Proof. This theorem easily follows from Theorem 1 by the use of well-known methods (see, for instance, [1] and [4]).

THEOREM 3. *Let q be a prime number satisfying the condition* $\exp(r^{\varepsilon_0}) \leqslant q \leqslant N\exp(-r^{\varepsilon_0})$. *Then we have*

$$R^{(s)}_{l,\beta}(N) = \beta R^{(s)}_{l,1}(N) + O(N\Delta_1),$$

where

$$\Delta_1 = (1/q + q/N)^{0.5-\varepsilon_1} + N^{-0.2+\varepsilon_1}.$$

Proof. If we put $a = 1$, $\Theta = 0$, $\tau = N\exp(-r^{\varepsilon_0})$ in Theorem 2, we obtain the equality

$$Z^{(s)}_{l,\beta}(N) = R^{(s)}_{l,\beta}(N).$$

The theorem is proved. In the case $\chi(a) = \left(\frac{a}{q}\right)$, $l = 1$ we obtain the result of paper [3].

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On the representation of integers by binary forms

by

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Let $F(x, y)$ be a binary form of degree $n \geqslant 3$ with integral coefficients of height a and with non-zero discriminant, and let m be an integer distinct from zero. H. Davenport and K. F. Roth, in 1955, proved a general theorem on Diophantine equations of which the following result is a particular case.

The equation $F(x, y) = m$ *cannot have more than*

$$(4a)^{2n^2}|m|^3 + \exp(643n^2)$$

integral solutions x, y.

This result is of great interest because it gives an explicit upper bound for the number of solutions. The proof depends on the deep ideas which Roth introduced into the Thue-Siegel theory of the approximations of algebraic numbers.

We establish in this paper a better upper bound for the number of solutions of $F(x, y) = m$. Our proof does not depend on Roth's method, but uses instead the p-adic generalization of the Thue-Siegel theorem discovered by one of us in 1932. We consider only primitive solutions x, y, i. e. solutions where x and y are relatively prime; but this is not an essential restriction.

Already in the original paper M_2 of 1933, it was proved that the equation $F(x, y) = m$ has not more than

$$c^{t+1}$$

solutions where $c > 0$ is a constant independent of m, and t denotes the number of distinct prime factors of m. Since $c^{t+1} = O(|m|^{\varepsilon})$ for every $\varepsilon > 0$, this estimate is better than that by Davenport and Roth for all sufficiently large $|m|$; but it does not show the dependance on the coefficients and the degree of $F(x, y)$ of the number of solutions.

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This lacuna will now be filled in the present paper. Our main result in that there are not more than

$$c_1(an)^{c_2\sqrt{n}}+(c_3n)^{t+1}$$

pairs of integers x, y with $x\neq 0$, $y>0$, $(x, y)=1$ for which $F(x, y)\neq 0$ has at most t given prime factors $p_1, \dots, p_t$. Here c_1, c_2, and c_3 are positive absolute constants which can be determined explicity and are not too large. In particular, if $|m|$ is greater than a certain limit which depends on the coefficients and the degree of $F(x, y)$, the number of primitive solutions of $F(x, y)=m$ is not greater than

$$(c_3n)^{t+1}.$$

This upper bound depends only on m and on the degree of $F(x, y)$, but is independent of the coefficients of this form.

Our proof makes very essential use of the ideas of the old papers M_1 and M_2. It is based on three new theorems (Lemmas 1 and 2 and Theorem 1) which perhaps have a little interest in themselves. Lemma 1 is an improvement of one by N. I. Feldman, while its p-adic analogue Lemma 2 is due to F. Kasch and B. Volkmann.

1. Throughtout this paper, the following notation will be used.

C is the field of complex numbers.
p is a prime.
P_p is the field of p-adic numbers.
$C_\mathfrak{p}$ is a finite algebraic extension of P_p, with the divisor $\mathfrak{p}$.
$|a|$ is the ordinary absolute value in C.
$|a|_p$ is the p-adic value in P_p normed such that $|p|_p=1/p$.
$|a|_\mathfrak{p}$ is the $\mathfrak{p}$-adic extension of $|a|_p$ in $C_\mathfrak{p}$; thus

$$|a|_\mathfrak{p}=|a|_p \quad \text{if} \quad a\in P_p.$$

$p_1, \dots, p_t$ are finitely many distinct primes.
$P_{p_\tau}, C_{\mathfrak{p}_\tau}, \mathfrak{p}_\tau, |a|_{p_\tau}$, and $|a|_{\mathfrak{p}_\tau}$, for $\tau=1, \dots, t$, are defined in analogy to $P_p, C_\mathfrak{p}, \mathfrak{p}, |a|_p$, and $|a|_\mathfrak{p}$, respectively.

Let

$$f(x_1, \dots, x_s)=\sum_{h_1=0}^{n_1}\dots\sum_{h_s=0}^{n_s} a_{h_1\dots h_s}x_1^{n_1-h_1}\dots x_s^{n_s-h_s}$$

be a polynomial in one or more variables with coefficients in C. Then

$$H(f)=\max_{\substack{0\leqslant h_1\leqslant n_1\\ \vdots\\ 0\leqslant h_s\leqslant n_s}}|a_{h_1\dots h_s}|$$

is called the *height* of f. Similarly, if the coefficients of the polynomial lie in P_p or $C_\mathfrak{p}$, we call

$$H_p(f)=\max_{\substack{0\leqslant h_1\leqslant n_1\\ \vdots\\ 0\leqslant h_s\leqslant n_s}}|a_{h_1\dots h_s}|_p \quad\text{and}\quad H_\mathfrak{p}(f)=\max_{\substack{0\leqslant h_1\leqslant n_1\\ \vdots\\ 0\leqslant h_s\leqslant n_s}}|a_{h_1\dots h_s}|_\mathfrak{p}$$

the *p-adic height*, and the *$\mathfrak{p}$-adic height*, of f, respectively. Further heights $H_{p_\tau}(f)$ and $H_{\mathfrak{p}_\tau}(f)$ are defined correspondingly.

The resultant $R(f, F)$ of two polynomials

$$f(x)=a_0x^n+a_1x^{n-1}+\dots+a_n \quad\text{and}\quad F(x)=A_0x^N+A_1x^{N-1}+\dots+A_N$$

with coefficients in an arbitrary field is defined as usual in terms of a determinant. Provided that $a_0\neq 0$, the descriminant $D(f)$ of $f(x)$ is then given by

$$D(f)=(-1)^{n(n-1)/2}a_0^{-1}R(f, f'),$$

where $f'(x)$ is the derivative of $f(x)$. A simple calculation allows to show that $D(f)$ may be written as the determinant

$$D(f)=$$

$$\mp n^{-(n-2)}\left|\begin{array}{cccccccc} na_0 & (n-1)a_1 & \dots & 2a_{n-2} & a_{n-1} & \dots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & \dots & na_0 & (n-1)a_1 & \dots & 2a_{n-2} & a_{n-1}\\ a_1 & 2a_2 & \dots & (n-1)a_{n-1} & na_n & \dots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & \dots & a_1 & 2a_2 & \dots & (n-1)a_{n-1} & na_n \end{array}\right| \begin{array}{l}\left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\}-1\text{ rows}\\ \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\}n-1\text{ rows}\end{array}$$

2. One can establish simple upper bounds for $|D(f)|$ and $|D(f)|_\mathfrak{p}$ when $f(x)$ has coefficients in C or $C_\mathfrak{p}$, respectively.

First let $f(x)$ be in $C[x]$. By Hadamard's theorem on determinants, it follows immediately from the last expression for $D(f)$ that

$$|D(f)|^2\leqslant n^{-2(n-2)}\{|na_0|^2+|(n-1)a_1|^2+\dots+|a_{n-1}|^2\}^{n-1}\times$$
$$\times\{|a_1|^2+|2a_2|^2+\dots+|na_n|^2\}^{n-1}.$$

Here

$$\left.\begin{array}{l}|na_0|^2+|(n-1)a_1|^2+\dots+|a_{n-1}|^2\\ |a_1|^2+|2a_2|^2+\dots+|na_n|^2\end{array}\right\}\leqslant H(f)^2(1^2+2^2+\dots+n^2)\leqslant H(f)^2\cdot n\cdot n^2.$$

Hence

$$|D(f)|^2\leqslant n^{-2(n-2)}\left(n^3H(f)^2\right)^{(n-1)+(n-1)},$$

and therefore

$$|D(f)|\leqslant n^{2n-1}H(f)^{2n-2}. \tag{1}$$

Secondly let $f(x)$ be in $C_\mathfrak{p}[x]$. From its definition, $D(f)$ is a homogeneous polynomial in $a_0, a_1, \ldots, a_n$ of dimension $2(n-1)$, with numerical coefficients which are rational integers. Hence

(2) $$|D(f)|_\mathfrak{p} \leqslant H_\mathfrak{p}(f)^{2n-2}.$$

3. For the moment, let $f(x)$ have coefficients in an arbitrary field K, and let ζ be a zero of $f(x)$ in K. Then $f(x)$ is divisible by $x-\zeta$; denote by

$$g(x) = \frac{1}{x-\zeta}\cdot f(x) = b_0x^{n-1}+b_1x^{n-2}+\ldots+b_{n-1}$$

the quotient polynomial. Since, formally,

$$\frac{1}{x-\zeta} = \frac{1}{x}+\frac{\zeta}{x^2}+\frac{\zeta^2}{x^3}+\ldots = -\left(\frac{1}{\zeta}+\frac{x}{\zeta^2}+\frac{x^2}{\zeta^3}+\ldots\right),$$

it is easily seen that

(3) $$b_k = \sum_{\varkappa=0}^{k} a_\varkappa \zeta^{k-\varkappa} = -\sum_{\varkappa=k+1}^{n} a_\varkappa \zeta^{k-\varkappa} \qquad (k = 0, 1, \ldots, n-1).$$

First assume that both ζ and the coefficients of $f(x)$ lie in C. On applying the first or the second formulae (3) according as $|\zeta| \leqslant 1$ or $|\zeta| > 1$, it follows immediately that

(4) $$H(g) \leqslant nH(f),$$

a result due to C. L. Siegel.

