 1} p_n^\lambda[s_N(x)] \qquad (\lambda = 1, 2, \ldots)$$

for all x in $\mathcal{E}_n$. Then $\{q_n^\lambda\}_{\lambda=1}^{\infty}$ is a sequence of semi-norms defining a locally convex topology t_n' on $\mathcal{E}_n$. The resulting topology is finer than that induced on $\mathcal{E}_n$ by t_n, since hypothesis (ii) yields $p_n^\lambda(x) \leqslant q_n^\lambda(x)$ for x in $\mathcal{E}_n$ and $\lambda = 1, 2, \ldots$ This implies ([5], p. 97) that $\mathcal{E}_n$ is metrizable under the topology t_n'.

Furthermore, it is evident that $\varphi_i(x) x_i$ lies in $\mathcal{U}_n$ for all x in $\mathcal{E}_n$, so that

$$p_n^\lambda[\varphi_i(x) x_i] \leqslant 2 q_n^\lambda(x) \qquad (\lambda = 1, 2, \ldots).$$

There are then two possibilities: given any index i, either x_i belongs to $\mathcal{U}_n$ or $\varphi_i = 0$ on $\mathcal{E}_n$. In the former case we have

$$|\varphi_i(x)|\, p_n^\lambda(x_i) \leqslant 2 q_n^\lambda(x) \qquad (\lambda = 1, 2, \ldots) \tag{6.1}$$

and also $p_n^\lambda(x_i) \neq 0$ for some λ. Hence, in all cases the functionals φ_i $(i = 1, 2, \ldots)$ are t_n'-continuous on $\mathcal{E}_n$.

Our immediate task is to prove that $\mathcal{E}_n$ is complete in the topology t_n'. Thus, suppose that $\{z_k\}$ is a t_n'-Cauchy sequence in $\mathcal{E}_n$. The inequality (6.1) then shows that $\{\varphi_i(z_k)\}_{k=1}^{\infty}$ is a Cauchy sequence of scalars, so that the quantity

$$a_i = \lim_{k\to\infty} \varphi_i(z_k)$$

exists finitely for each i. Since $\varphi_i = 0$ holds on $\mathcal{E}_n$ whenever $x_i \notin \mathcal{U}_n$, it follows that the sums

$$\sigma_N = \sum_{i=1}^{N} a_i x_i \qquad (N = 1, 2, \ldots)$$

all lie in $\mathcal{U}_n$. By t_n'-continuity of the coefficient functionals on $\mathcal{E}_n$ we see also that

$$\lim_{k\to\infty} s_N(z_k) = \sigma_N \qquad (N = 1, 2, \ldots)$$

for the topology t_n'.

Let λ, j, k, and ν $(\leqslant N)$ be arbitrary positive integers. Then

$$p_n^\lambda\Big\{\sum_{i=\nu}^{N} [\varphi_i(z_j) - \varphi_i(z_k)] x_i\Big\} \leqslant 2 q_n^\lambda(z_j - z_k),$$

and in the t_n'-limit as $j \to \infty$ this yields

$$p_n^\lambda\Big\{\sum_{i=\nu}^{N} [a_i - \varphi_i(z_k)] x_i\Big\} \leqslant \varepsilon_n^\lambda(k), \tag{6.2}$$

where

$$\varepsilon_n^\lambda(k) = \limsup_{j\to\infty} q_n^\lambda(z_j - z_k) \xrightarrow[k\to\infty]{} 0.$$

Thus,

$$p_n^\lambda\Big(\sum_{i=\nu}^{N} a_i x_i\Big) \leqslant \varepsilon_n^\lambda(k) + p_n^\lambda\Big[\sum_{i=\nu}^{N} \varphi_i(z_k) x_i\Big]. \tag{6.3}$$

Using (ii) together with the basis expansion for z_k and the completeness of $\mathcal{U}_n$, we ascertain the existence of a point z in $\mathcal{U}_n$ such that

$$z = \sum_{i=1}^{\infty} a_i x_i,$$

the series converging in $\mathcal{U}_n$ and thereby in $\mathcal{U}$. The basis properties then ensure that $a_i = \varphi_i(z)$ $(i = 1, 2, \ldots)$, and a glance at (6.3) shows that this makes z lie in $\mathcal{E}_n$. Convergence of $\{z_k\}$ to z in the topology t'_n is now apparent from (6.2), proving that $\mathcal{E}_n$ is complete in this topology.

