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Some properties of two-norm spaces and a characterization of reflexivity of Banach spaces

by

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In this paper we continue the investigations of [2] and [3]. The contents are divided into three sections. In the first we introduce the notion of quasi-normal two-norm spaces generalizing the concept of normal two-norm spaces. We show that all the main properties of normal two-norm spaces are preserved in this case. Introducing the notion of γ-semireflexivity we show that a two-norm space is γ-reflexive if and only if it is γ-semireflexive and quasi-normal.

In the second section we study the structure of the spaces Ξ^* and Ξ_γ. Since W. Orlicz and V. Pták ([14], p. 57) have proved that the space Ξ_γ is always closed in the space $\langle \Xi, \| \| \rangle$ conjugate to $\langle X, \| \| \rangle$ but not every strongly closed subspace of Ξ is a possible space Ξ_γ, it seems worth while to study the structure of Ξ_γ. We consider the space $\langle X, \| \| \rangle$, whence also $\langle \Xi, \| \| \rangle$, as fixed, and we show at first that, in general, there is no finest or coarsest starred norm $\| \|^*$ giving a fixed space Ξ_γ. Next, we give a characterization of starred norms $\| \|^*$ and of the possible spaces Ξ^*; however, we have not succeeded in giving such a characterization of its closures Ξ_γ, so we content ourselves with giving some sufficient conditions for a subspace of Ξ to be a possible space Ξ_γ, and with studying some examples of the possible situation of Ξ_γ in Ξ. It may happen for normal two-norm spaces that only one space Ξ_γ exists (it is always so if the space $\langle X, \| \| \rangle$ is reflexive); it may happen that exactly two possible spaces Ξ_γ exist. There may also exist spaces Ξ_γ of finite deficiency in Ξ; on the other hand, if $\langle X, \| \| \rangle$ is weakly complete, then the possible spaces Ξ_γ have either infinite or null deficiency.

In the third section we apply the results obtained to give a characterization of reflexivity: a Banach space $\langle X, \| \| \rangle$ is reflexive if and only if, for every norm $\| \|^*$ coarser than $\| \|$, the space Ξ^* conjugate to $\langle X, \| \|^* \rangle$ is strongly dense in the space $\langle \Xi, \| \| \rangle$ conjugate to $\langle X, \| \| \rangle$.

Throughout this paper we adopt the terminology of [3]. In particular, the following notations and notions will be used without further reference.

Given a two-norm space $\langle X, \|\,\|, \|\,\|^*\rangle$, we shall always suppose the following condition to be satisfied:

$$(\mathrm{n}_0) \qquad \|x\|^* \leqslant \|x\| \quad \text{for} \quad x \in X.$$

A sequence $\{x_n\}$ is called *γ-convergent to* x_0 (written $x_n \xrightarrow{\gamma} x_0$) if $\|x_n - x_0\|^* \to 0$ together with $\sup\limits_{n=1,2,\ldots} \|x_n\| < \infty$. A functional ξ is called *γ-linear* if it is distributive and if $x_n \xrightarrow{\gamma} x_0$ implies $\xi(x_n) \to \xi(x_0)$. The totality of the γ-linear functionals will be denoted by Ξ_γ. An operation Φ from a two-norm space into another will be called *γ-γ-continuous* if it transforms γ-convergent sequences into γ-convergent sequences; Φ is called a *γ-isomorphism* between X and Y if it is a distributive one-to-one operation from X onto Y, γ-continuous together with the inverse operation. The distributive functionals on X will be denoted by ξ, η, ζ; $\langle \Xi, \|\,\|\rangle$ and $\langle \Xi^*, \|\,\|^*\rangle$ will denote the spaces conjugate to $\langle X, \|\,\|\rangle$ and $\langle X, \|\,\|^*\rangle$, respectively. Thus

$$\|\xi\| = \sup\{\xi(x): x \in X, \|x\| \leqslant 1\},$$
$$\|\xi\|^* = \sup\{\xi(x): x \in X, \|x\|^* \leqslant 1\}.$$

We shall write

$$S = \{x: x \in X, \|x\| \leqslant 1\}, \qquad S^* = \{x: x \in X, \|x\|^* \leqslant 1\},$$
$$\Sigma = \{\xi: \xi \in \Xi, \|\xi\| \leqslant 1\}, \qquad \Sigma^* = \{\xi: \xi \in \Xi^*, \|\xi\|^* \leqslant 1\}.$$

Evidently $S \subset S^*$, $\Xi^* \subset \Xi_\gamma \subset \Xi$, $\Sigma^* \subset \Sigma$.

A subset Ω of Ξ is called *norming for* $\langle X, \|\,\|\rangle$ if there is a constant $r > 0$ such that the functional

$$\|x\|_1 = \sup\{|\xi(x)|: \xi \in \Omega \cap r\Sigma\}$$

is a norm equivalent to $\|\,\|$. A subset Ω of Ξ is called *strictly norming* if each set $A \subset X$ satisfying $\sup\{|\xi(x)|: x \in A\} < \infty$ for every $\xi \in \Omega$ is necessarily bounded with respect to the norm $\|\,\|$. Every linear strictly norming set is norming; every linear, closed and norming set is strictly norming ([1], p. 109).

A two-norm space $\langle X, \|\,\|, \|\,\|^*\rangle$ is called *normal* if $\lim\limits_{n\to\infty} \|x_n - x_0\|^* = 0$ implies $\|x_0\| \leqslant \varliminf\limits_{n\to\infty} \|x_n\|$.

1. Quasi-normal two-norm spaces. The following proposition justifies the definition to be set below:

1.1. Proposition. *Let K be a positive constant; the following statements are equivalent:*

1° $\|x_n - x_0\|^* \to 0$ *implies* $\|x_0\| \leqslant K \varliminf\limits_{n\to\infty} \|x_n\|$,

2° *there exists a norm* $\|\,\|_1$ *such that* $\dfrac{1}{K}\|x\| \leqslant \|x\|_1 \leqslant \|x\|$ *and such that the space* $\langle X, \|\,\|_1, \|\,\|^*\rangle$ *is normal*,

3° $\sup\{\xi(x): \xi \in \Xi^* \cap \Sigma\} \geqslant \dfrac{1}{K}\|x\|$,

4° $\sup\{\xi(x): \xi \in \Xi_\gamma \cap \Sigma\} \geqslant \dfrac{1}{K}\|x\|$,

5° $\Xi^* \cap \Sigma$ *is dense in* $\dfrac{1}{K}\Sigma$ *for the topology* $\sigma(\Xi, X)$,

6° *for every* $x_0 \in X$ *and* $\varepsilon > 0$ *there exists a constant* M *such that* $\|x_0 + z\| \geqslant \dfrac{1}{K}\|x_0\| - M\|z\|^* - \varepsilon$ *for any* $z \in X$.

Proof. 1° $\Rightarrow$ 2°. Let us write

$$\|x\|_1 = \sup\{\xi(x): \xi \in \Xi^* \cap \Sigma\};$$

then obviously $\|x\|_1 \leqslant \|x\|$ (1). Let R denote the closure of the ball S in the space $\langle X, \|\,\|^*\rangle$; the set R is convex and, by condition 1°, it is contained in the ball $K \cdot S = \{x: \|x\| \leqslant K\}$. Given any element $x \neq 0$ of X and arbitrary $\varepsilon > 0$, then the element $y = (K+\varepsilon)\|x\|^{-1}x$ does not belong to R, whence, by a theorem of Eidelheit ([8], [5], p. 22), there exists a functional $\xi \in \Xi^*$ such that $\xi(z) \leqslant 1 \leqslant \xi(y)$ for $z \in R$. Thus $\|\xi\| \leqslant 1$, since $\|z\| \leqslant 1$ implies $z \in R$. On the other hand, the inequality $1 \leqslant \xi\big(\|x\|^{-1}(K+\varepsilon)x\big)$ implies

$$\sup\{\xi(x): \xi \in \Xi^* \cap \Sigma\} \geqslant (K+\varepsilon)^{-1}\|x\|,$$

whence $\|x\|_1 \geqslant K^{-1}\|x\|$.