Secondly, let both ζ and the coefficients of $f(x)$ belong to $C_\mathfrak{p}$. The same method now leads to the inequality,

(5) $$H_\mathfrak{p}(g) \leqslant H_\mathfrak{p}(f).$$

Next, these formulae, together with (1) and (2), immediately give the estimates

(6) $$|D(g)| \leqslant (n-1)^{2n-3}H(g)^{2n-4} \leqslant n^{4n-7}H(f)^{2n-4} \quad \text{if} \quad f(x)\in C[x],\ \zeta\in C$$

and

(7) $$|D(g)|_\mathfrak{p} \leqslant H_\mathfrak{p}(g)^{2n-4} \leqslant H_\mathfrak{p}(f)^{2n-4} \quad \text{if} \quad f(x)\in C_\mathfrak{p}[x],\ \zeta\in C_\mathfrak{p}.$$

The discriminants of $f(x)$ and $g(x)$ are connected by the identity

$$D(f) = D(g)f'(\zeta)^2,$$

as follows at once on expressing the two discriminants in terms of the zeros

of $f(x)$ and $g(x)$, respectively. By means of (6) and (7) we arrive then at the estimates,

(8) $$|f'(\zeta)| \geqslant \frac{\left(|D(f)|\right)^{1/2}}{n^{2n-7/2}H(f)^{n-2}} \quad \text{if} \quad f(x)\in C[x],\ \zeta\in C,\ f(\zeta)=0,$$

and

(9) $$|f'(\zeta)|_\mathfrak{p} \geqslant \frac{\left(|D(f)|_\mathfrak{p}\right)^{1/2}}{H_\mathfrak{p}(f)^{n-2}} \quad \text{if} \quad f(x)\in C_\mathfrak{p}[x],\ \zeta\in C_\mathfrak{p},\ f(\zeta)=0.$$

4. These two lower bounds imply the following two lemmas.

LEMMA 1. *Let $f(x)$ be a polynomial in $C[x]$ of the exact degree n and with the discriminant $D(f)$, the height $H(f)$, and the zeros $\zeta_1, \ldots, \zeta_n$ in C. For every z in C,*

$$|f(z)| \geqslant \frac{\left(|D(f)|\right)^{1/2}}{2^{n-1}n^{2n-7/2}H(f)^{n-2}} \min_{1\leqslant\nu\leqslant n}|z-\zeta_\nu|.$$

LEMMA 2. *Let $f(x)$ be a polynomial in $C_\mathfrak{p}[x]$ of the exact degree n and with the discriminant $D(f)$, the $\mathfrak{p}$-adic height $H_\mathfrak{p}(f)$, and the zeros $\zeta_1, \ldots, \zeta_n$ in $C_\mathfrak{p}$. For every z in $C_\mathfrak{p}$,*

$$|f(z)|_\mathfrak{p} \geqslant \frac{\left(|D(f)|_\mathfrak{p}\right)^{1/2}}{H_\mathfrak{p}(f)^{n-2}} \min_{1\leqslant\nu\leqslant n}|z-\zeta_\nu|_\mathfrak{p}.$$

Both lemmas will be proved in the same manner, using the inequalities (8) and (9).

Proof of Lemma 1. Without loss of generality, the minimum

$$\delta = \min_{1\leqslant\nu\leqslant n}|z-\zeta_\nu|$$

is attained for the zero $\zeta = \zeta_n$, hence

$$\delta = |z-\zeta_n| = |z-\zeta|.$$

The decomposition

$$f(x) = a_0(x-\zeta_1)\ldots(x-\zeta_{n-1})(x-\zeta)$$

implies therefore that

$$|f(z)| = |a_0|\,\delta\prod_{\nu=1}^{n-1}|z-\zeta_\nu|.$$

Renumber now the zeros $\zeta_1, \ldots, \zeta_{n-1}$ such that, say,

$$|\zeta-\zeta_\nu| \begin{cases} \leqslant 2\delta & \text{if} \quad \nu = 1, 2, \ldots, N, \\ > 2\delta & \text{if} \quad \nu = N+1, N+2, \ldots, n-1, \end{cases}$$

where we put $N = 0$ if none of the first inequalities hold, and $N = n-1$ if none of the second ones is satisfied.

By the definition of δ,

$$|z-\zeta_\nu| \geqslant \delta \quad (\nu = 1, 2, \ldots, n-1),$$

whence

$$\prod_{\nu=1}^{N} |z-\zeta_\nu| \geqslant \delta^N \geqslant 2^{-N} \prod_{\nu=1}^{N} |\zeta-\zeta_\nu|.$$

Further, if $\nu = N+1, N+2, \ldots, n-1$, hence $|\zeta-\zeta_\nu| \geqslant 2\delta = 2|z-\zeta|$, then

$$|z-\zeta_\nu| = |(z-\zeta)+(\zeta-\zeta_\nu)| \geqslant |\zeta-\zeta_\nu| - |z-\zeta| \geqslant \tfrac{1}{2}|\zeta-\zeta_\nu|$$

and therefore

$$\prod_{\nu=N+1}^{n-1} |z-\zeta_\nu| \geqslant 2^{-(n-N-1)} \prod_{\nu=N+1}^{n-1} |\zeta-\zeta_\nu|.$$

Hence

$$\prod_{\nu=1}^{n-1} |z-\zeta_\nu| \geqslant 2^{-(n-1)} \prod_{\nu=1}^{n-1} |\zeta-\zeta_\nu|.$$

Here the identity

$$a_0 \prod_{\nu=1}^{n-1} (\zeta-\zeta_\nu) = f'(\zeta) \tag{10}$$

holds, and so the assertion follows immediately from (8).

Proof of Lemma 2. Now, without loss of generality, the minimum

$$\delta_{\mathfrak{p}} = \min_{1\leqslant\nu\leqslant n} |z-\zeta_\nu|_{\mathfrak{p}}$$

is attained for the zero $\zeta = \zeta_n$, hence

$$\delta_{\mathfrak{p}} = |z-\zeta_n|_{\mathfrak{p}} = |z-\zeta|_{\mathfrak{p}}.$$

Therefore, by the same decomposition of $f(x)$ as above,

$$|f(z)|_{\mathfrak{p}} = |a_0|_{\mathfrak{p}}\, \delta_{\mathfrak{p}} \prod_{\nu=1}^{n-1} |z-\zeta_\nu|_{\mathfrak{p}}.$$

Renumber again the zeros $\zeta_1, \ldots, \zeta_{n-1}$ such that, say,

$$|\zeta-\zeta_\nu|_{\mathfrak{p}} \begin{cases} \leqslant \delta_{\mathfrak{p}} & \text{if} \quad \nu = 1, 2, \ldots, N, \\ > \delta_{\mathfrak{p}} & \text{if} \quad \nu = N+1, N+2, \ldots, n-1, \end{cases}$$

with conventions for N similar to those above.

As in that proof,

$$|z-\zeta_\nu|_{\mathfrak{p}} \geqslant \delta_{\mathfrak{p}} \quad (\nu = 1, 2, \ldots, n-1),$$

so that

$$\prod_{\nu=1}^{N} |z-\zeta_\nu|_{\mathfrak{p}} \geqslant \delta_{\mathfrak{p}}^N \geqslant \prod_{\nu=1}^{N} |\zeta-\zeta_\nu|_{\mathfrak{p}}.$$

Further, if $\nu = N+1, N+2, \ldots, n-1$, hence $|\zeta-\zeta_\nu|_{\mathfrak{p}} > \delta_{\mathfrak{p}} = |z-\zeta|_{\mathfrak{p}}$, then

$$|z-\zeta_\nu|_{\mathfrak{p}} = |(z-\zeta)+(\zeta-\zeta_\nu)|_{\mathfrak{p}} = |\zeta-\zeta_\nu|_{\mathfrak{p}},$$

and hence

$$\prod_{\nu=N+1}^{n-1} |z-\zeta_\nu|_{\mathfrak{p}} = \prod_{\nu=N+1}^{n-1} |\zeta-\zeta_\nu|_{\mathfrak{p}}.$$

Therefore

$$\prod_{\nu=1}^{n-1} |z-\zeta_\nu|_{\mathfrak{p}} \geqslant \prod_{\nu=1}^{n-1} |\zeta-\zeta_\nu|_{\mathfrak{p}}.$$

The assertion follows now immediately from (9) and (10).

5. From now on we impose on $f(x)$ the restrictions that its coefficients are rational integers and that

$$a_0 \neq 0 \quad \text{and} \quad a_n \neq 0. \tag{11}$$

Then not only $f(x)$, but also

$$f^*(x) = a_0 + a_1 x + \ldots + a_n x^n$$

is of exact degree n. Let

$$F(x, y) = a_0 x^n + a_1 x^{n-1} y + \ldots + a_n y^n$$

be the binary form associated with $f(x)$. Evidently

$$F(x, y) = y^n f\left(\frac{x}{y}\right) = x^n f^*\left(\frac{y}{x}\right), \tag{12}$$

and, conversely,

$$f(x) = F(x, 1), \qquad f^*(x) = F(1, x).$$

It is obvious that

$$H(F) = H(f) = H(f^*).$$

Also, as is easily verified, $f(x)$ and $f^*(x)$ have the same discriminant. We therefore put

$$D(F) = D(f) = D(f^*)$$

and demand from now on that

$$D(F) \neq 0. \tag{13}$$

Thus $D(F)$ is a rational integer distinct from zero.

