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Singularities of harmonic functions of three variables generated by Whittaker-Bergman operators

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Introduction. As has been shown in investigations of Bergman (see [3] and the bibliography cited there), it is useful to introduce various integral operators transforming analytic functions of one and several complex varaibles into solutions of linear partial differential equations of even order. These operators, representing a generalization of the real part operator generating harmonic functions of two variables, preserve some properties of analytic functions. In this way they give a new insight into the theory of differential equations. Various results obtained in this way are based on the fact that the operators preserve the location and character of the singularities.

The general way of proceeding is based on the fact that, under the Bergman integral operator, there is a one-to-one correspondence in some linear spaces between harmonic functions of three real variables defined in a neighborhood of the origin and analytic functions of two complex variables [1, p. 463]. This is a correspondence in the local sense, in the small. But applying the method of continuation, one passes to a correspondence in the large. Thus, some singularities of analytic functions are translated into singularities of harmonic functions in the large. Investigations in this direction were conducted by Bergman [2], [3], Kreyzig [6], Mitchell [9], and others.

The present paper is devoted to the study of these relations in the case of harmonic functions in three real variables generated by rational functions, in a form as general and exact as possible.

In the first part are investigated the sets of regularity and the locus of singular points of the Bergman integral operator considered as a one valued harmonic function.

In the second part are considered the singular points of the multi-valued harmonic functions generated by the above mentioned one valued functions.

In the third part are discussed the loci of possible singular points of the multi-valued harmonic function generated by a harmonic element expressed as a series of Legendre polynomials in the neighborhood of the origin. The relation between the behavior of the coefficients of the series development and the singularities of the harmonic function is investigated.

Particular examples of the general theory are given.

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1. One valued harmonic functions. Let

(1.1) $$u = u(x, y, z; \zeta) = x + \tfrac{1}{2}(iy+z)\zeta + \tfrac{1}{2}(iy-z)\cdot 1/\zeta$$

where x, y, z are real variables in the whole space and ζ is a complex variable. Let $P(u, \zeta)$ and $Q(u, \zeta)$ be polynomials in ζ and u.

We consider the following Whittaker-Bergman integral operator which transforms analytic functions of u, ζ into uniform harmonic functions [1, p. 467], [15]

(1.2) $$F(x, y, z) = \frac{1}{2\pi i}\int_{|\zeta|=1} \frac{P(u, \zeta)}{Q(u, \zeta)}\, d\zeta .$$

This integral is defined for all (x, y, z) for which the equation in ζ

(1.3) $$Q(u, \zeta) = 0$$

has no root on the unit circumference $|\zeta| = 1$.

Denote by D the set of points (x, y, z) for which (1.3) has no root on the unit circumference and denote by D^* the boundary of D. One sees that the boundary points $(x^*, y^*, z^*) \in D^*$ can be characterized by a simple necessary condition.

For every $(x^*, y^*, z^*) \in D^*$ there must exist a root ζ^* of $Q(u^*, \zeta)$ on the circle $|\zeta| = 1$, where

$$u^* = x^* + \tfrac{1}{2}(iy^*+z^*)\zeta + \tfrac{1}{2}(iy^*-z^*)\cdot 1/\zeta .$$

This condition can be expressed in an algebraic form as follows. In view of (1.1), we put

(1.4) $$Q(u, \zeta) = [a_0\zeta^n + a_1\zeta^{n-1} + \ldots + a_{n-1}\zeta + a_n]\cdot 1/\zeta^{\nu}$$

where

$$a_j = a_j(x, y, z), \quad j = 1, 2, \ldots, n$$

are polynomials in x, y, z, and ν is the smallest integer for which Q can be written in form (1.4). For a point $(x^*, y^*, z^*) \in D^*$ and the corresponding root ζ^* on the unit circumference