We proceed to show that any given $\mathcal{U}_m$ is contained in some $\mathcal{E}_n$. In the first place, $\mathcal{U}_m$ is the union of its subspaces $\mathcal{U}_m \cap \mathcal{E}_n$ $(n = 1, 2, \ldots)$. Also, each of these subspaces is complete under the topology induced by t'_n. Indeed, if $\{z_k\}$ is a Cauchy sequence in $\mathcal{U}_m \cap \mathcal{E}_n$ under this topology, then there exists a point z in $\mathcal{E}_n$ such that $z_k \to z$ for t'_n and *a fortiori* for t_n. Moreover, all z_k belong to the closed subspace $\mathcal{U}_m$ of $\mathcal{U}$, and the convergence of $\{z_k\}$ to z in $\mathcal{U}$ forces z to lie in $\mathcal{U}_m$. The subspaces $\mathcal{U}_m \cap \mathcal{E}_n$ are thus Fréchet spaces under the topology induced by t'_n. Now, the injection mapping of $\mathcal{U}_m \cap \mathcal{E}_n$ (so topologized) into $\mathcal{U}_m$ is easily seen to have a closed graph (12) and is therefore continuous (by the closed graph theorem ([5], p. 37). There remains but to apply Exercice 13 a), p. 36, of Bourbaki [6] to conclude that $\mathcal{U}_m = \mathcal{U}_m \cap \mathcal{E}_n$ for some index n.

We have now at our disposal the following facts: (a) for any positive integer m there is a corresponding positive integer n such that $\mathcal{U}_m \subset \mathcal{E}_n$, and (b) each φ_i is continuous on $\mathcal{U}_m$ under the topology induced by t'_n. The theorem will follow if we can show that this topology is coarser than t_m. This, however, amounts to showing that the injection mapping of $\mathcal{U}_m$ into $\mathcal{E}_n$ (under t'_n) is continuous, and the proof is again an easy consequence of the closed graph theorem (12). This completes the demonstration that the coefficient functionals φ_i are all continuous on $\mathcal{U}$.

Condition (i) is satisfied whenever $\mathcal{U}$ is sequentially complete. To see this, suppose that B is a bounded subset of $\mathcal{U}$, and let A be the closed, convex, and circled envelope of B in $\mathcal{U}$. Arguments suggested by Bourbaki [6], p. 36, Exercice 13 a), b), show that there exists an index n for which $A \subset \mathcal{U}_n$ and such that A is absorbed by each t_n-neighbourhood of 0 in $\mathcal{U}_n$ (i. e. such that A is t_n-bounded). *A fortiori*, then, B is t_n-bounded. Bourbaki's suggested arguments depend on his Lemma 1 ([6], p. 21) and it is readily verified that completeness of A in this lemma can be replaced by sequential completeness (13).

Condition (ii) is automatically satisfied whenever $\mathcal{U}$ is the *strict* inductive limit of $\{\mathcal{U}_n\}$. Moreover, as pointed out by Bourbaki [6], p. 9, the strict inductive limit $\mathcal{U}$ of a sequence of quasi-complete spaces closed in $\mathcal{U}$ is itself quasi-complete. Since quasi-completeness obviously implies sequential completeness, we have

Corollary 12.1. *If $\mathcal{U}$ is the strict inductive limit of an increasing sequence of Fréchet spaces, then every basis in $\mathcal{U}$ is a Schauder basis.*

(12) Let $\{z_k\}$ be any sequence of points in $\mathcal{U}_m \cap \mathcal{E}_n$, and suppose that $\{z_k\}$ converges to a point z relative to the topology induced by t_m, and to a point z' relative to the topology induced by t'_n. Since $z_k \to z'$ in $\mathcal{U}_n$ and thereby in $\mathcal{U}$, condition (ii) guarantees that $z_k \to z'$ in $\mathcal{U}_m$. Hence $z' = z$.

(13) The precise result required is stated as (b) of Theorem A in [10].

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