It remains to prove that the space $\langle X, \|\,\|_1, \|\,\|\rangle$ is normal. Let $\|x_n - x_0\|^* \to 0$, $\varliminf\limits_{n\to\infty} \|x_n\|_1 < \infty$; then x_n is γ-convergent in the space $\langle X, \|\,\|, \|\,\|^*\rangle$. Given $\varepsilon > 0$, there exists a functional ξ_0 in the unit ball Σ_1 of the space $\langle \Xi, \|\,\|_1\rangle$ conjugate to $\langle X, \|\,\|_1\rangle$, such that $\|x_0\|_1 - \varepsilon \leqslant \xi_0(x_0) = \xi_0(x_n) - \xi_0(x_n - x_0) \leqslant \|\xi_0\|_1\|x_n\|_1 + |\xi_0(x_n - x_0)|$, and this implies $\|x_0\| - \varepsilon \leqslant \varliminf\limits_{n\to\infty} \|x_n\|_1$, since $\|\xi_0\|_1 \leqslant 1$ and ξ_0 is γ-linear. The above inequality is trivial if $\varliminf\limits_{n\to\infty} \|x_n\|_1 = \infty$.

(1) The existence of a norm equivalent to $\|\,\|$ and satisfying the equality below was deduced by J. Dixmier ([6], p. 1064) from general considerations.

$2^\circ \Rightarrow 3^\circ$. In this case $\|x_n - x_0\|^* \to 0$ implies $\|x_0\|_1 \leqslant \varliminf_{n\to\infty} \|x_n\|_1$. Rewriting the first part of the proof of the implication $1^\circ \Rightarrow 2^\circ$ with $K = 1$ and with $\|\ \|$ replaced by $\|\ \|_1$, we obtain

$$\sup\{\xi(x): \xi \in \Xi^* \cap \Sigma_1\} \geqslant (1+\varepsilon)^{-1}\|x\|_1 \geqslant K^{-1}(1+\varepsilon)^{-1}\|x\|,$$

where Σ_1 denotes the unit ball of the space conjugate to $\langle X, \|\ \|_1\rangle$, and 3° follows, since obviously $\Sigma_1 \subset \Sigma$.

$3^\circ \Rightarrow 4^\circ$. Obvious.

$4^\circ \Rightarrow 5^\circ$. Follows by a theorem of Dixmier ([6], p. 1062).

$5^\circ \Rightarrow 6^\circ$. Let $x_0 \in X$, $\varepsilon' > 0$; by the definition of the topology $\sigma(\Xi, X)$ and by 5°, for any $\xi \in \frac{1}{K}\Sigma$ there exists functional $\zeta \in \Xi^* \cap \Sigma$ such that $|\zeta(x_0) - \xi(x_0)| < \varepsilon'$. Thus ζ satisfies the inequalities

$$\zeta(x) \leqslant \|x\| \quad \text{for every } x \in X,$$

$$\zeta(x) \geqslant -M\|x\|^* \quad \text{for every } x \in X,$$

$$\zeta(x_0) \geqslant \xi(x_0) - \varepsilon',$$

M being a constant. By a theorem of Mazur and Orlicz ([11], p. 147), the inequality

$$\xi(x_0) - \varepsilon' - M\|z\|^* \leqslant \|x_0 + z\|$$

must be satisfied for all $z \in X$. Choosing ξ and ε' so as $\xi(x_0) = \frac{1}{K}\|x_0\|$ and $\varepsilon' = \varepsilon/K$ we obtain condition 6°.

$6^\circ \Rightarrow 1^\circ$. Setting $z = x_n - x_0$ into 6° we get

$$\|x_0\| \leqslant K\|x_n\| + KM\|x_n - x_0\|^* + K\varepsilon,$$

whence 1° follows.

In the spaces satisfying condition 1° with $K = 1$ we recognize normal two-norm spaces; hence, for $K = 1$, proposition 1.1 gives a characterization of normal two-norm spaces. The spaces for which there exists a constant K such that 1° is satisfied will be called *quasi-normal*. By Proposition 1.1, part 2°, all the properties of two-norm spaces, invariant under equivalent norms $\|\ \|$, possessed by normal two-norm spaces, are possessed also by quasi-normal spaces. In particular we have

THEOREM A. *For quasi-normal spaces the set Ξ_γ is identical with the closure (in the space $\langle \Xi, \|\ \|\rangle$) of the set Ξ^*.*

THEOREM B. *Let $\langle X, \|\ \|, \|\ \|^*\rangle$ be quasi-normal. Then $\Xi_\gamma = \Xi^*$ if and only if the norms $\|\ \|$ and $\|\ \|^*$ are equivalent.*

Let us notice that Proposition 1.1 is also valid except for part 6° in the case when $\langle X, \|\ \|^*\rangle$ is a B_0-space in which

$$\|x\|^* = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{[x]_n}{1+[x]_n},$$

$[\ ]_n$ denoting homogeneous pseudonorms. Since in [2] the space $\langle X, \|\ \|^*\rangle$ was supposed to be a B_0-space, we infer that our Theorems A and B are valid also in this case.

Now let $\langle \Xi^*, \|\ \|^*, \|\ \|\rangle$ and $\langle \mathfrak{X}^{(\gamma)}, \|\ \|, \|\ \|^*\rangle$ be the first and the second γ-conjugate two-norm spaces of $\langle X, \|\ \|, \|\ \|^*\rangle$ ([3], p. 278). The normed spaces $\langle \Xi^*, \|\ \|^*\rangle$ and $\langle \mathfrak{X}^{(\gamma)}, \|\ \|\rangle$ are always complete. The γ-canonical mapping of X into $\mathfrak{X}^{(\gamma)}$ is defined by

$$x \to \mathfrak{y}_x \quad \text{where} \quad \mathfrak{y}_x(\xi) = \xi(x) \quad \text{for} \quad \xi \in \Xi^*$$

(if ξ varies over the whole of Ξ, this mapping is recognized to be the canonical embedding of the space $\langle X, \|\ \|\rangle$ into its second conjugate space $\langle \mathfrak{X}, \|\ \|\rangle$). By the definition of the norm $\|\ \|^*$ in $\mathfrak{X}^{(\gamma)}$, we have

$$\|\mathfrak{y}_x\|^* = \sup\{\xi(x): \xi \in \Xi^* \cap \Sigma^*\} = \|x\|^*;$$

moreover,

$$\|\mathfrak{y}_x\| = \sup\{\xi(x): \xi \in \Xi^* \cap \Sigma\} \leqslant \sup\{\xi(x): \xi \in \Xi \cap \Sigma\} = \|x\|,$$

whence the γ-canonical mapping of $\langle X, \|\ \|, \|\ \|^*\rangle$ into $\langle \mathfrak{X}^{(\gamma)}, \|\ \|, \|\ \|^*\rangle$ is γ-γ-continuous. It is a γ-isomorphism if and only if $\|\mathfrak{y}_x\| \geqslant a\|x\|$ for some $a > 0$. Hence

1.2. PROPOSITION. *The γ-canonical mapping is a γ-isomorphism if and only if the space $\langle X, \|\ \|, \|\ \|^*\rangle$ is quasi-normal.*

1.3. PROPOSITION. *Let the space $\langle X, \|\ \|\rangle$ be complete; then $\langle X, \|\ \|, \|\ \|^*\rangle$ is quasi-normal if and only if the γ-canonical image $\mathfrak{X}_0$ of X is closed in $\langle \mathfrak{X}^{(\gamma)}, \|\ \|\rangle$.*

Proof. Necessity follows by 1.2, sufficiency by Banach's inversion theorem applied to the γ-canonical mapping considered as an operation from $\langle X, \|\ \|\rangle$ onto $\langle \mathfrak{X}_0, \|\ \|\rangle$.

The space $\langle X, \|\ \|, \|\ \|^*\rangle$ will be called *γ-reflexive* if the γ-canonical mapping transforms X onto $\mathfrak{X}^{(\gamma)}$ in a γ-isomorphical fashion ([2]). Thus,

([2]) This definition is more general than an analogous one in [3], since in that case the space $\langle X, \|\ \|, \|\ \|^*\rangle$ was assumed a priori to be normal.