(1.5) $$a_0^*\zeta^{*n} + a_1^*\zeta^{*n-1} + \ldots + a_{n-1}^*\zeta^* + a_n^* = 0$$

where

$$a_j^* = a_j(x^*, y^*, z^*), \quad j = 1, 2, \ldots, n .$$

From the conjugate of (1.5), since

$$\bar{\zeta}^* = 1/\zeta^* ,$$

we obtain

(1.6) $$\bar{a}_n^*\zeta^{*n} + \bar{a}_{n-1}^*\zeta^{*n-1} + \ldots + \bar{a}_1^*\zeta + \bar{a}_0^* = 0$$

where $\bar{a}_j^*$ is the conjugate of a_j. Eliminating ζ^* from equations (1.5) and (1.6) by Sylvester's dialytic method we obtain

(1.7) $$\begin{vmatrix} a_0^* & a_1^* & \ldots & a_n^* & 0 & \ldots & 0 \\ 0 & a_0^* & \ldots & a_{n-1}^* & a_n^* & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \bar{a}_n^* & \bar{a}_{n-1}^* & \ldots & \bar{a}_0^* & 0 & \ldots & 0 \\ 0 & \bar{a}_n^* & \ldots & \bar{a}_1^* & \bar{a}_0^* & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0 .$$

This equation defines the locus of points of the boundary D^*.

Consider an arbitrary point (x^*, y^*, z^*) of the space. We shall call this point *regular with respect to the operator* (1.2) if this operator can be continued through this point along each curve $L^{\cdot}$ which goes from D to this point. More exactly, this means that there is a harmonic function $F^*(x, y, z)$ defined in a neighborhood of (x^*, y^*, z^*) so that for every point $(x^{\cdot}, y^{\cdot}, z^{\cdot}) \in L^{\cdot}$ sufficiently near (x^*, y^*, z^*) we have

$$F^*(x, y, z) = F(x, y, z)$$

for every (x, y, z) sufficiently near $(x^{\cdot}, y^{\cdot}, z^{\cdot})$.

If a point is not regular it will be called *singular*. We shall need in the future the discriminant of $Q(u, \zeta)$ by which we shall understand the discriminant of the expression in brackets in (1.4), which means the following determinant

(1.8) $$\begin{vmatrix} a_0 & a_1 & \ldots & a_n & 0 & \ldots & 0 \\ 0 & a_0 & \ldots & a_{n-1} & a_n & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ na_0 & (n-1)a_1 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & na_0 & \ldots & a_{n-1} & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix},$$

where $a_j = a_j(x, y, z)$ are polynomials in x, y, z.

Suppose that the discriminant of $Q(u, \zeta)$ with respect to ζ does not vanish identically and denote by S^* the set of all points x, y, z for which (1.8) vanishes. Then we can prove the following:

THEOREM 1. *A necessary condition that a point (x^*, y^*, z^*) will be a singular point of the operator (1.2) is that the determinant (1.7) and the discriminant (1.8) vanish simultaneously at this point.*

In other words, *the locus possible of the singular points of the operator* (1.2) *is the intersection of the sets* D^* *and* S^*.

Proof. The condition that determinant (1.7) vanishes is evident. Hence it suffices to show that at the singular point $(x^*, y^*, z^*) \in D^*$ the discriminant (1.8) must also vanish.

Suppose that the discriminant (1.8) does not vanish at a point $(x^*, y^*, z^*) \in D^*$. We shall show that in this case (x^*, y^*, z^*) is a regular point. Indeed, denote by

$$\zeta_1^*, \zeta_2^*, \ldots, \zeta_n^* \tag{1.9}$$

all the roots of the equation in ζ

$$Q(u^*, \zeta) = 0, \quad u^* = x^* + \tfrac{1}{2}(iy^* + z^*)\zeta + \tfrac{1}{2}(iy^* - z^*)\cdot 1/\zeta. \tag{1.10}$$

The roots (1.9) are all simple in view of the assumption that the discriminant does not vanish.