Denote by $p_1, \ldots, p_t$ finitely many distinct primes. Then, for each suffix $\tau = 1, \ldots, t$, let P_{p_τ} be the p_τ-adic field and $|a|_{p_\tau}$ the p_τ-adic value. Further denote by $C_{\mathfrak{p}_\tau}$ a finite algebraic extension of P_{p_τ} in which $f(x)$ and hence also $f^*(x)$ and $F(x, y)$ split into products of linear factors. Also let $\mathfrak{p}_\tau$ be the prime divisor of $C_{\mathfrak{p}_\tau}$, and let $|a|_{\mathfrak{p}_\tau}$ be the $\mathfrak{p}_\tau$-adic continuation of $|a|_{p_\tau}$ in $C_{\mathfrak{p}_\tau}$ so that

$$|a|_{\mathfrak{p}_\tau} = |a|_{p_\tau} \quad \text{if} \quad a \in P_{p_\tau}.$$

Finally write $\zeta_1, \ldots, \zeta_n$ for the zeros of $f(x)$ in C and $\zeta_{\tau 1}, \ldots, \zeta_{\tau n}$ for its zeros in $C_{\mathfrak{p}_\tau}$; all these zeros are distinct from 0 because it is assumed that $a_n \neq 0$. It follows that

$$\left.\begin{aligned} f(x) &= a_0 \prod_{\nu=1}^{n} (x - \zeta_\nu) = a_0 \prod_{\nu=1}^{n} (x - \zeta_{\tau\nu}) \\ f^*(x) &= a_n \prod_{\nu=1}^{n} \left(x - \frac{1}{\zeta_\nu}\right) = a_n \prod_{\nu=1}^{n} \left(x - \frac{1}{\zeta_{\tau\nu}}\right) \\ F(x, y) &= a_0 \prod_{\nu=1}^{n} (x - \zeta_\nu y) = a_0 \prod_{\nu=1}^{n} (x - \zeta_{\tau\nu} y) \end{aligned}\right\} \quad (\tau = 1, 2, \ldots, t)$$

for all rational numbers x and y since such numbers lie in all $t+1$ fields $C, C_{\mathfrak{p}_1}, \ldots, C_{\mathfrak{p}_t}$.

6. Let from now on x and y be rational integers distinct from zero. By means of the two Lemmas 1 and 2 we shall establish simple lower bounds for $|F(x, y)|$ and $|F(x, y)|_{p_\tau}$ in terms of x and y. We begin with the absolute value.

For shortness, put

$$\Lambda = \frac{(|D(F)|)^{1/2}}{2^{n-1} n^{2n-7/2} H(F)^{n-2}}$$

and write

$$|x, y| = \max(|x|, |y|), \qquad \sigma = \max(1, |\zeta_1|, \ldots, |\zeta_n|).$$

From Lemma 1 and by the identities (12),

$$\text{(14A)} \qquad |F(x, y)| \geqslant \Lambda |y|^n \min_{1 \leqslant \nu \leqslant n} \left| \frac{x}{y} - \zeta_\nu \right|,$$

$$\text{(14B)} \qquad |F(x, y)| \geqslant \Lambda |x|^n \min_{1 \leqslant \nu \leqslant n} \left| \frac{y}{x} - \frac{1}{\zeta_\nu} \right|.$$

We must now distinguish several cases.

If $|x| \leqslant |y|$ and hence $|x, y| = |y|$, from (14A)

$$\text{(15A)} \qquad |F(x, y)| \geqslant \Lambda |x, y|^n \min_{1 \leqslant \nu \leqslant n} \left(1, \left| \frac{x}{y} - \zeta_\nu \right| \right).$$

Next let $|x| > |y|$ and therefore $|x, y| = |x|$. First assume that

$$\left| \frac{y}{x} - \frac{1}{\zeta_\nu} \right| > \frac{1}{2\sigma} \quad \text{for all suffixes} \quad \nu = 1, 2, \ldots, n.$$

Then (14B) implies that

$$\text{(15B)} \qquad |F(x, y)| \geqslant \Lambda |x, y|^n \cdot \frac{1}{2\sigma} \geqslant \frac{\Lambda}{2\sigma} |x, y|^n \min_{1 \leqslant \nu \leqslant n} \left(1, \left| \frac{x}{y} - \zeta_\nu \right| \right).$$

Secondly, let

$$\min_{1 \leqslant \nu \leqslant n} \left| \frac{y}{x} - \frac{1}{\zeta_\nu} \right|, = \left| \frac{y}{x} - \frac{1}{\zeta_N} \right| \text{ say, be } \leqslant \frac{1}{2\sigma}.$$

Since

$$\left| \frac{1}{\zeta_N} \right| \geqslant \frac{1}{\sigma},$$

we have

$$\left| \frac{y}{x} \right| = \left| \frac{1}{\zeta_N} + \left(\frac{y}{x} - \frac{1}{\zeta_N} \right) \right| \geqslant \left| \frac{1}{\zeta_N} \right| - \left| \frac{y}{x} - \frac{1}{\zeta_N} \right| \geqslant \frac{1}{\sigma} - \frac{1}{2\sigma} = \frac{1}{2\sigma},$$

and hence

$$\left| \frac{y}{x} - \frac{1}{\zeta_N} \right| = \left| \frac{y}{x} \cdot \frac{1}{\zeta_N} \cdot \left(\frac{x}{y} - \zeta_N \right) \right| \geqslant \frac{1}{2\sigma} \cdot \frac{1}{\sigma} \cdot \left| \frac{x}{y} - \zeta_N \right|.$$

Therefore in the present case,

$$\text{(15C)} \qquad |F(x, y)| \geqslant \frac{\Lambda}{2\sigma^2} |x, y|^n \min_{1 \leqslant \nu \leqslant n} \left(1, \left| \frac{x}{y} - \zeta_\nu \right| \right).$$

For all integers $x \neq 0$ and $y \neq 0$ one of the estimates (15) holds; furthermore, $\sigma \geqslant 1$. Hence it follows that

$$\text{(16)} \qquad |F(x, y)| \geqslant \frac{\Lambda}{2\sigma^2} |x, y|^n \min_{1 \leqslant \nu \leqslant n} \left(1, \left| \frac{x}{y} - \zeta_\nu \right| \right) \quad \text{for all integers} \quad x \neq 0,\ y \neq 0.$$

7. A lower bound for $|F(x, y)|_{p_\tau}$ may be obtained in a very similar way. It suffices, for our purpose, to consider integers $x \neq 0$ and $y \neq 0$ that are relatively prime.

Since $F(x, y)$ has rational integral coefficients, the p_τ-adic heights

$$H_{p_\tau}(F) = H_{p_\tau}(f) = H_{p_\tau}(f^*) \qquad (\tau = 1, 2, \ldots, t)$$

are all at most 1. For shortness, put

$$\Lambda_\tau = \left(|D(F)|_{p_\tau}\right)^{1/2} \qquad (\tau = 1, 2, \ldots, t)$$

and

$$\sigma_\tau = \max(1, |\zeta_{\tau 1}|_{\mathfrak{p}_\tau}, \ldots, |\zeta_{\tau n}|_{\mathfrak{p}_\tau}) \qquad (\tau = 1, 2, \ldots, t).$$

From Lemma 2 and by the identities (12),

$$\text{(17A)} \qquad |F(x, y)|_{p_\tau} \geqslant \Lambda_\tau (|y|_{p_\tau})^n \min_{1\leqslant \nu \leqslant n} \left|\frac{x}{y} - \zeta_{\tau\nu}\right|_{\mathfrak{p}_\tau},$$

$$\text{(17B)} \qquad |F(x, y)|_{p_\tau} \geqslant \Lambda_\tau (|x|_{p_\tau})^n \min_{1\leqslant \nu \leqslant n} \left|\frac{y}{x} - \frac{1}{\zeta_{\tau\nu}}\right|_{\mathfrak{p}_\tau}.$$

Again several cases will be distinguished.

If p_τ does not divide y, (17A) implies that

$$\text{(18A)} \qquad |F(x, y)|_{p_\tau} \geqslant \Lambda_\tau \min_{1\leqslant \nu \leqslant n} \left(1, \left|\frac{x}{y} - \zeta_{\tau\nu}\right|_{\mathfrak{p}_\tau}\right).$$

Next let p_τ divide y and hence not divide x. First assume that

$$\left|\frac{y}{x} - \frac{1}{\zeta_{\tau\nu}}\right|_{\mathfrak{p}_\tau} \geqslant \frac{1}{\sigma_\tau} \quad \text{for all suffixes} \quad \nu = 1, 2, \ldots, n.$$

Then, by (17B),

$$\text{(18B)} \qquad |F(x, y)|_{p_\tau} \geqslant \Lambda_\tau \cdot \frac{1}{\sigma_\tau} \geqslant \frac{\Lambda_\tau}{\sigma_\tau} \min_{1\leqslant \nu \leqslant n} \left(1, \left|\frac{x}{y} - \zeta_{\tau\nu}\right|_{\mathfrak{p}_\tau}\right).$$

Secondly, let

$$\min_{1\leqslant \nu \leqslant n} \left|\frac{y}{x} - \frac{1}{\zeta_{\tau\nu}}\right|_{\mathfrak{p}_\tau}, = \left|\frac{y}{x} - \frac{1}{\zeta_{\tau N}}\right|_{\mathfrak{p}_\tau} \text{ say, be } < \frac{1}{\sigma_\tau}.$$

Then

$$\left|\frac{y}{x}\right|_{p_\tau} = \left|\frac{1}{\zeta_{\tau N}} + \left(\frac{y}{x} - \frac{1}{\zeta_{\tau N}}\right)\right|_{\mathfrak{p}_\tau} = \left|\frac{1}{\zeta_{\tau N}}\right|_{\mathfrak{p}_\tau} \geqslant \frac{1}{\sigma_\tau}$$

so that

$$\left|\frac{y}{x} - \frac{1}{\zeta_{\tau N}}\right|_{\mathfrak{p}_\tau} = \left|\frac{y}{x} \cdot \frac{1}{\zeta_{\tau N}} \cdot \left(\frac{x}{y} - \zeta_{\tau N}\right)\right|_{\mathfrak{p}_\tau} \geqslant \frac{1}{\sigma_\tau} \cdot \frac{1}{\sigma_\tau} \cdot \left|\frac{x}{y} - \zeta_{\tau N}\right|_{\mathfrak{p}_\tau}.$$

Therefore in the present case,

$$\text{(18C)} \qquad |F(x, y)|_{p_\tau} \geqslant \frac{\Lambda_\tau}{\sigma_\tau^2} \min_{1\leqslant \nu \leqslant n} \left(1, \left|\frac{x}{y} - \zeta_{\tau\nu}\right|_{\mathfrak{p}_\tau}\right).$$

For all integers $x \neq 0$ and $y \neq 0$ that are relatively prime one of the estimates (18) holds; furthermore $\sigma_\tau \geqslant 1$. Hence

$$\text{(19)} \qquad |F(x, y)|_{p_\tau} \geqslant \frac{\Lambda_\tau}{\sigma_\tau^2} \min_{1\leqslant \nu \leqslant n} \left(1, \left|\frac{x}{y} - \zeta_{\tau\nu}\right|_{\mathfrak{p}_\tau}\right)$$

for all such integers, and for all suffixes $\tau = 1, 2, \ldots, t$.