γ-reflexivity implies completeness of the normed space $\langle X, \| \|\rangle$ and quasi-normality of $\langle X, \| \|, \| \|^*\rangle$. The space $\langle X, \| \|, \| \|^*\rangle$ will be called *γ-semireflexive* if the γ-canonical mapping transforms X *onto* $\mathfrak{X}^{(\gamma)}$.

1.4. Proposition. *The following statements are equivalent:*

1° *the space* $\langle X, \| \|, \| \|^*\rangle$ *is γ-reflexive,*

2° *the space* $\langle X, \| \|, \| \|^*\rangle$ *is quasi-normal and γ-semireflexive,*

3° *the space* $\langle X, \| \|\rangle$ *is complete and the space* $\langle X, \| \|, \| \|^*\rangle$ *is γ-semireflexive,*

4° *the space* $\langle X, \| \|, \| \|^*\rangle$ *is quasi-normal and the ball S is conditionally compact for the topology $\sigma(X, \Xi_\gamma)$ (or for $\sigma(X, \Xi^*)$, which amounts to the same).*

Proof. The implications $1^\circ \Rightarrow 2^\circ \Rightarrow 3^\circ \Rightarrow 1^\circ$ follow by the foregoing argument. The equivalency $1^\circ \Leftrightarrow 4^\circ$ may be proved in the same way as Theorem 3.2 of [3], since γ-reflexivity implies quasi-normality, and the set Ξ_γ is in this case strictly norming.

Let us now give a (non-effective) example of a space $\langle X, \| \|, \| \|^*\rangle$ γ-semireflexive and not γ-reflexive. Given an infinitely dimensional reflexive Banach space $\langle X, \| \|^*\rangle$, let ξ_0 be a distributive discontinuous functional on X. Let us introduce the norm $\|x\| = \|x\|^* + |\xi_0(x)|$, and let us consider the spaces $\langle X, \| \|, \| \|^*\rangle$ and $\langle \Xi^*, \| \|^*, \| \|\rangle$. Any functional $\mathfrak{y}$ linear over $\langle \Xi^*, \| \|\rangle$ is linear with respect to the finer norm $\| \|^*$, whence, by reflexivity of $\langle X, \| \|^*\rangle$, $\mathfrak{y}$ is of the form $\mathfrak{y}(\xi) = \xi(x)$. Thus $\langle X, \| \|, \| \|^*\rangle$ is γ-semireflexive. However, by 1.4, the space $\langle X, \| \|, \| \|^*\rangle$ is not γ-reflexive, the space $\langle X, \| \|\rangle$ not being complete. Let us notice that, by a well-known theorem, every linear functional on $\langle X, \| \|\rangle$ is of the form $\zeta(x) + a\xi_0(x)$ with $\zeta \in \Xi^*$; since Ξ_γ is closed in $\langle \Xi, \| \|\rangle$ and since $\Xi^* \subset \Xi_\gamma$, two possibilities may occur: $\Xi_\gamma = \Xi^*$ or $\Xi_\gamma = \Xi$. The second case must be excluded because it implies ([3], p. 290) the normality of $\langle X, \| \|, \| \|^*\rangle$ and, in turn, the γ-reflexivity of $\langle X, \| \|, \| \|^*\rangle$, by Proposition 1.4. This shows that the hypothesis of quasi-normality cannot be omitted in Theorem B.

Next, we shall give an (effective) example of a non-quasi-normal two-norm space. The construction will be based upon an idea of Mazurkiewicz [13]. Let X be the space c_0 of null-convergent sequences $x = \{x_\nu\}$ with $\|x\| = \sup_{\nu=1,2,\ldots} |x_\nu|$. In the space Ξ, conjugate to X, let us consider the functionals

$$\xi_{ik}(x) = \frac{x_1}{2^1} + \frac{x_3}{2^2} + \ldots + \frac{x_{2i-1}}{2^i} + i x_{2N(i,k)},$$

where $(i, k) \to N(i, k)$ is a one-to-one mapping of the set of pairs of

positive integers onto the set of positive integers. Then let us write

$$\|x\|^* = \sum_{i,k=1}^{\infty} \frac{1}{2^{i+k}} |\xi_{ik}(x)|.$$

One can easily show (as in the proof of Theorem 2.5 below) that the space $\langle X, \| \|, \| \|^*\rangle$ is not quasi-normal, for the linear span of the elements ξ_{ik} is a total but not a norming set (Mazurkiewicz [13]).

2. On the structure of the space Ξ^* and Ξ_γ. In this section we shall deal with the problem which are the possible spaces Ξ^* and Ξ_γ, the space $\langle X, \| \|\rangle$ being given.

Given a total subset Ω of the ball Σ, the functional

$$\|x\|^* = \sup\{|\omega(x)| : \omega \in \Omega\}$$

is easily seen to be a norm in X satisfying (n_0). W. Orlicz and V. Pták ([14], p. 63) introduce in this way coarser norms in $\langle X, \| \|\rangle$, restricting themselves, however, to strongly compact subsets Ω of $\langle \Xi, \| \|\rangle$.

Let us observe that all the possible norms may be obtained in such a manner. Indeed

$$\|x\|^* = \sup\{|\zeta(x)| : \zeta \in \Sigma^*\},$$

$\Sigma^* \subset \Sigma$, and, evidently, Σ^* is weakly closed in Ξ (with respect to the topology $\sigma(\Xi, X)$), whence it is weakly compact. Let us denote by Ω_1 the weak closure of the smallest symmetric convex set containing Ω. Then

$$\sup\{|\omega(x)| : \omega \in \Omega\} = \sup\{\omega(x) : \omega \in \Omega_1\}$$

and, since Ω_1 is weakly closed, we get

2.1. Proposition. *Every norm $\| \|^*$ in $\langle X, \| \|\rangle$ satisfying condition (n_0) is of the form*

$$\|x\|^* = \sup\{\omega(x) : \omega \in \Omega\},$$

where Ω is a total convex symmetric subset of Σ, closed with respect to the topology $\sigma(\Xi, X)$. Conversely, every functional of this form is a norm satisfying (n_0).

The structure of all possible spaces Ξ^* may simply be deduced from that of Ω.

2.2. Proposition. *Let Ω and $\| \|^*$ be as in Proposition 2.1. Then the set Ξ^* is equal to the smallest linear set $\mathcal{L}(\Omega)$ spanned upon Ω, i. e. $\Xi^* = \bigcup_{n=1}^{\infty} n\Omega$. Moreover, the unit ball Σ^* induced by $\| \|^*$ is identical with Ω.*

Proof. The inclusion $\Omega \subset \Sigma^*$ is obvious. Let $\zeta \in \Sigma^*$; then

$$|\zeta(x)| \leqslant \|x\|^* = \sup\{\omega(x): \omega \in \Omega\}$$

for every $x \in X$. Suppose, if possible, that $\zeta \notin \Omega$. The closedness of Ω for the topology $\sigma(\Xi, X)$ implies (see e. g. [5], p. 22) the existence of an $x_0 \in X$ such that

$$\sup\{\omega(x_0): \omega \in \Omega\} < \zeta(x_0),$$

which is impossible. Thus $\Sigma^* = \Omega$, and this implies $\Xi^* = \bigcup_{n=1}^{\infty} n\Omega$.

Now we shall be concerned with the study of the possible spaces Ξ_γ, the space $\langle X, \|\,\|\rangle$ being fixed. First we shall be concerned with the norms $\|\,\|^*$ yielding a given space Ξ_γ. Simple examples show that several norms which are non-equivalent (even on S) may exist leading to the same class Ξ_γ of γ-linear functionals. Thus, the question arises whether or not there exists a finest and a coarsest norm $\|\,\|^*$ leading to a given set Ξ_γ. Both questions will be answered in the negative: the first by 2.3 and the second by 2.4.