Consider an arbitrary root ζ_k^*. It follows from the theorem of the continuity of the roots of algebraic equations [14, p. 148] that for every point (x, y, z) in a sufficiently small neighborhood δ_k of (x^*, y^*, z^*) there is a root

$$\zeta_k = \varphi_k(x, y, z) \tag{1.11}$$

of the equation

$$Q(u, \zeta) = 0, \quad u = x + \tfrac{1}{2}(iy + z)\zeta + \tfrac{1}{2}(iy - z)\cdot 1/\zeta, \tag{1.12}$$

where $(x, y, z) \in \delta_k$, so that this root is an analytic function of x, y, z and that

$$\zeta_k^* = \varphi_k(x^*, y^*, z^*). \tag{1.13}$$

Now consider an arbitrary curve

$$L^{\cdot}: \quad x(t), y(t), z(t), \quad 0 \leqslant t \leqslant 1, \tag{1.14}$$

which goes through the interior of D to the point (x^*, y^*, z^*) as t increases from 0 to 1; $L^{\cdot}$ does not cross the set D^*. Then for the points $(x^{\cdot}, y^{\cdot}, z^{\cdot}) \in L^{\cdot}$ which lie in the interior of the intersection δ of the neighborhoods $\delta_1, \delta_2, \ldots, \delta_n$, the corresponding roots

$$\zeta_k^{\cdot} = \varphi_k(x^{\cdot}, y^{\cdot}, z^{\cdot}), \quad k = 1, 2, \ldots, n \tag{1.15}$$

of the equation

$$Q(u^{\cdot}, \zeta) = 0, \quad u^{\cdot} = x^{\cdot} + \tfrac{1}{2}(iy^{\cdot} + z^{\cdot})\zeta + \tfrac{1}{2}(iy^{\cdot} - z^{\cdot})\cdot 1/\zeta \tag{1.16}$$

never lie on the unit circumference; that is, for every k, either

$$|\zeta_k^{\cdot}| < 1 \quad \text{or} \quad |\zeta_k^{\cdot}| > 1.$$

This follows from the assumption that the line $L^{\cdot}$ does not cross the boundary set D^*.

Suppose that for the above mentioned points $(x^{\cdot}, y^{\cdot}, z^{\cdot})$ the first γ roots

$$\zeta_k = \varphi_k(x^{\cdot}, y^{\cdot}. z^{\cdot}), \quad k = 1, 2, \ldots, \gamma \leqslant n \tag{1.15'}$$

lie in the interior of the unit circle. Then we have, according to [16, ch. VI], [14, p. 148]

$$\begin{gathered} F(x, y, z) = \sum_{k=1}^{\gamma} \frac{P(u_k, \varphi_k(x, y, z))}{Q'_\zeta(u_k, \varphi_k(x, y, z))}, \\ u_k = x + \tfrac{1}{2}(iy + z)\zeta_k + \tfrac{1}{2}(iy - z)\cdot 1/\zeta_k, \end{gathered} \tag{1.17}$$

where $\zeta_k = \varphi_k(x, y, z)$ in a sufficiently small neighborhood of $(x^{\cdot}, y^{\cdot}, z^{\cdot})$ contained in δ.

On the other hand, for $(x, y, z) \in \delta$, we can define the harmonic function $F^*(x, y, z)$ by the formula

$$F^*(x, y, z) = \sum_{k=1}^{\gamma} \frac{P(u_k, \zeta_k)}{Q'_\zeta(u_k, \zeta_k)}, \tag{1.18}$$

where

$$\zeta_k = \varphi_k(x, y, z), \quad u_k = x + \tfrac{1}{2}(iy + z)\zeta_k + \tfrac{1}{2}(iy - z)\cdot 1/\zeta_k.$$

Then in each sufficiently small neighborhood of every point $(x^{\cdot}, y^{\cdot}, z^{\cdot})$ of $L^{\cdot}$ sufficiently close to (x^*, y^*, z^*), we have, according to (1.17), (1.18)

$$F^*(x, y, z) = F(x, y, z). \tag{1.19}$$

Hence, this shows that (1.2) can be continued along the arbitrary curve $L^{\cdot}$ through the point (x^*, y^*, z^*). Thus (x^*, y^*, z^*) is a regular point, and our assertion that the only singular points of D^* are those for which the discriminant vanishes is proved.

In the following examples we determine the locus of the boundary surfaces D^* and of the possible singular points on this boundary for some particular cases.