8. On forming the product of the relation (16) and the t relations (19), we obtain the inequality

$$\text{(20)} \qquad |F(x, y)| \prod_{\tau=1}^{t} |F(x, y)|_{p_\tau} \geqslant$$

$$\geqslant M|x, y|^n \min_{1\leqslant \nu \leqslant n} \left(1, \left|\frac{x}{y} - \zeta_\nu\right|\right) \prod_{\tau=1}^{t} \min_{1\leqslant \nu \leqslant n} \left(1, \left|\frac{x}{y} - \zeta_{\tau\nu}\right|_{\mathfrak{p}_\tau}\right),$$

where M denotes the expression

$$M = \frac{\Lambda \Lambda_1 \ldots \Lambda_t}{2\sigma^2 \sigma_1^2 \ldots \sigma_t^2} = \frac{(|D(F)|)^{1/2} \prod_{\tau=1}^{t} (|D(F)|_{p_t})^{1/2}}{2^n n^{2n-7/2} H(F)^{n-2} (\sigma\sigma_1 \ldots \sigma_t)^2}.$$

It has advantages to replace M by a simpler, although slightly smaller number, as follows.

First, $D(F)$ is a rational integer not zero; hence

$$\text{(21)} \qquad |D(F)| \prod_{\tau=1}^{t} |D(F)|_{p_\tau} \geqslant 1.$$

Secondly,

$$\text{(22)} \qquad \sigma \leqslant \frac{H(F)}{|a_0|} + 1 \leqslant \frac{2H(F)}{|a_0|}; \qquad \sigma_\tau \leqslant \left|\frac{1}{a_0}\right|_{p_\tau} \qquad (\tau = 1, 2, \ldots, t).$$

For in the case of the complex zeros ζ_ν of $f(x)$,

$$\zeta_\nu = -a_0^{-1}(a_1 + a_2 \zeta_\nu^{-1} + \ldots + a_n \zeta_\nu^{-(n-1)}).$$

Hence, if $|\zeta_\nu| > 1$,

$$|\zeta_\nu| < \frac{H(F)}{|a_0|}(1 + |\zeta_\nu|^{-1} + |\zeta_\nu|^{-2} + \ldots) = \frac{H(F)\,|\zeta_\nu|}{|a_0|\,(|\zeta_\nu| - 1)},$$

giving the assertion for σ.

Next let $\zeta_{\tau\nu}$ be a $\mathfrak{p}_\tau$-adic zero of $f(x)$, and let $\eta = a_0 \zeta_{\tau\nu}$. Then

$$\eta^n + a_1 \eta^{n-1} + a_0 a_2 \eta^{n-2} + \ldots + a_0^{n-1} a_n = 0,$$

and so η is an algebraic integer and hence also a $\mathfrak{p}_\tau$-adic integer, whence the assertion for σ_τ.

By hypothesis, a_0 is a rational integer not zero; therefore

$$|a_0|\prod_{\tau=1}^{t}|a_0|_{p_\tau}\geqslant 1.$$

The estimates (22) imply then that

$$\sigma\sigma_1\ldots\sigma_t\leqslant 2H(F).$$

On combining this with (21), it follows that

$$M\geqslant\frac{1}{\big(2H(F)\big)^2\cdot 2^n n^{2n-7/2}H(F)^{n-2}}.$$

From now on we shall be concerned only with the case when $n\geqslant 3$ and therefore certainly

$$n^{7/2}>2^3.$$

Hence M allows the lower bound

$$M>\big(2n^2H(F)\big)^{-n},$$

and we arrive at the following result.

THEOREM 1. *Let*

$$F(x,y)=a_0x^n+a_1x^{n-1}y+\ldots+a_ny^n,\quad \textit{where}\quad a_0\neq 0\ \textit{and}\ a_n\neq 0,$$

be a binary form of degree $n\geqslant 3$ with rational integral coefficients and discriminant distinct from zero; denote by $a=H(F)$ the height of $F(x,y)$. Let $p_1,\ldots,p_t$ be finitely many distinct primes; let P_{p_τ}, for $\tau=1,\ldots,t$, be the p_τ-adic field, and let $C_{\mathfrak{p}_\tau}$ be a finite algebraic extension of P_{p_τ} in which the equation $F(x,1)=0$ has n roots $\zeta_{\tau1},\ldots,\zeta_{\tau n}$; let further $\zeta_1,\ldots,\zeta_n$ be the n roots of the same equation in the complex field C. If x and y are any two rational integers which are relatively prime and distinct from zero, then

$$|F(x,y)|\prod_{\tau=1}^{t}|F(x,y)|_{p_\tau}\geqslant$$

$$\geqslant(2n^2a)^{-n}|x,y|^n\min_{1\leqslant\nu\leqslant n}\left(1,\left|\frac{x}{y}-\zeta_\nu\right|\right)\prod_{\tau=1}^{t}\min_{1\leqslant\nu\leqslant n}\left(1,\left|\frac{x}{y}-\zeta_{\tau\nu}\right|_{\mathfrak{p}_\tau}\right).$$

9. For shortness, put

$$\Phi(x,y)=|F(x,y)|\prod_{\tau=1}^{t}|F(x,y)|_{p_\tau},\qquad k=(2n^2a)^n.$$

Let further γ and δ be two constants depending on n which will be chosen later and are such that

$$\gamma>0,\qquad\delta\geqslant 0,\qquad\gamma+\delta=n.$$

Any pair of integers x, y is said to be *admissible* if

$$x\neq 0,\ y\neq 0,\ (x,y)=1,\ F(x,y)\neq 0\quad\text{and hence}\quad f\left(\frac{x}{y}\right)\neq 0.$$

Our aim is to find an upper bound for the number of admissible pairs x, y for which

$$\Phi(x,y)=1,$$

thus which have the property that the integer $F(x,y)\neq 0$ possesses only the given prime factors $p_1,\ldots,p_t$. It has some advantage to study a slightly more general problem, and we shall therefore also establish an upper bound for the number of admissible pairs x, y satisfying

$$\Phi(x,y)\leqslant|x,y|^\delta. \tag{23}$$

By Theorem 1, such pairs have also the property

$$\min_{1\leqslant\nu\leqslant n}\left(1,\left|\frac{x}{y}-\zeta_\nu\right|\right)\prod_{\tau=1}^{t}\min_{1\leqslant\nu\leqslant n}\left(1,\left|\frac{x}{y}-\zeta_{\tau\nu}\right|_{\mathfrak{p}_\tau}\right)\leqslant k|x,y|^{-\gamma}$$

and hence even more the property

$$\min_{1\leqslant\nu\leqslant n}\left(1,\left|\frac{x}{y}-\zeta_\nu\right|\right)\prod_{\tau=1}^{t}\min_{1\leqslant\nu\leqslant n}\left(1,|x-y\zeta_{\tau\nu}|_{\mathfrak{p}_\tau}\right)\leqslant k|x,y|^{-\gamma}. \tag{24}$$

For the latter inequality is weaker than the first because

$$|x-y\zeta_{\tau\nu}|_{\mathfrak{p}_\tau}=|y|_{p_\tau}\left|\frac{x}{y}-\zeta_{\tau\nu}\right|_{\mathfrak{p}_\tau}\leqslant\left|\frac{x}{y}-\zeta_{\tau\nu}\right|_{\mathfrak{p}_\tau}.$$

10. The solutions of (24) can be subdivided into n^{t+1} classes which, in general, need not all be disjoint.

Let ζ stand for any one of the n zeros $\zeta_1,\ldots,\zeta_n$ of $f(x)$ in C; also, if $\tau=1,\ldots,t$, let $\zeta^{(\tau)}$ stand for any one of the n zeros $\zeta_{\tau1},\ldots,\zeta_{\tau n}$ of $f(x)$ in $C_{\mathfrak{p}_\tau}$. Thus there are n^{t+1} distinct sets of $t+1$ zeros

$$(\zeta,\zeta^{(1)},\ldots,\zeta^{(t)}).$$

It is obvious that every solution x, y of (24) satisfies at least one of the n^{t+1} inequalities

$$\min\left(1,\left|\frac{x}{y}-\zeta\right|\right)\prod_{\tau=1}^{t}\min\left(1,|x-y\zeta^{(\tau)}|_{\mathfrak{p}_\tau}\right)\leqslant k|x,y|^{-\gamma} \tag{25}$$

that correspond to these sets of $t+1$ zeros.