2.3. Proposition. *Let the space $\langle X, \|\,\|, \|\,\|^*\rangle$ be quasi-normal and let the norms $\|\,\|$ and $\|\,\|^*$ be non-equivalent. Then there exists a norm $\|\,\|_1^*$, essentially finer than $\|\,\|^*$, giving rise to the same set Ξ_γ as $\|\,\|^*$ and such that the convergences γ in the spaces $\langle X, \|\,\|, \|\,\|^*\rangle$ and $\langle X, \|\,\|, \|\,\|_1^*\rangle$ are equivalent.*

Proof. By Theorem B, the sets Ξ_γ and Ξ^* are different. Let $\xi_0 \in \Xi_\gamma \setminus \Xi^*$, $\|\xi_0\| = 1$ and let us introduce a new coarser norm $\|x\|_1^* = \frac{1}{2}(\|x\|^* + |\xi_0(x)|)$. Then the convergences γ in $\langle X, \|\,\|, \|\,\|^*\rangle$ and in $\langle X, \|\,\|, \|\,\|_1^*\rangle$ are identical. Since ξ_0 is continuous with respect to $\|\,\|_1^*$ but not with respect to $\|\,\|^*$, the norms $\|\,\|^*$ and $\|\,\|_1^*$ are not equivalent.

The negative answer to the second question will be obtained by considering the following example.

Let $\boldsymbol{c}$ denote the space of all convergent sequences $x = \{x_\nu\}$, let $\|x\| = \sup_{\nu=1,2,\ldots} |x_\nu|$, and let $\|\,\|^*$ be any coarser norm in $\langle \boldsymbol{c}, \|\,\|\rangle$. We shall denote by e_n the n-th unit vector in $\boldsymbol{c}$ and by η_n the n-th functional on $\boldsymbol{c}$, biorthogonal to $\{e_\nu\}$. The space $\langle \Xi, \|\,\|\rangle$ conjugate to $\langle \boldsymbol{c}, \|\,\|\rangle$ is equivalent to the space $\boldsymbol{l}^1$ ([4], p. 66), and η_n is the n-th unit vector in $\boldsymbol{l}^1$.

Let Λ denote the set of all functionals on $\langle \boldsymbol{c}, \|\,\|\rangle$ of the form $\xi(x) = \sum_{n=1}^{\infty} a_n x_n$ with $\sum_{n=1}^{\infty} |a_n| < \infty$.

2.4. Proposition. *There exists no coarsest norm $\|\,\|^*$ in the space $\langle \boldsymbol{c}, \|\,\|\rangle$ such that $\Xi_\gamma = \Lambda$.*

Proof. At first, let us remark that if, for a coarser norm $\|\,\|^*$, the set Ξ_γ is equal to Λ, then $\langle \boldsymbol{c}, \|\,\|, \|\,\|^*\rangle$ is normal (by 1.1). Let $\|\,\|^*$ be such a norm; we shall prove the existence of a norm $\|\,\|^{**}$ in $\boldsymbol{c}$ such that $\|x\|^{**} \leqslant \|x\|^*$ for all x, yielding the set Λ as the corresponding space Ξ_γ, and essentially coarser than $\|\,\|^*$. By Theorem A, the set Ξ^* is dense in Λ with respect to the norm $\|\xi\| = \sum_{n=1}^{\infty} |a_n|$. Thus, for every n, m, there exists a functional ζ_{nm} such that $\|\zeta_{nm}\| < 1$ and $\|\zeta_{nm} - \eta_n\| < 1/m$. Every linear functional on $\langle \boldsymbol{c}, \|\,\|\rangle$ is of the form ([4], p. 66)

$$\xi(x) = \alpha \lim_{\nu\to\infty} x_\nu + \sum_{\nu=1}^{\infty} \xi(e_\nu) x_\nu,$$

$\alpha = \alpha(\xi)$ being independent of x, and $\|\xi\| = |\alpha| + \sum_{\nu=1}^{\infty} |\xi(e_\nu)|$. Hence

$$|\zeta_{nm}(e_n) - \eta_n(e_n)| < \frac{1}{m} \quad \text{and} \quad |\alpha(\zeta_{nm})| + \sum_{\nu \neq n} |\zeta_{nm}(e_\nu)| < \frac{1}{m}.$$

Obviously the set $\{\zeta_{n,m}: n, m = 1, 2, \ldots\}$ is total. Let us choose positive numbers $a_{\mu\nu}$ so that

$$\sum_{\mu,\nu=1}^{\infty} a_{\mu\nu} < 1, \qquad \sum_{\mu,\nu=1}^{\infty} a_{\mu\nu} \|\zeta_{\mu\nu}\| < 1 \quad \text{and} \quad \sum_{\mu,\nu=1}^{\infty} a_{\mu\nu} \|\zeta_{\mu\nu}\|^* < 1.$$

Then $\|x\|^{**} = \sum_{\mu,\nu=1}^{\infty} a_{\mu\nu} |\zeta_{\mu\nu}(x)|$ is a norm in $\boldsymbol{c}$ (since $\{\zeta_{nm}\}$ is total), and

$$\|x\|^{**} \leqslant \sum_{\mu,\nu=1}^{\infty} a_{\mu\nu} \|\zeta_{\mu\nu}\|^* \|x\|^* \leqslant \|x\|^* \leqslant \|x\|.$$

We shall prove that the norm $\|\,\|^{**}$ is essentially coarser than $\|\,\|^*$. Indeed, let $z_n = e_n/\|e_n\|^*$. Then $\|z_n\|^* = 1$ and, on the other hand,

$$\|z_n\|^{**} = \sum_{\mu=1}^{\infty} \sum_{\nu=1}^{\infty} a_{\mu\nu} |\zeta_{\mu\nu}(z_n)| = \sum_{\mu=1}^{m-1} \sum_{\nu=1}^{m-1} + \sum_{\mu=1}^{m-1} \sum_{\nu=m}^{\infty} + \sum_{\mu=m}^{\infty} \sum_{\nu=1}^{\infty} = I_1 + I_2 + I_3$$

and

$$I_2 = \sum_{\mu=1}^{m-1} \sum_{\nu=1}^{\infty} a_{\mu\nu} |\zeta_{\mu\nu}(z_n)| \leqslant \sum_{\mu=1}^{m} \sum_{\nu=m}^{\infty} a_{\mu\nu} \|\zeta_{\mu\nu}\|^* \to 0 \quad \text{as } m \to \infty,$$

$$I_3 = \sum_{\mu=m}^{\infty} \sum_{\nu=1}^{\infty} a_{\mu\nu} |\zeta_{\mu\nu}(z_n)| \leqslant \sum_{\mu=m}^{\infty} \sum_{\nu=1}^{\infty} a_{\mu\nu} \|\zeta_{\mu\nu}\|^* \to 0 \quad \text{as } m \to \infty,$$

uniformly with respect to n. Since $1 \geqslant \|\zeta_{\mu\nu}\| \geqslant \sum_{n=1}^{\infty} |\zeta_{\mu\nu}(e_n)|$ for every μ and ν, we have $\lim_{n\to\infty} \zeta_{\mu\nu}(e_n) = 0$, whence $I_1 \to 0$ as $n \to \infty$. We have thus proved that $\|z_n\|^{**} \to 0$.

It remains to prove that the set of all γ-linear functionals on $\langle \mathbf{c}, \|\ \|, \|\ \|^{**}\rangle$ is identical with Λ. Since $\|\ \|^{**}$ is coarser than $\|\ \|^*$, the space Ξ^{**} conjugate to $\langle \mathbf{c}, \|\ \|^{**}\rangle$ is contained in Λ. On the other hand, the functionals ζ_{mn} belong to Λ, their strong limits η_n belong to Ξ_γ (by Theorem A), whence $\Xi_\gamma = \Lambda$.

We shall now give a sufficient condition for a subset of Ξ to be a possible space Ξ_γ.

2.5. THEOREM. *Let Υ be a linear, closed, norming and separable subset of $\langle \Xi, \|\ \|\rangle$. Then there exists a coarser norm $\|\ \|^*$ in X such that the space $\langle X, \|\ \|, \|\ \|^*\rangle$ is normal, γ-precompact* (3) *and such that $\Xi_\gamma = \Upsilon$.*

Proof. Let $\xi_1, \xi_2, \ldots$ be a sequence strongly dense in $\Upsilon \cap \Sigma$ and let us write

$$\|x\|^* = \sum_{n=1}^{\infty} \frac{1}{2^n} |\xi_n(x)|, \qquad \|x\|_1^* = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|\xi_n(x)|}{1+|\xi_n(x)|}.$$

The space $\langle X, \|\ \|_1^*\rangle$ is of B_0^*-type. The γ-convergences in $\langle X, \|\ \|, \|\ \|^*\rangle$ and in $\langle X, \|\ \|, \|\ \|_1^*\rangle$ coincide, whence Ξ_γ is the same for both spaces. Evidently $\Xi_\gamma \supset \Upsilon$, whence, by 1.1, the spaces $\langle X, \|\ \|, \|\ \|^*\rangle$ and $\langle X, \|\ \|, \|\ \|_1^*\rangle$ are quasi-normal. By a theorem of Mazur and Orlicz ([12], p. 139), the space Ξ_1^* conjugate to $\langle X, \|\ \|_1^*\rangle$ is identical with the smallest linear set spanned upon the functionals $\xi_1, \xi_2, \ldots$, whence, by Theorem A, $\Xi_\gamma = \Upsilon$. The usual diagonal method shows that $\langle X, \|\ \|, \|\ \|^*\rangle$ is γ-precompact.