Example 1. Consider an operator of the form

$$F(x, y, z) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{d\zeta}{(u - a)\zeta}, \quad u = x + \tfrac{1}{2}(iy + z)\zeta + \tfrac{1}{2}(iy - z)\cdot 1/\zeta, \tag{1.20}$$

where $a = \lambda + i\mu$ is a complex number such that the right hand side is defined.

According to the earlier remarks, the locus of the boundary points of D^* is the set of points (x, y, z) satisfying equation (1.7). However, in some special cases it is more convenient to use another condition.

For the operator (1.20) the boundary D^* is the set of points (x, y, z) where the equation

$$(u-a)\zeta = 0 \tag{1.21a}$$

has a root ζ on the unit circumference.

Writing (1.21a) in expanded form we obtain

$$\tfrac{1}{2}(iy+z)\zeta^2+(x-\lambda-i\mu)\zeta+\tfrac{1}{2}(iy-z) = 0\,. \tag{1.21b}$$

Suppose that (1.21) has a root $\zeta_1 = e^{i\theta_1}$. The modulus of the product of the roots of (1.21) is

$$\left|\frac{iy-z}{iy+z}\right| = 1\,. \tag{1.22}$$

Since the product of the roots of (1.21) is of modulus unity, there must be another root on the unit circumference. Call this second root $\zeta_2 = e^{i\theta_2}$.

It follows then, that

$$\frac{2(x-\lambda-i\mu)}{iy+z} = -(e^{i\theta_1}+e^{i\theta_2})\,, \qquad \sqrt{\frac{iy-z}{iy+z}} = e^{i(\theta_1+\theta_2)/2}\,, \tag{1.23}$$

and dividing the first expression by the second we obtain

$$\frac{2(x-\lambda-i\mu)}{\sqrt{-y^2-z^2}} = -(e^{i(\theta_1-\theta_2)/2}+e^{i(\theta_2-\theta_1)/2}) = -2\cos\tfrac{1}{2}(\theta_2-\theta_1)\,. \tag{1.24}$$

Thus, in general, necessary conditions following from the assumption that $(x, y, z)\in D^*$ in this case are that

$$\frac{x-\lambda-i\mu}{\sqrt{-y^2-z^2}} \text{ is real, and } \left|\frac{x-\lambda-i\mu}{\sqrt{-y^2-z^2}}\right| \leqslant 1\,. \tag{1.25}$$

Thus

$$\operatorname{Im}\left\{\frac{x-\lambda-i\mu}{i\sqrt{y^2+z^2}}\right\} = 0\,, \tag{1.26}$$

and

$$\left|\frac{x-\lambda-i\mu}{i\sqrt{y^2+z^2}}\right|^2 = \frac{(x-\lambda)^2+\mu^2}{y^2+z^2} \leqslant 1\,, \tag{1.27}$$

which implies

$$x = \lambda\,, \qquad y^2+z^2 \geqslant \mu^2\,. \tag{1.28}$$

The locus of the boundary set D^* in this case is the plane $x = \lambda$ excluding the disc $y^2+z^2 < \mu^2$. We shall call this *the set of apparent singularities.*

Note that for $\mu = 0$ the disc reduces to the point $(\lambda, 0, 0)$.

EXAMPLE 2. We will now consider the operator of the form

$$F(x, y, z) = \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{d\zeta}{au^2\zeta^2+bu\zeta+c}\,, \tag{1.29}$$

where a, b, c are some complex numbers, and u is given by (1.1).

The locus of the boundary set D^* is the set of points in x, y, z space for which the equation

$$au^2\zeta^2+bu\zeta+c = 0 \tag{1.30}$$

has a root ζ on the unit circumference.