11. Let β denote a further constant depending on n which will be chosen later and is such that

$$0<\beta<\gamma.$$

Put

$$\sigma=\frac{\gamma-\beta}{\beta}$$

and denote by v the smallest positive integer for which

$$v\geqslant\frac{1}{\sigma}(t+1).$$

Assume now that x, y is any admissible solution of (25) with

$$|x,y|>k^{1/\beta}. \tag{26}$$

Since $\sigma>0$ and $k>1$, we have

$$k|x,y|^{-\gamma}=k^{-\sigma}(k|x,y|^{-\beta})^{1+\sigma}\leqslant(k|x,y|^{-\beta})^{1+\sigma}.$$

Hence there exist $t+1$ non-negative numbers $\varphi_0,\varphi_1,\ldots,\varphi_t$ depending on x and y such that

$$\begin{cases}\min\left(1,\left|\dfrac{x}{y}-\zeta\right|\right)=(k|x,y|^{-\gamma})^{\varphi_0}\\ \min(1,|x-y\zeta^{(\tau)}|_{\mathfrak{p}_\tau})=(k|x,y|^{-\gamma})^{\varphi_\tau}\qquad(\tau=1,2,\ldots,t)\end{cases} \tag{27}$$

and therefore also

$$\begin{cases}\min\left(1,\left|\dfrac{x}{y}-\zeta\right|\right)\leqslant(k|x,y|^{-\beta})^{\varphi_0(1+\sigma)}\\ \min(1,|x-y\zeta^{(\tau)}|_{\mathfrak{p}_\tau})\leqslant(k|x,y|^{-\beta})^{\varphi_\tau(1+\sigma)}\qquad(\tau=1,2,\ldots,t).\end{cases} \tag{28}$$

From (25) and (27) it follows that

$$\varphi_0+\varphi_1+\ldots+\varphi_t\geqslant1.$$

Write

$$v(1+\sigma)\varphi_\tau=g_\tau+\gamma_\tau\qquad(\tau=0,1,\ldots,t)$$

where $g_0,g_1,\ldots,g_t$ are non-negative integers, while $\gamma_0,\gamma_1,\ldots,\gamma_t$ are real numbers such that

$$0\leqslant\gamma_\tau<1\qquad(\tau=0,1,\ldots,t).$$

Then

$$\sum_{\tau=0}^{t}g_\tau=v(1+\sigma)\sum_{\tau=0}^{t}\varphi_\tau-\sum_{\tau=0}^{t}\gamma_\tau\geqslant v(1+\sigma)-(t+1)\geqslant v.$$

This means that there exists at least one set of $t+1$ non-negative integers $f_0,f_1,\ldots,f_t$ for which

$$f_0+f_1+\ldots+f_t=v,\qquad f_\tau\leqslant g_\tau\qquad(\tau=0,1,\ldots,t)$$

and therefore also

$$\frac{f_\tau}{v}\leqslant\frac{g_\tau}{v}\leqslant(1+\sigma)\varphi_\tau\qquad(\tau=0,1,\ldots,t).$$

The inequalities (28) imply then that also

$$\begin{cases}\min\left(1,\left|\dfrac{x}{y}-\zeta\right|\right)\leqslant(k|x,y|^{-\beta})^{f_0/v}\\ \min(1,|x-y\zeta^{(\tau)}|_{\mathfrak{p}_\tau})\leqslant(k|x,y|^{-\beta})^{f_\tau/v}\qquad(\tau=1,2,\ldots,t).\end{cases} \tag{29}$$

From its definition, the set of $t+1$ integers $f_0,f_1,\ldots,f_t$ has only

$$\binom{v+t}{t}$$

possibilities. Therefore every solution x, y of the two conditions (25) and (26) satisfies one of the $\binom{v+t}{t}$ possible sets of inequalities (29).

On combining this result with that of § 10, we find:

LEMMA 3. *Every admissible pair x, y satisfying the two inequalities* (23) *and* (26) *is a solution of at least one of the*

$$N=\binom{v+t}{t}n^{t+1}$$

sets of inequalities (29) *that are obtained if* (i) *the set of zeros* $(\zeta,\zeta^{(1)},\ldots,\zeta^{(t)})$ *of $f(x)$ runs over all its n^{t+1} possibilities, and* (ii) *the integers $f_0,f_1,\ldots,f_t$ run over all $\binom{v+t}{t}$ solutions of*

$$f_0\geqslant0,\ f_1\geqslant0,\ \ldots,\ f_t\geqslant0,\qquad f_0+f_1+\ldots+f_t=v.$$

12. The following result holds.

LEMMA 4. *Let the notation be as before, and let further s be one of the integers $1,2,\ldots,n-1$ while $\mathrm{B},\Theta,\vartheta$ and $\varkappa$ are four constants such that*

$$\mathrm{B}=\frac{n}{s+1}+s+\Theta\leqslant n,\qquad0<\vartheta\leqslant\tfrac12,\qquad\Theta>\mathrm{B}\vartheta,\qquad\varkappa\geqslant1.$$

Put

$$K=(4a)^{\frac{\left(\frac{n}{s+1}+\theta\right)\left(s+\frac{n}{\vartheta}\right)}{\min(1,\Theta-\mathrm{B}\vartheta)}}\varkappa^{\frac{1+\theta-\frac{s}{\mathrm{B}}}{\Theta-\mathrm{B}\vartheta}},$$

and denote by $\Gamma_0, \Gamma_1, \ldots, \Gamma_t$ *non-negative constants such that*

$$\Gamma_0+\Gamma_1+\ldots+\Gamma_t = 1.$$

Let there exist admissible pairs of integers x, y *for which*

$$(30)\qquad |x, y| > K, \begin{cases} \min\left(1, \left|\dfrac{x}{y}-\zeta\right|\right) \leqslant (\varkappa|x, y|^{-\mathrm{B}})^{\Gamma_0} \\ \min(1, |x-y\zeta^{(\tau)}|_{\mathfrak{p}_\tau}) \leqslant (\varkappa|x, y|^{-\mathrm{B}})^{\Gamma_\tau} \quad (\tau = 1, 2, \ldots, t), \end{cases}$$

and let x_0, y_0 *be such a pair with smallest* $|x_0, y_0|$. *Every admissible solution* x, y *of* (30) *then satisfies the inequalities*

$$(31)\qquad |x_0, y_0| \leqslant |x, y| < (\varkappa^{1/\mathrm{B}}|x_0, y_0|)^{2n^3/\sigma}.$$

With a slight change of notation, this lemma is essentially the Hilfssatz 3 of the paper M_1, pp. 709-10. However, this Hilfssatz is proved in M_1 only with the following two restrictions.

RESTRICTION A: *The zero* ζ *of* $f(x)$ *is a real number; further, for* $\tau = 1, \ldots, t$, *the zero* $\zeta^{(\tau)}$ *of* $f(x)$ *is a* p_τ*-adic number.*

RESTRICTION B: *The polynomial* $f(x)$ *is irreducible over the rational field.*

The lemma remains valid without these restrictions. In fact, the proof of Hilfssatz 3 is given on pp. 693-709 of M_1. An inspection of this proof shows that the Restriction A is entirely unnecessary and is used nowhere. It was imposed for the insufficient reason that non-real numbers in C and non-p_τ-adic numbers in $C_{\mathfrak{p}_\tau}$ cannot be approximated arbitrarily closely by rational numbers.

The Restriction B is required in the paper M_1 only once, in the proof of Hilfssatz 1 on pp. 696-699. However, a very slight alteration of this proof makes it again valid for all polynomials $f(x)$ with integral coefficients that have non-zero discriminant. The proof so changed can be found, with all its details, in the paper P, pp. 22-25, where it is used to prove an even more general result than Lemma 4.

13. We also require the following result.

LEMMA 5. *Let the notation be as in Lemma 4. Let further* x_1, y_1 *and* x_2, y_2 *be two admissible pairs satisfying the conditions*

$$\frac{x_1}{y_1} \neq \frac{x_2}{y_2}, \qquad |x_1, y_1| \leqslant |x_2, y_2|,$$

and, for $j = 1$ *and* $j = 2$,

$$(32)\qquad \begin{cases} \min\left(1, \left|\dfrac{x_j}{y_j}-\zeta\right|\right) \leqslant (\varkappa|x_j, y_j|^{-\mathrm{B}})^{\Gamma_0}, \\ \min(1, |x_j-y_j\zeta^{(\tau)}|_{\mathfrak{p}_\tau}) \leqslant (\varkappa|x_j, y_j|^{-\mathrm{B}})^{\Gamma_\tau} \quad (\tau = 1, 2, \ldots, t). \end{cases}$$

Then

$$|x_2, y_2| \geqslant \frac{1}{2\varkappa}|x_1, y_1|^{\mathrm{B}-1}.$$

The proof of this lemma is given in the paper M_2, pp. 39-40. Although this proof again imposes the Restriction A, this restriction once more is not required and may again be omitted.

From now on, assume that

$$\mathrm{B} > 2.$$

The assertion of the lemma takes then the form,

$$(33)\qquad (2\varkappa)^{-\frac{1}{\mathrm{B}-2}}|x_2, y_2| \geqslant \{(2\varkappa)^{-\frac{1}{\mathrm{B}-2}}|x_1, y_1|\}^{\mathrm{B}-1}$$

which is more convenient for the following application.