2.6. PROPOSITION. *Let $\langle X, \|\ \|\rangle$ be separable, and let $\langle \Xi, \|\ \|\rangle$ be non-separable. Then there exist uncountably many coarser norms $\|\ \|_\tau^*$ such that $\langle X, \|\ \|, \|\ \|_\tau^*\rangle$ are normal for all τ and such that the spaces Ξ_γ are different for different τ.*

Proof. Let $x_1, x_2, \ldots$ be a sequence dense in S with respect to $\|\ \|$, and let $\zeta_1, \zeta_2, \ldots$ be functionals linear on $\langle X, \|\ \|\rangle$ satisfying $\zeta_n(x_n) = \|x_n\|$ and $\|\zeta_n\| = 1$ for $n = 1, 2, \ldots$ Obviously, the smallest strongly closed linear set Υ spanned upon $\zeta_1, \zeta_2, \ldots$ satisfies the assumptions of Theorem 2.5 and the family of all separable over-sets of Υ, inducing various starred norms in $\langle X, \|\ \|\rangle$ following Theorem 2.5, is, by non-separability of $\langle \Xi, \|\ \|\rangle$, uncountable.

(3) $\langle X, \|\ \|, \|\ \|^*\rangle$ is called *γ-precompact* if every bounded set in $\langle X, \|\ \|\rangle$ is conditionally compact with respect to the metric $\varrho(x, y) = \|x-y\|^*$. This theorem is closely related to a theorem of Orlicz and Pták ([14], p. 64).

The next example shows that, as regards a normal two-norm space, in some cases there may exist exactly one non-trivial space Ξ_γ, and that the space Ξ_γ may have deficiency 1.

2.7. THEOREM. *Let $\|\ \|^*$ be a coarser norm in $\langle \mathbf{c}, \|\ \|\rangle$. Then $\langle \mathbf{c}, \|\ \|, \|\ \|^*\rangle$ is normal if and only if the set Ξ_γ contains all the functionals η_n; thus all the possible spaces Ξ_γ for normal $\langle \mathbf{c}, \|\ \|, \|\ \|^*\rangle$ are either $\Xi_\gamma = \Lambda$ or $\Xi_\gamma = \Xi$.*

For every n there exists a coarser norm $\|\ \|_n^*$ in $\mathbf{c}$ such that the corresponding space Ξ_γ has deficiency n (in the case $n > 1$, of course, the space $\langle \mathbf{c}, \|\ \|, \|\ \|^*\rangle$ is not normal but only quasi-normal). Moreover, there exists a coarser norm $\|\ \|_\infty^*$ in $\mathbf{c}$ such that the corresponding space Ξ_γ has infinite deficiency.

Proof. We prove first that $\eta_n \in \Xi_\gamma$ if $\langle \mathbf{c}, \|\ \|, \|\ \|^*\rangle$ is normal, $n = 1, 2, \ldots$ By Proposition 1.1, $\Xi^* \cap \Sigma$ is dense in Σ for the topology $\sigma(\Xi, X)$, whence, for every $\varepsilon > 0$ and k, there exists a functional

$$\zeta(x) = b_0 \lim_{n\to\infty} x_n + \sum_{n=1}^{\infty} b_n x_n$$

belonging to $\Xi^* \cap \Sigma$ and such that $|1-b_k| = |\eta_k(e_k) - \zeta(e_k)| < \varepsilon/2$. Since $1 \geqslant \|\zeta\| = |b_k| + \sum_{\nu\neq k} |b_\nu|$, we get

$$\sum_{\nu\neq k} |b_\nu| \leqslant 1-|b_k| \leqslant |1-b_k| < \frac{\varepsilon}{2}.$$

Hence, for any $x \in S$,

$$|\eta_k(x) - \zeta(x)| = |(1-b_k)x_k - \sum_{\nu\neq k} b_\nu x_\nu| \leqslant |1-b_k| \cdot \|x\| + \|x\| \sum_{\nu\neq k} |b_\nu| < \varepsilon;$$

it follows that $\|\eta_k - \zeta\| < \varepsilon$. The number $\varepsilon > 0$ being arbitrary, η_k belongs to the closure of Ξ^* in $\langle \Xi, \|\ \|\rangle$, i. e. to Ξ_γ.

The proof of the second part will be preceded by the following considerations. Let $\langle X, \|\ \|, \|\ \|^*\rangle$ and $\langle Y, \|\ \|, \|\ \|^*\rangle$ be two quasi-normal spaces. Their Cartesian product $Z = X \times Y$, with the norms $\|z\| = \|x\| + \|y\|$ and $\|z\|^* = \|x\|^* + \|y\|^*$ for $z = (x, y)$, is easily seen to be a quasi-normal two-norm space (4). One may easily prove that the general form

(4) If the initial spaces are normal, their product, although quasi-normal, does not need to be normal. On the other hand, if the finer norm in Z is defined by $\|z\| = \max(\|x\|, \|y\|)$ and if $\langle X, \|\ \|, \|\ \|^*\rangle$ and $\langle Y, \|\ \|, \|\ \|^*\rangle$ are normal, then, for any starred norm $\|\ \|^*$ such that $\|z_n\|^* \to 0$ is equivalent to $\|x_n\|^* \to 0$ with $\|y_n\|^* \to 0$, the space $\langle Z, \|\ \|, \|\ \|^*\rangle$ is also normal.

of γ-linear functionals on $\langle Z, \| \|, \| \|^*\rangle$ is

$$\zeta(z) = \xi(x) + \eta(y),$$

where ξ is a γ-linear functional on $\langle X, \| \|, \| \|^*\rangle$ and η is a γ-linear functional on $\langle Y, \| \|, \| \|^*\rangle$.

Let n be finite. Then the space c is isomorphic to its n-tuple Cartesian product (see [4], p. 182). Thus, by the foregoing argument, there is a coarser norm $\| \|_1^*$ in c such that the corresponding space Ξ_γ is of deficiency n.

Now let us pass to the case $n = \aleph_0$. Let E be the set of the ordinals $\leqslant \omega^2$, with the order topology. The space $C(E)$ of real continuous functions on E is a Banach space with the norm $\|x\| = \sup\{|x(t)| : t \in E\}$ and any linear functional on $C(E)$ is of the form

$$\xi(x) = \sum_{\varphi \leqslant \omega^2} a_\varphi x(\varphi) \quad \text{and} \quad \|\xi\| = \sum_{\varphi \leqslant \omega^2} |a_\varphi| < \infty .$$

Next, let E_0 be the set of the isolated points of E (i.e. the set of non-limit ordinals), let $t_1, t_2, \ldots$ be any arrangement of all elements of E_0 into a sequence, and let

$$\|x\|^* = \sum_{m=1}^{\infty} \frac{1}{2^m} |x(t_m)| .$$

It is obvious that $\langle C(E), \| \|, \| \|^*\rangle$ is a normal two-norm space, and every γ-linear functional on $\langle C(E), \| \|, \| \|^*\rangle$ is of the form

$$\xi(x) = \sum_{m=1}^{\infty} a_m x(t_m) \quad \text{with} \quad \sum_{m=1}^{\infty} |a_m| < \infty .$$

Thus, in this case Ξ_γ is of infinite deficiency in Ξ. It is known that the space $\langle C(E), \| \|\rangle$ is isomorphic to the space c; this isomorphism induces in c a norm $\| \|_\infty^*$ (corresponding to the norm $\| \|^*$ defined above). Obviously $\| \|_\infty^*$ is the required one.