Suppose that the equation (1.30) has a root ζ on the unit circumference. We then have

$$au^2\zeta^2+bu\zeta+c = 0\,, \qquad |\zeta| = 1\,. \tag{1.31}$$

It follows that

$$u\zeta = \frac{-b+\sqrt{b^2-4ac}}{2a} = p_1 \quad\text{or}\quad u\zeta = \frac{-b-\sqrt{b^2-4ac}}{2a} = p_2\,. \tag{1.32}$$

The conditions (1.32) can be written in the form

$$\tfrac{1}{2}(iy+z)\zeta^2+x\zeta+\tfrac{1}{2}(iy-z)-p_k = 0\,, \qquad k = 1 \text{ or } 2\,. \tag{1.33}$$

Since $|\zeta| = 1$, the conjugate of (1.33) yields

$$[\tfrac{1}{2}(-z-iy)-\overline{p}_k]\zeta^2+x\zeta+\tfrac{1}{2}(z-iy) = 0\,, \qquad k = 1 \text{ or } 2\,. \tag{1.34}$$

A necessary condition that (1.33) and (1.34) have a common root is that the resultant of the two equations should vanish.

The resultant is as follows:

$$\begin{aligned}&\{\tfrac{1}{4}(y^2+z^2)-[\tfrac{1}{2}(iy-z)-p_k][\tfrac{1}{2}(-iy-z)-\overline{p}_k]\}^2-\\ &-\{\tfrac{1}{2}(iy+z)x-x[\tfrac{1}{2}(-y-z)-\overline{p}_k]\}\{\tfrac{1}{2}(-iy+z)x-x[\tfrac{1}{2}(iy-z)-p_k]\}\\ &\qquad= (y\mu_k-z\lambda_k)^2-2(y\mu_k-z\lambda_k)(\lambda_k^2+\mu_k^2)+(\lambda_k^2+\mu_k^2)^2-\\ &\qquad\quad -x^2[y^2+z^2-2(y\mu_k-z\lambda_k)+(\lambda_k^2+\mu_k^2)]\,,\end{aligned} \tag{1.35}$$

where

$$p_k = \lambda_k-i\mu_k\,, \qquad k = 1 \text{ or } 2\,. \tag{1.36}$$

The surfaces obtained by setting the resultant (1.35) equal to zero, for $k = 1$ and 2 give the expected locus of the points of D^*.

They are ruled surfaces. This can easily be demonstrated as follows.

Writing equation (1.32) in an expanded form, we have

$$u = x+\frac{1}{2}iy\left(\zeta+\frac{1}{\zeta}\right)+\frac{1}{2}z\left(\zeta-\frac{1}{\zeta}\right) = \frac{1}{\zeta}(\lambda_k+i\mu_k)\,, \qquad k = 1 \text{ or } 2\,. \tag{1.37}$$

For every fixed ζ (1.37) defines a straight line in the x, y, z space. If ζ ranges over the circle $|\zeta| = 1$, the family (1.37) generates a ruled surface which we will show to be identical to the corresponding surface given by (1.35).

Put

(1.38) $$\zeta = \cos\theta + i\sin\theta\,, \qquad 1/\zeta = \cos\theta - i\sin\theta\,.$$

Then we have

(1.39) $$x + i(y\cos\theta + z\sin\theta) = (\lambda_k + i\mu_k)(\cos\theta - i\sin\theta)\,, \qquad k = 1 \text{ or } 2\,.$$

Equating real and imaginary parts, the ruled surface is found by eliminating θ from the system

(1.40) $$\begin{gathered}\lambda_k\cos\theta + \mu_k\sin\theta = x\,,\\ (y-\mu_k)\cos\theta + (z+\lambda_k)\sin\theta = 0\,,\\ \cos^2\theta + \sin^2\theta = 1\,.\end{gathered}$$

We obtain

(1.41) $$[x(z+\lambda_k)]^2 + [x(y-\mu_k)]^2 = [(y\mu_k - z\lambda_k) - (\lambda_k^2 + \mu_k^2)]^2$$

from which follows

(1.42) $$\begin{gathered}(y\mu_k - z\lambda_k)^2 - 2(y\mu_k - z\lambda_k)(\lambda_k^2+\mu_k^2) + (\lambda_k^2+\mu_k)^2 -\\ - x^2[y^2 + z^2 - 2(y\mu_k - z\lambda_k) + (\lambda_k^2 + \mu_k^2)] = 0\,, \qquad k = 1 \text{ or } 2\,.\end{gathered}$$

The left side of (1.42) is identical to the resultant (1.35).