Let

$$x_0, y_0;\; x_1, y_1;\; \ldots,\; x_r, y_r$$

be finitely many admissible pairs satisfying the inequalities (32) and with the additional properties that

$$\frac{x_i}{y_i} \neq \frac{x_j}{y_j} \quad \text{if} \quad 0 \leqslant i < j \leqslant r$$

and

$$A \leqslant |x_0, y_0| \leqslant |x_1, y_1| \leqslant \ldots \leqslant |x_r, y_r| \leqslant B$$

where A and B are two constants such that

$$(2\varkappa)^{\frac{1}{\mathrm{B}-2}} < A < B.$$

By (33),

$$(2\varkappa)^{-\frac{1}{\mathrm{B}-2}}|x_{j+1}, y_{j+1}| \geqslant \{(2\varkappa)^{-\frac{1}{\mathrm{B}-2}}|x_j, y_j|\}^{\mathrm{B}-1} \quad (j = 0, 1, \ldots, r-1).$$

Therefore,

$$(2\varkappa)^{-\frac{1}{\mathrm{B}-2}}|x_r, y_r| \geqslant \{(2\varkappa)^{-\frac{1}{\mathrm{B}-2}}|x_0, y_0|\}^{(\mathrm{B}-1)^r}$$

and so also

$$(2\varkappa)^{-\frac{1}{\mathrm{B}-2}}B \geqslant \{(2\varkappa)^{-\frac{1}{\mathrm{B}-2}}A\}^{(\mathrm{B}-1)^r} > 1.$$

Hence

$$(34)\qquad r \leqslant \frac{\log \dfrac{\log\left((2\varkappa)^{-\frac{1}{\mathrm{B}-2}}B\right)}{\log\left((2\varkappa)^{-\frac{1}{\mathrm{B}-2}}A\right)}}{\log(\mathrm{B}-1)}.$$

14. We procede now to the closer study of the number of admissible pairs x, y that satisfy one of the systems of inequalities (29) to which our problem has already been reduced. To do so, we apply the Lemmas 4 and 5 where we put

$$B = \beta, \quad \varkappa = k, \quad \Gamma_0 = \frac{f_0}{v}, \quad \Gamma_1 = \frac{f_1}{v}, \ldots, \Gamma_t = \frac{f_t}{v}.$$

This choice of parameters is valid because β will soon be fixed as a quantity greater than 2, and it is obvious from the definition that k is greater than 1.

For convenience, we shall from now on not distinguish between two admissible pairs of the form

$$x, y \quad \text{and} \quad -x, -y,$$

and of two such pairs only one will be counted, say that with $y > 0$. It follows that if x_1, y_1 and x_2, y_2 are two distinct admissible pairs, the rational numbers x_1/y_1 and x_2/y_2 are likewise distinct.

Denote by

$$S = S\left(\frac{f_0}{v}, \frac{f_1}{v}, \ldots, \frac{f_t}{v}\right)$$

the set of distinct admissible pairs x, y that satisfy (29). This set we divide in three disjoint subsets S_1, S_2, and S_3, as follows.

S_1 consists of those admissible pairs in S for which

$$|x, y| < (2k)^{\frac{2}{\beta-2}},$$

S_2 of those pairs for which

$$(2k)^{\frac{2}{\beta-2}} \leqslant |x, y| \leqslant K,$$

and S_3 of those pairs for which

$$|x, y| > K.$$

Let N_1, N_2, and N_3 denote the numbers of elements of S_1, S_2 and S_3, respectively.

We note that the pairs x, y in S_2 and S_3 satisfy the inequality (26) because

$$(2k)^{\frac{2}{\beta-2}} > k^{\frac{1}{\beta}}.$$

15. The Thue-Siegel method does not seem to lead to any non-trivial estimate for N_1. It is, however, obvious that

$$N_1 < 2(2k)^{\frac{4}{\beta-2}}. \tag{35}$$

For every pair x, y in S_1 has coordinates of the form

$$x, y = \mp 1, \mp 2, \ldots, [(\mp 2k)^{\frac{2}{\beta-2}}],$$

and only the pairs with positive y need be counted; also $|x, y| < (2k)^{\frac{2}{\beta-2}}$.

Evidently

$$|F(x, y)| \leqslant (n+1)a|x, y|^n \quad \text{for all } x \text{ and } y \text{ in } C.$$

It follows that, if m can be written in at least one way as

$$m = F(x, y) \quad \text{where } x, y \text{ is a pair in } S_1,$$

necessarily

$$|m| < (n+1)a(2k)^{\frac{2n}{\beta-2}}, \quad = C \text{ say}.$$

Conversely, if $|m| \geqslant C$, all admissible representations of m in the form $m = F(x, y)$ belong to either S_2 or S_3.

16. For the two remaining numbers N_2 and N_3 upper bounds are obtained by means of the formula (34). Its right-hand side augmented by 1 evidently is an upper bound for the number of admissible pairs x, y for which $|x, y|$ lies between A and B and which satisfy (29).

First put

$$A = (2k)^{\frac{2}{\beta-2}}, \quad B = K.$$

Then $A > (2k)^{\frac{1}{\beta-2}}$, and we shall soon fix the parameters such that the second condition $A < B$ is also satisfied. It follows then from (34) that

$$N_2 \leqslant \frac{\log \dfrac{\log\{(2k)^{-\frac{1}{\beta-2}}K\}}{\log\{(2k)^{\frac{1}{\beta-2}}\}}}{\log(\beta-1)} + 1. \tag{36}$$

In a similar way, Lemma 4 enables us to find an upper bound for N_3. By the lemma, every pair x, y in S_3 satisfies the inequality

$$|x_0, y_0| \leqslant |x, y| < (k^{\frac{1}{\beta}}|x_0, y_0|)^{\frac{2n^3}{\vartheta}},$$

where $|x_0, y_0|$ is an integer greater than K. We may therefore put

$$A = |x_0, y_0|, \quad B = (k^{\frac{1}{\beta}}|x_0, y_0|)^{\frac{2n^3}{\vartheta}},$$

and then the inequalities $(2k)^{\frac{1}{\beta-2}} < A < B$ are again satisfied. Hence by (34),

$$N_3 \leqslant \frac{\log \dfrac{\log\{(2k)^{-\frac{1}{\beta-2}}(k^{\frac{1}{\beta}}|x_0, y_0|)^{\frac{2n^3}{\vartheta}}\}}{\log\{(2k)^{-\frac{1}{\beta-2}}|x_0, y_0|\}}}{\log(\beta-1)} + 1. \tag{37}$$

17. Both estimates (36) and (37) take a more explicit form on fixing the parameters. We shall discuss two different choices of these parameters, one corresponding to $\delta = 0$, and one to a rather large value of δ.

For shortness, put

$$s = \left[\frac{\sqrt{4n+1}-1}{2}\right] \quad \text{and} \quad a = \frac{n}{s+1}+s.$$

Then

$$a = \min_{h=1,2,\ldots,n-1}\left(\frac{n}{h+1}+h\right)$$

and

$$2\sqrt{n}-1 \leqslant a \leqslant \sqrt{4n+1}-1.$$

As a first choice of the parameters, put

$$\beta = a+\frac{1}{n}, \quad \gamma = n, \quad \delta = 0, \quad \Theta = \frac{1}{n}, \quad \vartheta = \frac{1}{2(an+1)}$$

so that

$$\Theta-\beta\vartheta = \frac{1}{2n} < 1.$$

The constant K of Lemma 4 becomes then

$$K = (4a)^{\frac{\left(\frac{n}{s+1}+\vartheta\right)\left(3+\frac{n}{\vartheta}\right)}{\Theta-\beta\vartheta}} k^{\frac{1+\vartheta-\frac{s}{\beta}}{\Theta-\beta\vartheta}}$$

where

$$k = (2n^2a)^n.$$

Now $\sqrt{4n+1} < 2\sqrt{n}+1$, hence

$$s \leqslant \frac{\sqrt{4n+1}-1}{2} \leqslant \sqrt{n} \leqslant \frac{a+1}{2},$$

whence

$$\frac{n}{s+1}+\vartheta > \frac{n}{s+1} = a-s \geqslant \frac{a-1}{2}.$$

On the other hand,

$$\frac{n}{s+1}+\vartheta = a-(s-\vartheta) \leqslant a \quad \text{because} \quad \vartheta < 1 \leqslant s.$$

It follows that

$$(a-1)n \leqslant \frac{\dfrac{n}{s+1}+\vartheta}{\Theta-\beta\vartheta} \leqslant 2an.$$

Further

$$3+\frac{n}{\vartheta} = 2an^2+2n+3 = 2an^2\left(1+\frac{1}{an}+\frac{3}{2an^2}\right),$$

so that

$$2an^2 \leqslant 3+\frac{n}{\vartheta} \leqslant 2an^2\left(1+\frac{1}{\frac{5}{2}\times 3}+\frac{3}{2\times\frac{5}{2}\times 3^2}\right) = \tfrac{12}{5}an^2.$$

For a assumes its smallest value when $n = 3$, $s = 1$, and then $a = \frac{5}{2}$. For the same reason,

$$a-1 = a\left(1-\frac{1}{a}\right) \geqslant a\left(1-\frac{2}{5}\right) = \frac{3a}{5}.$$

Therefore, finally,

$$\tfrac{3}{5}an\times 2an^2 \leqslant \frac{\left(\dfrac{n}{s+1}+\vartheta\right)\left(3+\dfrac{n}{\vartheta}\right)}{\Theta-\beta\vartheta} \leqslant 2an\times\tfrac{12}{5}an^2.$$

The exponent of k has the trivial lower and upper bounds,

$$0 \leqslant \left(1-\frac{s}{a}\right)\times 2n \leqslant \frac{1+\vartheta-(s/\beta)}{\Theta-\beta\vartheta} \leqslant \left(1+\frac{1}{2an}-\frac{1}{n}\right)\times 2n \leqslant 2n.$$

Since $k \geqslant 1$, it follows then that

$$(4a)^{\frac{6}{5}a^2n^3} \leqslant K \leqslant (4a)^{\frac{24}{5}a^2n^3} k^{2n}.$$