For the space c_0 of null-convergent sequences we obtain by the preceding considerations:

2.8. PROPOSITION. *Let $\| \|^*$ be a coarser norm in $\langle \mathbf{c}_0, \| \|\rangle$ such that $\langle \mathbf{c}_0, \| \|, \| \|^*\rangle$ is normal. Then $\Xi_\gamma = \Xi$.*

The following propositions give further information about the possible spaces Ξ_γ.

2.9. PROPOSITION. *Suppose $\langle X, \| \|, \| \|^*\rangle$ to be quasi-normal and let the deficiency of Ξ_γ be finite and greater than $m-1$; then there exist at least $m+1$ coarser norms $\| \|^*, \| \|_1^*, \ldots, \| \|_m^*$ in $\langle X, \| \|\rangle$ leading to different classes $\Xi_\gamma, \Xi_\gamma^{(1)}, \ldots, \Xi_\gamma^{(m)}$ of γ-linear functionals.*

Proof. Let $\eta \in \Xi \setminus \Xi_\gamma$ and let $\|\eta\| = 1$. Then the norm

$$\|x\|_1^* = \max(\|x\|^*, |\eta(x)|)$$

is finer than $\| \|^*$ and coarser than $\| \|$, and the space $\langle X, \| \|, \| \|_1^*\rangle$ is quasi-normal. Let ξ be any γ-linear functional on $\langle X, \| \|, \| \|_1^*\rangle$. Then, by Theorem A, there exist linear functionals ξ_n on $\langle X, \| \|_1^*\rangle$ such that $\|\xi - \xi_n\| \to 0$. We may write $\xi_n(x) = \zeta_n(x) + \lambda_n \eta(x)$, where ζ_n are linear functionals on $\langle X, \| \|^*\rangle$. If $\lambda_n \neq 0$, then the closedness of Ξ_γ in Ξ implies

$$\|\xi_n\| = \|\zeta_n + \lambda_n \eta\| = |\lambda_n| \, \|\lambda_n^{-1}\zeta_n + \eta\| \geqslant |\lambda_n| \inf\{\|\eta - \zeta\| : \zeta \in \Xi_\gamma\} = |\lambda_n| \cdot \delta,$$

whence $\sup_{n=1,2,\ldots} |\lambda_n| \leqslant \delta^{-1} \sup_{n=1,2,\ldots} \|\xi_n\| < \infty$, for $\|\xi_n\| \to \|\xi\|$. Thus there exists a subsequence $\lambda_{n_k} \to \lambda_0$ such that

$$\zeta_{n_k} = \xi_{n_k} - \lambda_{n_k}\eta \to \xi - \lambda_0 \eta = \zeta_0$$

and $\zeta_0 \in \Xi_\gamma$, which means that the space $\Xi_\gamma^{(1)}$ of all γ-linear functionals on $\langle X, \| \|, \| \|_1^*\rangle$ is identical with the smallest linear set spanned upon Ξ_γ and η.

In this way we may construct starred norms

$$\|x\|_1^* \leqslant \|x\|_2^* \leqslant \ldots \leqslant \|x\|_m^* \leqslant \|x\|$$

such that $\Xi_\gamma^{(1)} \subset \Xi_\gamma^{(2)} \subset \ldots \subset \Xi_\gamma^{(m)}$ and $\Xi_\gamma^{(i)} \neq \Xi_\gamma^{(i+1)}$.

2.10. THEOREM. *Let $\langle X, \| \|\rangle$ be a weakly complete* (5) *Banach space. Then, for any coarser norm $\| \|^*$, either $\Xi_\gamma = \Xi$ or the quotient space Ξ/Ξ_γ is non-separable.*

In particular, if $\langle X, \| \|\rangle$ is weakly complete and if Ξ_γ has finite deficiency in Ξ, then $\Xi_\gamma = \Xi$.

Proof. Let us assume Ξ/Ξ_γ to be separable. Ξ/Ξ_γ consists of the cosets with respect to the relation;

$$\xi_1 \sim \xi_2 \quad \text{if} \quad \xi_1 - \xi_2 \in \Xi_\gamma .$$

The norm of the coset $\bar{\xi}$ corresponding to a functional ξ is defined by

$$\|\bar{\xi}\| = \inf\{\|\xi + \eta\| : \eta \in \Xi_\gamma\} .$$

The coset corresponding to 0 is identical with Ξ_γ. Let ξ_n be a sequence of linear functionals on $\langle X, \| \|\rangle$ such that the corresponding cosets $\bar{\xi}_1 = \xi_1 + \Xi_\gamma$, $\bar{\xi}_2 = \xi_2 + \Xi_\gamma, \ldots$ are dense in $\langle \Xi/\Xi_\gamma, \| \|\rangle$. Let x_n

(5) i. e. sequentially weakly complete.

be a sequence γ-convergent to 0, and let $x_n' = x_{k_n}$ be an arbitrary subsequence. The sequence x_n' contains a subsequence $x_n'' = x_{l_n}'$ such that every sequence $\xi_k(x_n'')$ is convergent $(k = 1, 2, \ldots)$. Since $x_n'' \xrightarrow{\gamma} 0$, $\zeta(x_n'') \to 0$ for every $\zeta \in \Xi_\gamma$. It is easily seen that the set Γ of all functionals of the form $\zeta + \xi_k$ with $\zeta \in \Xi_\gamma$ and $k = 1, 2, \ldots$ is strongly dense in $\langle \Xi, \| \, \| \rangle$. Since the sequence $\eta(x_n'')$ is convergent for every $\eta \in \Gamma$ and $\sup \|x_n''\| < \infty$, by the Banach-Steinhaus theorem, the sequence x_n'' is weakly fundamental. $\langle X, \| \, \| \rangle$ being weakly complete, there exists an element x_0 such that $\xi(x_n'' - x_0) \to 0$ for every $\xi \in \Xi$. Since Ξ_γ is total and $\zeta(x_0) = 0$ for all $\zeta \in \Xi_\gamma$, x_0 must be equal to 0.

Thus, we have proved that x_n'' tends weakly to 0; x_n' being an arbitrary subsequence of x_n, x_n tends weakly to 0, and, by Proposition 6.2 of [3], we obtain $\Xi_\gamma = \Xi$.

2.11. PROPOSITION. *Let $\langle X, \| \, \| \rangle$ be a Banach space such that the canonical image $\mathfrak{X}_0$ of X in the biconjugate space is of finite* ([6]) *deficiency k in $\mathfrak{X}$. Then, for any coarser norm $\| \, \|^*$ in $\langle X, \| \, \| \rangle$, either Ξ_γ is of deficiency not greater than k in Ξ, or $\Xi_\gamma = \Xi$.*

Proof. Let us suppose that $\| \, \|^*$ is a coarser norm in $\langle X, \| \, \| \rangle$ such that the space Ξ_γ is of deficiency greater than k in Ξ. Since Ξ_γ is closed in $\langle \Xi, \| \, \| \rangle$, this implies the existence of a closed subspace $\Gamma \subset \Xi$ of deficiency $p = k+1$ with $\Xi_\gamma \subset \Gamma$. Thus every linear functional $\xi \in \Xi$ may be uniquely represented as $\xi(x) = \eta(x) + \alpha_1\zeta_1(x) + \ldots + \alpha_p\zeta_p(x)$, where $\eta \in \Gamma$ and $\zeta_1, \ldots, \zeta_p$ are fixed. Let us write $\mathfrak{z}_\nu(\xi) = \alpha_\nu$ $(\nu = 1, 2, \ldots \ldots, p)$; $\mathfrak{z}_\nu$ are obviously linear functionals on $\langle \Xi, \| \, \| \rangle$ and they are linearly independent. No functional $\mathfrak{y}_x \neq 0$ is equal to a linear combination $\beta_1\mathfrak{z}_1 + \ldots + \beta_p\mathfrak{z}_p$ of $\mathfrak{z}_1, \ldots, \mathfrak{z}_p$, since if it were so, we should have $\mathfrak{z}_r(\eta) = 0$ and consequently $\mathfrak{y}_x(\eta) = \eta(x) = 0$ for $\eta \in \Gamma$, which implies $x = 0$, Γ being total. This means that the deficiency of $\mathfrak{X}_0$ in $\mathfrak{X}$ is greater than k, contrarily to our hypothesis.