EXAMPLE 1′. We shall now investigate the locus of possible singular points of the one valued harmonic function defined by the integral (1.20):

$$F(x, y, z) = \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{d\zeta}{(u-a)\zeta}\,.$$

Since the locus of the points (x, y, z) of D^* for (1.20) has already been determined, in order to find the locus of singular points it suffices to obtain the locus of points (x, y, z) where the discriminant of the denominator of (1.20) vanishes (assuming that the discriminant does not vanish identically). The intersection of the two loci will be the locus of the possible singular points of (1.20).

The discriminant of the denominator of (1.20) is

(1.43) $$(x - \lambda - i\mu)^2 + (y^2 + z^2)\,.$$

The vanishing of (1.43) yields

(1.44) $$x = \lambda\,, \qquad y^2 + z^2 = \mu^2$$

as the locus of the points where the discriminant vanishes.

The locus of the points of D^* for (1.20) is given by (1.28):

$$x = \lambda\,, \qquad y^2 + z^2 \geqslant \mu^2\,.$$

The intersection of (1.44) and (1.28) is obviously (1.44).

EXAMPLE 2′. We shall now investigate the locus of possible singular points of the one valued harmonic function defined by the integral (1.29):

$$F(x, y, z) = \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{d\zeta}{au^2\zeta^2 + bu\zeta + c}\,.$$

Similarly as in example 1′, the locus of the points (x, y, z) of D^* for (1.29) has already been determined in example 2. Thus to find the locus of singular points it suffices to obtain the locus of the points where the discriminant of the denominator of (1.29) vanishes (assuming, as before, that the discriminant does not vanish identically). The intersection of the two locii is the locus of the singular points of (1.29).

The discriminant of the denominator is obtained by eliminating ζ from the equation

(1.45) $$au^2\zeta^2 + bu\zeta + c = 0$$

and its derivative equation with respect to ζ

(1.46) $$(2au\zeta + b)([iy + z]\zeta + x) = 0\,.$$

Now there are two cases of (1.45), (1.46):

(i) $$au^2\zeta^2 + bu\zeta + c = 0 \quad \text{and} \quad 2au\zeta + b = 0\,.$$

Eliminating ζ from (i), we obtain

(1.47) $$b^2 + 4ac = 0$$

Hence, case (i) is impossible, because we assumed that the discriminant does not vanish identically.

(ii) $$au^2\zeta^2 + bu\zeta + c = 0 \quad \text{and} \quad [iy + z]\zeta + x = 0\,.$$

From the second equation of (ii) we obtain

(1.48) $$\zeta = -\frac{x}{iy + z}\,,$$

and consequently substituting (1.48) into the expression for u we obtain

(1.49) $$u\zeta = -\frac{R^2}{2(iy+z)}\,, \qquad R^2 = x^2 + y^2 + z^2\,,$$

which yields, using (1.45) and (1.32)

(1.50)
$$-\frac{R^2}{2(iy+z)}=\frac{R^2(iy-z)}{2(y^2+z^2)}=p_1$$

or

(1.51)
$$-\frac{R^2}{2(iy+z)}=\frac{R^2(iy-z)}{2(y^2+z^2)}=p_2\,.$$

That means, exhibiting the real and imaginary parts, and using (1.36)

(1.52)
$$\frac{R^2z}{y^2+z^2}=-2\lambda_1 \quad\text{and}\quad \frac{R^2y}{y^2+z^2}=2\mu_1\,,$$

or

(1.53)
$$\frac{R^2z}{y^2+z^2}=-2\lambda_2 \quad\text{and}\quad \frac{R^2y}{y^2+z^2}=2\mu_2\,.$$

By division, we obtain from (1.52) or (1.53)