A simple upper bound for k is obtained as follows. Since $n \geqslant 3$,

$$\frac{\log n}{n} \leqslant \frac{\log 3}{3}$$

because

$$\frac{d}{dx}\left(\frac{\log x}{x}\right) < 0 \quad \text{if} \quad x > e.$$

Hence

$$n \leqslant 4^{\log n/\log 4} \leqslant 4^{(n\log 3)/(3\log 4)} \leqslant 4^{n/3} \leqslant (4a)^{n/3},$$

and so

$$k = (4a)^n \left(\frac{n^2}{2}\right)^n \leqslant (4a)^n n^{2n} \leqslant (4a)^{n+2n^2/3} \leqslant (4a)^{n^2/3+2n^2/3} = (4a)^{n^2}.$$

This inequality implies that

$$k^{2n} \leqslant (4a)^{2n^3} = (4a)^{a^2n^3\times\frac{2}{a^2}} \leqslant (4a)^{\frac{8}{25}a^2n^3},$$

and so, since

$$\tfrac{24}{5}+\tfrac{8}{25} < 6,$$

finally

$$(4a)^{\frac{6}{5}a^2n^3} \leqslant K \leqslant (4a)^{6a^2n^3}.$$

18. The right-hand sides of (36) and (37) can now easily be evaluated. In the formula (36),

$$(2k)^{\frac{-1}{\beta-2}} K \leqslant K \leqslant (4a)^{6a^2n^3}.$$

Also

$$k = (4a)^n \left(\frac{n^2}{2}\right)^n \geqslant (4a)^n$$

and hence

$$(2k)^{\frac{1}{\beta-2}} \geqslant 4a$$

because $\beta-2 < n$. It follows that

$$N_2 \leqslant \frac{\log \dfrac{\log\{(4a)^{6a^2n^3}\}}{\log(4a)}}{\log(\beta-1)} + 1.$$

Since $\beta > a$, we find that

(38) $$N_2 \leqslant \frac{\log(6a^2n^3)}{\log(a-1)} + 1.$$

19. Put

$$L = k^{1/\beta}|x_0, y_0|;$$

then

$$L > K.$$

The upper bound for N_3 may be written as

$$N_3 \leqslant \frac{\log \dfrac{\log\{(2k)^{\frac{-1}{\beta-2}} L^{2n^3/\vartheta}\}}{\log\{(2k)^{\frac{-1}{\beta-2}} k^{-1/\beta} L\}}}{\log(\beta-1)} + 1.$$

Also this expression will now be simplified.

Since $\beta > a \geqslant \frac{5}{2}$, we have

$$(2k)^{\frac{1}{\beta-2}} k^{\frac{1}{\beta}} \leqslant 4k^2 \cdot k^{\frac{2}{-}} \leqslant 4(4a)^{\left(2+\frac{2}{5}\right)n^2} \leqslant (4a)^{\frac{12}{5}n^2+1} \leqslant (4a)^{\left(\frac{12}{5}+\frac{1}{n}\right)n^2} \leqslant (4a)^{3n^2},$$

so that, by the lower bound for K,

$$L^{\frac{1}{2}} \geqslant K^{\frac{1}{2}} \geqslant (4a)^{\frac{3}{5}a^2n^3} = (4a)^{3n^2\times\frac{a^2n}{5}} \geqslant (4a)^{3n^2\times\frac{(5/2)^2\cdot 3}{5}} \geqslant (4a)^{3n^2}$$
$$\geqslant (2k)^{\frac{1}{\beta-2}} k^{\frac{1}{\beta}}.$$

It follows then that

$$(2k)^{\frac{-1}{\beta-2}} k^{-\frac{1}{\beta}} L \geqslant L^{1/2},$$

hence that

$$N_3 \leqslant \frac{\log \dfrac{\log(L^{2n^3/\vartheta})}{\log(L^{1/2})}}{\log(\beta-1)} + 1,$$

whence

$$N_3 \leqslant \frac{\log\left(\dfrac{4n^3}{\vartheta}\right)}{\log(\beta-1)} + 1 \leqslant \frac{\log\{8n^3(an+1)\}}{\log(a-1)} + 1.$$

Here

$$a = \min_{h=1,2,\ldots,n-1}\left(\frac{n}{h+1}+h\right) \leqslant \frac{n}{2}+1 = n - \frac{n-2}{2} \leqslant n - \frac{1}{2},$$

and hence

$$an+1 \leqslant (n-\tfrac{1}{2})n+1 = n^2 - \frac{n-2}{2} \leqslant n^2.$$

Thus, finally,

(39) $$N_3 \leqslant \frac{\log(8n^5)}{\log(a-1)} + 1.$$

On adding (38) and (39), we obtain the further estimate,

(40) $$N_2 + N_3 \leqslant \frac{\log(48a^2n^8)}{\log(a-1)} + 2.$$

20. We had chosen

$$\beta = a + \frac{1}{n}, \qquad \gamma = n.$$

The quantity σ is then given by

$$\sigma = \frac{\gamma-\beta}{\beta} = \frac{n^2 - an - 1}{an+1},$$

and v is the smallest positive integer satisfying

$$v \geqslant \frac{1}{\sigma}(t+1).$$

Let us now apply Lemma 3 to the equation

$$\Phi(x, y) = 1.$$

But instead of considering only the admissible pairs x, y with

$$(26) \qquad |x, y| > k^{2/\beta},$$

et us impose the stronger condition

$$(41) \qquad |x, y| \geqslant k^{\frac{2}{\beta-2}}.$$

In other word, we assume that x, y belongs to one of the subsets S_2 or S_3 of S, and we exclude the elements of the subset S_1.

The inequality (40) gives an upper bound for the number of such admissible pairs. We have exactly the same bound for all $\binom{v+t}{t}$ choices of the $t+1$ integers $f_0, f_1, \ldots, f_t$, and for all n^{t+1} choices of the $t+1$ zeros $\zeta, \zeta^{(1)}, \ldots, \zeta^{(t)}$ of $f(x)$.

We obtain thus the result that there are not more than

$$(42) \qquad \left[\frac{\log(48a^2n^8)}{\log(\alpha-1)}+2\right]\binom{v+t}{t} n^{t+1}$$

admissible pairs x, y for which

$$\Phi(x, y) = 1, \qquad |x, y| \geqslant (2n^2a)^{\frac{2n}{\beta-2}}.$$

Here again only one of the two pairs x, y and $-x, -y$, say the pair with $y > 0$, has been counted.

21. The integer v was chosen such that

$$\frac{1}{\sigma}(t+1) \leqslant v < \frac{1}{\sigma}(t+1)+1$$

and hence that

$$(43) \qquad v+t \leqslant \left(\frac{1}{\sigma}+1\right)(t+1).$$

Hence, when t is small, it is advantageous to use the obvious estimate

$$(44) \qquad 0 < \binom{v+t}{t} \leqslant \frac{(v+t)^t}{t!} \leqslant \left(\frac{1}{\sigma}+1\right)^t \frac{(t+1)^t}{t!}$$

for the binomial coefficient. If, however, t is large, there is a better estimate which is obtained as follows.

By Cauchy's theorem, applied to the function $(1+z)^{v+t}$,

$$\binom{v+t}{t} = \frac{1}{2\pi i}\int_C \frac{(1+z)^{v+t}}{z^{t+1}}\,dz$$

where C denotes, say, the circle of radius ϱ with centre at $z = 0$, described in positive direction. Therefore,

$$0 < \binom{v+t}{t} \leqslant \frac{1}{2\pi}\cdot 2\pi\varrho\cdot\frac{(1+\varrho)^{v+t}}{\varrho^{t+1}} = \frac{(1+\varrho)^{v+t}}{\varrho^t},$$

and on choosing $\varrho = t/v$,

$$0 < \binom{v+t}{t} \leqslant \frac{(v+t)^{v+t}}{v^v t^t}.$$

Hence, by (43),

$$0 < \binom{v+t}{t} \leqslant \left(1+\frac{1}{t}\right)^t (t+1)\left\{\left(\frac{1}{\sigma}+1\right)(\sigma+1)^{1/\sigma}\right\}^{t+1}.$$

Since

$$\left(1+\frac{1}{t}\right)^t \leqslant e$$

for all positive integers t, it follows then that

$$(45) \qquad 0 < \binom{v+t}{t} \leqslant e(t+1)\left\{\left(\frac{1}{\sigma}+1\right)(\sigma+1)^{1/\sigma}\right\}^{t+1}.$$

Here, by definition,

$$\sigma = \frac{n^2-an-1}{an+1}, \qquad \sigma+1 = \frac{n^2}{an+1}, \qquad \frac{1}{\sigma}+1 = \frac{n^2}{n^2-an-1}.$$

On substituting these upper bounds in (42), we obtain the following result.

Theorem 2. *Let $F(x, y)$ be a binary form of degree $n \geqslant 3$ with integral coefficients and non-zero discriminant satisfying*

$$F(1, 0) \neq 0 \quad \textit{and} \quad F(0, 1) \neq 0.$$

Let $a = H(F)$ be the height of $F(x, y)$; let

$$\alpha = \min_{h=1,2,\ldots,n-1}\left(\frac{n}{h+1}+h\right), \qquad \beta = \alpha+\frac{1}{n};$$

and let $p_1, \ldots, p_t$ be any finite number of distinct primes.

(i) *There are not more than*

$$2^{\frac{\beta+2}{\beta-2}}(2n^2a)^{\frac{4n}{\beta-2}}+$$

$$+e(t+1)\left[\frac{\log(48a^2n^8)}{\log(a-1)}+2\right]\left\{\frac{n^3}{n^2-an-1}\left(\frac{n^2}{an+1}\right)^{(an+1)/(n^2-an-1)}\right\}^{t+1}$$

pairs of integers x, y satisfying

$$x\neq 0,\quad y>0,\quad (x,y)=1,\quad F(x,y)\neq 0,$$

for which $F(x,y)$ has no prime factor distinct from $p_1,\dots,p_t$.