3. A characterization of reflexivity of Banach spaces. In this section $\langle X, \| \, \| \rangle$ will stand for a Banach space; $\langle \Xi, \| \, \| \rangle$, $\langle \mathfrak{X}, \| \, \| \rangle$, Σ, S, $\mathfrak{y}_x$, $\mathfrak{X}_0$ etc. will preserve their previous meaning.

3.1. LEMMA. *Let $\mathfrak{z} \in \mathfrak{X}$ be a linear functional on $\langle \Xi, \| \, \| \rangle$ not belonging to the canonical image $\mathfrak{X}_0$ of X. Then the set*

$$\Omega = \{\xi : \xi \in \Xi, \mathfrak{z}(\xi) = 0\}$$

is strictly norming for the space $\langle X, \| \, \| \rangle$ ([7]).

([6]) The existence of such spaces has been proved by R. C. James [10].

([7]) This lemma may be deduced from a general result of J. Dixmier ([6], p. 1064). We give here an elementary and effective proof.

Proof. We may suppose freely that $\|\mathfrak{z}\| = 1$. Ω being linear and closed in $\langle \Xi, \| \, \| \rangle$, it is sufficient to prove that Ω is norming. Let $\delta = \inf\{\|\mathfrak{z} - \mathfrak{y}\| : x \in X\}$; then $0 < \delta \leqslant 1$. We shall prove that, for every $x_0 \in X$, there exists a functional $\zeta \in \Omega \cap \Sigma$ such that $\zeta(x_0) = \frac{1}{4}\delta\|x_0\|$, or, which amounts to the same, that a functional $\zeta \in \Xi$ exists such that

$$\|\zeta\| \leqslant 1, \quad \mathfrak{z}(\zeta) = 0, \quad \mathfrak{y}_z(\zeta) = \tfrac{1}{4}\delta, \quad \text{where} \quad x_0 = z\cdot\|x_0\|, \ \|z\| = 1 = \|\mathfrak{y}_z\|.$$

By a theorem of Helly (see [9], [11], p. 171, [5], p. 38), such a functional exists if the inequality

$$|\lambda_1 \cdot \tfrac{1}{4}\delta + \lambda_2 \cdot 0| \leqslant (1-\varepsilon)\|\lambda_1\mathfrak{y}_z + \lambda_2\mathfrak{z}\|$$

is satisfied for every pair λ_1, λ_2 of real numbers and for an $\varepsilon > 0$. For $\lambda_1 = 0$ this inequality is obvious; if $\lambda_1 \neq 0$, then setting $t = \lambda_1^{-1}\lambda_2$ and $\varepsilon = \frac{1}{2}$ we obtain the inequality $\delta \leqslant 2\|\mathfrak{y}_z + t\mathfrak{z}\|$, which will be proved now. If $|t| \geqslant \frac{1}{2}$, we have

$$2\|\mathfrak{y}_z + t\mathfrak{z}\| = 2|t|\,\|t^{-1}\mathfrak{y}_z + \mathfrak{z}\| \geqslant 2|t|\cdot\delta \geqslant \delta;$$

if $|t| < \frac{1}{2}$, then $2\|\mathfrak{y}_z + t\mathfrak{z}\| \geqslant 2(\|\mathfrak{y}_z\| - |t|\,\|\mathfrak{z}\|) \geqslant 2(1-\frac{1}{2}) \geqslant \delta$.

Now, let X_0 be a closed subspace of $\langle X, \| \, \| \rangle$ and let $\| \, \|_0^*$ be a norm in X_0 satisfying $\|x\|_0^* \leqslant \|x\|$ for $x \in X_0$; let us write

$$S_0^* = \{x : x \in X_0, \|x\|_0^* \leqslant 1\}, \qquad S^* = \operatorname{conv}(S \cup S_0^*),$$

and let $\|x\|^*$ be the Minkowski functional of S^* in X.

3.2. LEMMA. *Under the above notation*

(a) *the functional $\| \, \|^*$ is a norm satisfying $\|x\|^* \leqslant \|x\|$ for $x \in X$ and $\|x\|^* = \|x\|_0^*$ for $x \in X_0$,*

(b) *every linear functional on $\langle X_0, \| \, \|_0^* \rangle$ may be extended onto X with the preservation of both norms: relative to $\langle X_0, \| \, \|^* \rangle$ and relative to $\langle X_0, \| \, \| \rangle$,*

(c) *the space $\langle X, \| \, \|, \| \, \|^* \rangle$ is quasi-normal if and only if $\langle X_0, \| \, \|, \| \, \|_0^* \rangle$ is so* ([8]).

Proof. Since S and S_0^* are convex, S^* is identical with the set of all elements of form $z = ts + (1-t)s_0$ with $s \in S$, $s_0 \in S_0^*$, $0 \leqslant t \leqslant 1$. We shall prove first that $S^* \cap X_0 = S_0^*$. The inclusion $S_0^* \subset S^* \cap X_0$ being obvious, let us assume that $z \in S^* \cap X_0$, whence $z = ts + (1-t)s_0$, $s \in S$, $s_0 \in S_0^*$, $0 \leqslant t \leqslant 1$. If $t = 0$, then $z \in S_0^*$; if $t \neq 0$, then $s = t^{-1}[z - (1-t)s_0]$ belongs to X_0, whence $s \in X_0 \cap S \subset S_0^*$, which implies $z \in S_0^*$.

The identity $S^* \cap X_0 = S_0^*$ implies $\|x\|^* = \|x\|_0^*$ for $x \in X_0$. The inclusion $S \subset S^*$ implies $\|x\|^* \leqslant \|x\|$ in X.

([8]) One can easily verify that the extension of norm $\| \, \|_0^*$ by this method coincides with that given by A. Sobczyk [15].

We shall prove now that $\| \|^*$ is a norm. Suppose that $\|z\|^* = 0$. Then $nz \in S^*$ for $n = 1, 2, \ldots$, whence $nz = \vartheta_n x_n + (1-\vartheta_n) y_n$ with $x_n \in S$, $y_n \in S_0^*$, $0 \leqslant \vartheta_n \leqslant 1$, which implies

$$\inf\{\|z-u\|: u \in X_0\} \leqslant \left\| z - \frac{(1-\vartheta_n)y_n}{n} \right\| = \frac{\vartheta_n}{n}\|x_n\| \leqslant \frac{1}{n}.$$

It follows that $z \in X_0$, since X_0 is closed in $\langle X, \| \|\rangle$. Therefore $\|z\|_0^* = \|z\|^* = 0$, which gives $z = 0$.

To prove (b) let ζ be any linear functional on $\langle X_0, \| \|_0^*\rangle$. We shall prove that its Hahn-Banach extension $\bar\zeta$ (preserving the norm $\|\zeta\|_0 = \sup\{\zeta(x): x \in X_0 \cap S\}$) satisfies the desired conditions. It is to be proved that the norm $\| \|^*$ defined by $\|\zeta\|^* = \sup\{\zeta(x): \|x\|^* \leqslant 1,\ x \in X_0\}$ does not increase as well. Indeed, if $z \in S^*$, then $z = ts + (1-t)s_0$, $s \in S$, $s_0 \in S_0^*$, $0 \leqslant t \leqslant 1$, whence, by $\|\bar\zeta\| = \|\zeta\| \leqslant \|\zeta\|^*$,

$$\bar\zeta(z) = t\bar\zeta(s) + (1-t)\zeta(s_0) \leqslant t\|\bar\zeta\|\|s\| + (1-t)\|\zeta\|^*\|s_0\|^* \leqslant \|\zeta\|^*.$$

Let Ξ_0^* be the space conjugate to $\langle X_0, \| \|_0^*\rangle$, let Ξ^* be the space conjugate to $\langle X, \| \|^*\rangle$. To prove (c) it is sufficient to show that the set Ξ^* is norming for $\langle X, \| \|^*\rangle$ if Ξ_0^* is norming for $\langle X_0, \| \|_0^*\rangle$. Thus, let us assume that there exists an $A > 0$ such that

$$\sup\{\zeta(x): \zeta \in \Xi_0^* \cap \Sigma\} \geqslant A\|x\| \quad \text{for} \quad x \in X_0.$$