(1.54)
$$z=-\frac{\lambda_1}{\mu_1}y\,,$$

or

(1.55)
$$z=-\frac{\lambda_2}{\mu_2}y\,.$$

Combining (1.52) and (1.54), or (1.53) and (1.55) we obtain

(1.56)
$$x^2+y^2\left(\frac{\lambda_1^2+\mu_1^2}{\mu_1^2}\right)-2\mu_1 y\left(\frac{\lambda_1^2+\mu_1^2}{\mu_1^2}\right)=0$$

or

(1.57)
$$x^2+y^2\left(\frac{\lambda_2^2+\mu_2^2}{\mu_2^2}\right)-2\mu_2 y\left(\frac{\lambda_2^2+\mu_2^2}{\mu_2^2}\right)=0\,.$$

Equations (1,54) and (1.56), or (1.55) and (1.57) represent the intersections of planes and elliptic cylinders which are, in general, ellipses, but which can sometimes be circles or pairs of straight lines.

This is the locus of the zero points of the corresponding discriminant.

The locus of the expected singular points of (1.29) is, of course, the intersection of the curves (1.54), (1.56) and (1.55), (1.57) with the surfaces (1.35). In general, it will consist of a finite number of points of x, y, z space.

2. Multi-valued harmonic functions. We shall consider the multi-valued harmonic functions generated by continuing the element defined in the small by the integral (1.2). First, of course, we must define continuation of harmonic elements, and harmonic functions generated by the continuation of elements in the x, y, z space. Our definitions are analogous to those of analytic functions.

A pair $\{F; p\}$, consisting of the point $p=(x, y, z)$ and the function F harmonic in a neighborhood of the point p, will be called a *harmonic element*; the point p will be called the *center* of the element $\{F; p\}$.

Let us consider the spheres with center at p to which the function F, harmonic in a neighborhood of p, can be extended as a harmonic function, that is, spheres in which there exists a harmonic function identical with F in the neighborhood of the point p. Among these spheres there exists a largest one, which we shall call the *sphere* of the element $\{F; p\}$. The harmonic function in the sphere of the element $\{F; p\}$, identical with F in the neighborhood of the point p, will be denoted by F_p.

Two elements $\{F; a\}$ and $\{G; b\}$ are considered identical, $\{F; a\} = \{G; b\}$ if $a = b$ and if the functions F and G are identical in a neighborhood of the point $a = b$. Of course, there will also be $F_a = F_b$.

Every harmonic element $\{F_a; b\}$, where b is an arbitrary point of the sphere K of the element $\{F; a\}$, will be called a *direct continuation* of the element $\{F; a\}$. The function F_a is defined in the entire sphere K, and hence the element $\{F_a; b\}$ is defined at each point $b \in K$.

A family of harmonic elements $\{\mathcal{R}(t)\}$, $a \leqslant t \leqslant b$, depending on a real parameter t ranging over the interval $\langle a, b\rangle$, is a chain of elements along the curve L given by the equations

$$x=x(t)\,,\quad y=y(t)\,,\quad z=z(t)\,,\quad a\leqslant t\leqslant b$$

if

1. for every $t \in \langle a, b\rangle$ the point $x(t), y(t), z(t)$ is the center of the element $\mathcal{R}(t)$,

2. to every $t \in \langle a, b\rangle$ there corresponds a number $\varepsilon > 0$ such that if $|h| < \varepsilon$ and $t+h \in \langle a, b\rangle$, then the element $\mathcal{R}(t+h)$ is a direct continuation of the element $\mathcal{R}(t)$.

A harmonic element $\mathcal{R}_2$ is called a *continuation* of the element $\mathcal{R}_1$ if there exists a chain of elements $\{\mathcal{R}(t)\}$, $a \leqslant t \leqslant b$ along some curve: $x = x(t)$, $y = y(t)$, $z = z(t)$ where $a \leqslant t \leqslant b$ such that $\mathcal{R}_1 = \mathcal{R}(a)$ and $\mathcal{R}_2 = \mathcal{R}(b)$. The chain $\{\mathcal{R}(t)\}$ joins the element $\mathcal{R}_2$ to $\mathcal{R}_1$.