(ii) *There are not more than*

$$e(t+1)\left[\frac{\log(48a^2n^8)}{\log(a-1)}+2\right]\left\{\frac{n^3}{n^2-an-1}\left(\frac{n^2}{an+1}\right)^{(an+1)/(n^2-an-1)}\right\}^{t+1},$$

pairs of integers x, y satisfying

$$x\neq 0,\quad y>0,\quad (x,y)=1,\quad F(x,y)\neq 0,\quad |x,y|\geqslant (2n^2a)^{\frac{2n}{\beta-2}},$$

for which $F(x,y)$ has no prime factors distinct from $p_1,\dots,p_t$.

(iii) *If p is a sufficiently large prime, there are not more than*

$$2\left[\frac{\log(48a^2n^8)}{\log(a-1)}+2\right]\left(\frac{n^2}{n^2-an-1}\right)n^2$$

pairs of integers x, y satisfying

$$x\neq 0,\quad y>0,\quad (x,y)=1,$$

for which $\mp F(x,y)$ is equal to p or a power of p.

22. The upper bounds in the second and the third parts of the theorem are of particular interest because they do not depend on the coefficients of the form, but only on its degree.

Computation shows that the factor

$$\left[\frac{\log(48a^2n^8)}{\log(a-1)}+2\right]$$

is equal to 37 for $n=3$, 26 for $n=4$, and 22 for $n=5$. With increasing n it first decreases to a minimum 16 and then increases again, first to 17 and 18 and then to 19. The latter value it retains for all sufficiently large n.

The expression

$$\frac{n^2}{n^2-an-1}\left(\frac{n^2}{an+1}\right)^{(an+1)/(n^2-an-1)},$$

that occurs both in the first and the second part of the theorem as a factor of the basis of the $(t+1)$ st power, is about 47.7 for $n=3$, 13.1 for $n=4$, and 9.1 for $n=5$. It has the limit 1 as n tends to infinity and is always less than 2 when $n\geqslant 43$.

In a weakened, but simpler form, the theorem may thus be stated as follows.

There exist four positive absolute constants c_1, c_2, c_3, and c_4, i. e. numbers which do not depend on the binary form $F(x,y)$, on the primes $p_1,\dots,p_t$, or on their number t, such that the upper bound in the first part of the theorem is not greater than

$$c_1(an)^{c_2\sqrt{n}}+(c_3n)^{t+1},$$

that in the second part is not greater than

$$(c_3n)^{t+1},$$

and that in the third part not greater than

$$c_4n^2.$$

We see, in particular, that if m is an integer of sufficiently large absolute value and with exactly t distinct prime factors, there cannot be more than

$$(c_3n)^{t+1}$$

pairs of integers x, y satisfying

$$x\neq 0,\quad y>0,\quad (x,y)=1,\quad F(x,y)=m.$$

23. As a second choice of the parameters, let

$$\beta=a+\frac{1}{n},\quad \gamma=a+\frac{4}{3n},\quad \delta=n-a-\frac{4}{3n},$$

$$\Theta=\frac{1}{n},\quad \vartheta=\frac{1}{2(an+1)}.$$

Since the consideration in §§ 17-19 do not depend on the values of γ and δ, we obtain the same upper bounds (35) for N_1 and (40) for N_2+N_3 as before.

On the other hand, σ now has the value

$$\sigma=\frac{\gamma-\beta}{\beta}=\frac{1}{3(an+1)}.$$

Hence, by (45), the binomial coefficient $\binom{v+t}{t}$ satisfies the inequality

$$0 < \binom{v+t}{t} \leqslant e(t+1)\left\{(3an+4)\left(1+\frac{1}{3(an+1)}\right)^{3(an+1)}\right\}^{t+1}.$$

Here

$$\left(1+\frac{1}{3(an+1)}\right)^{3(an+1)} \leqslant e.$$

The following result is then obtained by repeating the discussion in §§ 20-21.

THEOREM 3. *Let the notation be as in Theorem 2.*

(i) *There are not more than*

$$2^{\frac{\beta+2}{\beta-2}}(2n^2a)^{\frac{4n}{\beta-2}}+e(t+1)\left[\frac{\log(48a^2n^8)}{\log(a-1)}+2\right]\{en(3an+4)\}^{t+1}$$

pairs of integers x, y such that

$$x \neq 0, \quad y > 0, \quad (x, y) = 1,$$

$$0 < |F(x,y)|\prod_{\tau=1}^{t}|F(x,y)|_{p_\tau} \leqslant |x,y|^{n-a-\frac{4}{3n}}.$$

(ii) *There are not more than*

$$e(t+1)\left[\frac{\log(48a^2n^8)}{\log(a-1)}+2\right]\{en(3a+4)\}^{t+1}$$

pairs of integers x, y such that

$$x \neq 0, \quad y > 0, \quad (x, y) = 1, \quad |x,y| \geqslant (2n^2a)^{\frac{2n}{\beta-2}},$$

$$0 < |F(x,y)|\prod_{\tau=1}^{t}|F(x,y)|_{p_\tau} \leqslant |x,y|^{n-a-\frac{4}{3n}}.$$

If c_5 denotes a further positive absolute constant, the upper bound in (i) has the form

$$c_1(an)^{c_2\sqrt{n}}+(c_5n^{5/2})^{t+1},$$

while that in (ii) has the form

$$(c_5n^{5/2})^{t+1}.$$

24. We conclude this paper with an application of Theorem 2. Let

$$p_{11}, \ldots, p_{1r}, p_{21}, \ldots, p_{2s}, p_{31}, \ldots, p_{3t}$$

be $r+s+t$ fixed distinct primes of which the smallest and the largest are P and Q, say. Further let

$$\{x_{ij}\} = \{x_{11}, \ldots, x_{1r}, x_{21}, \ldots, x_{2s}, x_{31}, \ldots, x_{3t}\}$$

be a system of $r+s+t$ non-negative integers such that

$$(46) \qquad p_{11}^{x_{11}}\ldots p_{1r}^{x_{1r}}+p_{21}^{x_{21}}\ldots p_{2s}^{x_{2s}} = p_{31}^{x_{31}}\ldots p_{3t}^{x_{3t}}.$$

Our aim is to give an upper bound for the number of solutions $\{x_{ij}\}$ of this equation.

Denote by $n \geqslant 3$ an integer which will soon be chosen equal to 12. For each pair of suffixes i and $j = 1$ or 2 write

$$x_{ij} = nX_{ij}+Y_{ij}$$

where X_{ij} is a non-negative integer while Y_{ij} is one of the numbers $0, 1, \ldots, n-1$; further put

$$x = p_{11}^{X_{11}}\ldots p_{1r}^{X_{1r}}, \qquad y = p_{21}^{X_{21}}\ldots p_{2s}^{X_{2s}},$$

$$a_0 = p_{11}^{Y_{11}}\ldots p_{1r}^{Y_{1r}}, \qquad a_n = a_{21}^{Y_{21}}\ldots p_{2s}^{Y_{2s}}.$$

The equation (46) becomes then

$$(47) \qquad a_0x^n+a_ny^n = p_{31}^{x_{31}}\ldots p_{3t}^{x_{3t}}$$

where evidently

$$a_0 > 0, \quad a_n > 0, \quad x > 0, \quad y > 0, \quad (x,y) = 1, \quad a_0x^n+a_ny^n > 0.$$

The binary form on the left-hand side of (47) has the height

$$a = \max(a_0, a_n)$$

which satisfies the inequality

$$(48) \qquad a \leqslant Q^{(n-1)\max(r,s)}.$$

Also the pair of coefficients a_0 and a_n has only

$$n^{r+s}$$

possibilities.

For each pair of coefficients a_0 and a_n we divide now the solutions x, y of (47) into two classes C_1 and C_2 according as

$$|x,y| < (2n^2Q^{(n-1)\max(r,s)})^{\frac{2n}{\beta-2}} \quad \text{or} \quad |x,y| \geqslant (2n^2Q^{(n-1)\max(r,s)})^{\frac{2n}{\beta-2}},$$

and we denote by N_1 and N_2 the numbers of elements of C_1 and C_2, respectively. We further choose for n the value

$$n = 12, \quad \text{so that} \quad \alpha = 6, \quad \beta > 6, \quad \frac{2n}{\beta-2} < 6.$$

An upper bound for N_1 is found as follows. In explicit form,

$$\max(p_{11}^{X_{11}} \ldots p_{1r}^{X_{1r}}, p_{21}^{X_{21}} \ldots p_{2s}^{X_{2s}}) < 288^6 Q^{66\max(r,s)},$$

so that

$$\max(X_{11}+\ldots+X_{1r}, X_{21}+\ldots+X_{2s}) < \frac{6\log 288 + 66\max(r,s)\log Q}{\log P}.$$

This implies that each of the integers $X_{11}, \ldots, X_{1r}, X_{21}, \ldots, X_{2s}$ is smaller than the expression on the right-hand side and so has at most

$$\frac{1}{12} c_6 (r+s) \frac{\log Q}{\log P}$$

possibilities where c_6 is a positive absolute constant. It follows then that

$$N_1 \leqslant \left\{\frac{1}{12} c_6 (r+s) \frac{\log Q}{\log P}\right\}^{r+s}.$$

An upper bound for N_2 is obtained immediately from Theorem 2. It has the form

$$N_2 \leqslant c_7^{t+1}$$

where $c_7 = 12c_3$ is another positive absolute constant.

As the solutions of (46) satisfy 12^{r+s} equations (47), it follows finally that the equation (46) has not more than

$$\left\{c_6 (r+s) \frac{\log Q}{\log P}\right\}^{r+s} + c_8^{r+s+t+1}$$

solutions $\{x_{ij}\}$; here c_8 is a further positive absolute constant.

It would have great interest to decide whether this upper bound can be replaced by one that is independent of the given $r+s+t$ primes, thus of P and Q, and depends only on the number $r+s+t$ of the primes.

For the last result, compare also Chapter 1, §§ 1-4, and Chapter 3, § 3, of the book on transcendental numbers by Gelfond, and p. 724 of the paper M_1.

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