By Proposition 1.1, it is sufficient to prove that there is a constant K such that $\|x_n\| \leqslant 1$, $\|x_n - x_0\|^* \to 0$ implies $\|x_0\| \leqslant K$. Since $\|x_n\|^* \leqslant \|x_n\| \leqslant 1$, we infer that $\|x_0\|^* \leqslant 1$, i. e. that $x_0 \in S^*$, whence $x_0 = ts + (1-t)s_0$, $s \in S$, $s_0 \in S_0^*$, $0 \leqslant t \leqslant 1$. Let $\zeta \in \Xi_0^* \cap \Sigma$ be such that $\zeta(s_0) \geqslant A\|s_0\|$, and let $\bar\zeta$ be the extension of ζ as in (b). Then $|\bar\zeta(x_n)| \leqslant 1$, since $\bar\zeta \in \Sigma$, and $\bar\zeta(x_n) \to \bar\zeta(x_0)$, since $\bar\zeta \in \Xi^*$, whence $|\bar\zeta(x_0)| \leqslant 1$. On the other hand, $1 \geqslant \bar\zeta(x_0) = t\bar\zeta(s) + (1-t)\zeta(s_0)$ and $|\bar\zeta(s)| \leqslant 1$, for $\bar\zeta \in \Sigma$, $s \in S$. Thus $(1-t)A\|s_0\| \leqslant (1-t)\zeta(s_0) \leqslant 1 + t|\bar\zeta(s)| \leqslant 2$, and

$$\|x_0\| \leqslant \|ts + (1-t)s_0\| \leqslant t\|s\| + (1-t)\|s_0\| \leqslant t + \frac{2}{A} \leqslant 1 + \frac{2}{A} = K.$$

3.3. Theorem. *A Banach space $\langle X, \| \|\rangle$ is reflexive if and only if, for every norm $\| \|^*$ coarser than $\| \|$, the space Ξ^* conjugate to $\langle X, \| \|^*\rangle$ is dense in $\langle \Xi, \| \|\rangle$.*

Proof. Necessity is stated in Theorem 3.7 of [3]. Let $\langle X, \| \|\rangle$ be any non-reflexive Banach space. To prove the sufficiency we shall show that there exists a norm $\| \|^*$ in $\langle X, \| \|\rangle$ coarser than $\| \|$ and such that the space Ξ^* is not dense in $\langle \Xi, \| \|\rangle$ and is norming for $\langle X, \| \|\rangle$.

From the non-reflexivity of $\langle X, \| \|\rangle$ it follows, as a consequence of a theorem of Eberlein ([7], [5], p. 56), that there exists a closed, separable, non-reflexive subspace X_0 of X. Let Ξ_0 denote the conjugate space of $\langle X_0, \| \|\rangle$, and let $x_1, x_2, \ldots$ be a sequence dense in $\langle X_0, \| \|\rangle$. By assumption, there exists a linear functional $\mathfrak{z}$ on $\langle \Xi_0, \| \|\rangle$ which does not belong to the canonical image of $\langle X_0, \| \|\rangle$ in its second conjugate space. By Proposition 3.1, the set

$$\Omega = \{\xi: \xi \in \Xi_0,\ \mathfrak{z}(\xi) = 0\}$$

is norming for $\langle X_0, \| \|\rangle$. Hence there exist functionals $\zeta_n \in \Omega$ $(n = 1, 2, \ldots)$ and a constant $K > 0$ such that

$$\|\zeta_n\| = 1, \quad |\zeta_n(x_n)| \geqslant K\|x_n\| \quad (n = 1, 2, \ldots).$$

Let Υ denote the smallest linear and strongly closed subset of Ξ_0 spanned upon the functionals $\zeta_1, \zeta_2, \ldots$ Evidently, Υ is norming. By Theorem 2.5, there exists in X_0 a norm $\| \|_0^*$ coarser than $\| \|$, such that the space $\langle X_0, \| \|, \| \|_0^*\rangle$ is quasi-normal and such that the closure of the space Ξ_0^* conjugate to $\langle X_0, \| \|_0^*\rangle$ is identical with Υ.

We take into account the extension $\| \|^*$ of the norm $\| \|_0^*$, according to Lemma 3.2. The norm $\| \|^*$ is coarser than $\| \|$, and $\|x\|^* = \|x\|_0^*$ for $x \in X_0$. Let η be any fixed functional belonging to $\Xi_0 \setminus \Omega$ and let $\bar\eta$ be its Hahn-Banach extension on $\langle X, \| \|\rangle$. Then, for every $\xi \in \Xi^*$, the restricted functional $\xi | X_0$ belongs to Ξ_0^* and $\Xi_0^* \subset \Upsilon \subset \Omega$, whence

$$\|\xi - \bar\eta\| = \sup\{\xi(x) - \bar\eta(x): x \in S\} \geqslant \sup\{\xi(x) - \eta(x): x \in S \cap X_0\}$$
$$\geqslant \inf\{\|\zeta - \eta\|: \zeta \in \Omega\} = \delta > 0,$$

which means that $\bar\eta$ does not belong to the closure of Ξ^* in $\langle \Xi, \| \|\rangle$.

3.4. Remark. *If $\langle X, \| \|\rangle$ is a non-reflexive Banach space, then there exists a coarser norm $\| \|^*$ in X such that the space $\langle X, \| \|, \| \|^*\rangle$ is quasi-normal and $\Xi_\gamma \neq \Xi$.*

This immediately follows by condition (c) of Lemma 3.2. Let us notice that the statement that $\langle X, \| \|, \| \|^*\rangle$ *is quasi-normal* cannot be replaced by the statement that $\langle X, \| \|, \| \|^*\rangle$ *is normal*, in virtue of Proposition 2.8.

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Charakterisierung von Fourierkoeffizienten mit einem Summierbarkeitsfaktorentheorem und Multiplikatoren

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1. Einleitung (1). Die Absicht dieser Note ist es zu zeigen, wie der bekannte Satz von Schur-Bosanquet über Summierbarkeitsfaktoren mit einem Schlag verschiedene bekannte und neue Kriterien für Fourierkoeffizienten liefert, wenn man die in den Arbeiten [13] und [14] eingeführte Theorie komplementärer Fourierkoeffizientenräume heranzieht. Abschnitt 2 enthält einige Raumdefinitionen, Abschnitt 3 enthält Aussagen über Fourierkoeffizienten, welche das bekannte Kriterium von Kolmogoroff [26] und seine Verallgemeinerungen durch Moore [30], [31] und Cesari [7] enthalten. Abschnitt 4 enthält neue Aussagen über Multiplikatoren, welche teilweise auch mit dem Satz von Schur-Bosanquet bewiesen werden und Verallgemeinerungen bekannter Aussagen. Abschnitt 5 enthält ergänzende Bemerkungen und auch ergänzende Literaturhinweise zu den Arbeiten [11], [13] und [14].

2. Definitionen. Wir verwenden die in den Arbeiten [13] und [14] eingeführten Symbole und Vereinbarungen und verweisen auf die dortigen ausführlichen Raumdefinitionen für $E = L_p$ $(1 \leqslant p \leqslant \infty)$, L_Φ, C, V, A, die zugeordneten komplementären Räume E^* und die zugeordneten Stieltjes-Räume dE. Die in [13] und [14] mit $(C_1 - E)^*$ bezeichneten C_1-komplementären Räume bezeichnen wir hier — wie in [15] — kürzer durch das Symbol E^{1*}. Neu hinzu kommen die folgenden Räume:

1) Ist $E \subset P_\infty$ (P_∞ = Menge der trigonometrischen Reihen) und E ein BK-Raum, so ist E_{kN} $(0 \leqslant k < \infty)$ die Untermenge von E in der das trigonometrische Orthogonalsystem eine C_k-Basis bildet, d. h. es ist genau dann

$$\mathring{f} \equiv (a_j, b_j) \equiv \sum_{j=1}^{\infty} (a_j \cos jt + b_j \sin jt) \in E_{kN} \qquad (0 \leqslant k < \infty),$$

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