Every (non empty) family $\mathcal{F}$ of harmonic elements with centers in a region G will be called a *harmonic function* in the region G if

1. given any two elements of the family $\mathcal{F}$, one is a continuation of the other in the region G;

2. every harmonic element which is a continuation in the region G of a harmonic element belonging to $\mathcal{F}$, belongs to $\mathcal{F}$.

If $\mathcal{R}$ is an arbitrary harmonic element with center in a region G, then the set of all continuations of this element in the region G is a harmonic function in G containing $\mathcal{R}$. It is the only harmonic function in G which contains $\mathcal{R}$. Consequently, every harmonic element, with center in a region G, determines a harmonic function in this region.

If $\mathcal{F}$ is a harmonic function in the full space G and H is a subregion contained in G, then every element of $\mathcal{F}$ with center in H determines a harmonic function in H. Every such function is called a *branch* of the harmonic function $\mathcal{F}$ in the subregion H; all of its elements also belong to the function $\mathcal{F}$.

For a harmonic function $\mathcal{F}$ in G, a point g belonging to G will be called an *ordinary point* of the harmonic function $\mathcal{F}$ if it has a neighborhood in which every branch of the function $\mathcal{F}$ is arbitrarily continuable; in the contrary case the point g will be called a *singular point* of the function $\mathcal{F}$ (compare [12]).

Consider, again, two arbitrary polynomials in u, ζ which we denote as before by $P(u, \zeta)$ and $Q(u, \zeta)$. Further, putting $u = x + \frac{1}{2}(iy+z)\zeta + \frac{1}{2}(iy-z)\cdot 1/\zeta$, and representing $Q(u, \zeta)$ in the form (1.4), form the sets D^* and S^* described in theorem 1.

Then the Whittaker-Bergman integral operator

$$F_0(x, y, z) = \frac{1}{2\pi i}\int_{|\zeta|=1} \frac{P(u, \zeta)}{Q(u, \zeta)}\, d\zeta\,, \tag{2.1}$$

where $P(u, \zeta)$ and $Q(u, \zeta)$ are polynomials in ζ, defines a harmonic function in the neighborhood of a point $p_0 = (x_0, y_0, z_0)$ which does not belong to the set D^* and S^*. The pair $\{F_0; p_0\}$ consisting of the function (2.1) and the point p_0 is a harmonic element which generates the harmonic function $\mathcal{F}$. We prove the following:

THEOREM 2. *The singular points of the harmonic function are contained in the set S^*.*

Proof. Obviously it is sufficient to prove that the harmonic element $\{F_0; p_0\}$ can be continued along each curve in $R - S^*$ emanating from the point (x_0, y_0, z_0), where R is the full space. (See Remark 1.).

Suppose, for a moment, that x, y, z are complex variables. Consider the equation

$$a_0\zeta^n + a_1\zeta^{n-1} + \ldots + a_n = 0 \tag{2.2}$$

where the coefficients $a_j = a_j(x, y, z)$, $j = 1, 2, \ldots, n$, are analytic functions of x, y, z in a neighborhood of (x_0, y_0, z_0) defined in the same way as in (1.4). This equation defines a multi-valued algebraic function $\mathcal{A}$. For the point (x_0, y_0, z_0) equation (2.2) can have some simple roots which lie in the interior of the unit circle: $|\zeta| = 1$, which roots we denote by

$$\zeta_{k0}\,, \quad k = 1, 2, \ldots, m\,. \tag{2.3}$$

From well known theorems it follows there is a system of analytic functions

$$\zeta_{k0}(x, y, z)\,, \quad k = 1, 2, \ldots, m\,, \tag{2.4}$$

defined in a neighborhood of (x_0, y_0, z_0) so that

$$\zeta_{k0}(x_0, y_0, z_0) = \zeta_{k0}\,, \quad k = 1, 2, \ldots, m\,,$$

and

$$Q\big[u\big(x, y, z, \zeta_{k0}(x, y, z)\big), \zeta_{k0}(x, y, z)\big] = 0\,. \tag{2.5}$$

Hence, the pairs

$$\{\zeta_{k0}(x, y, z); (x_0, y_0, z_0)\}\,, \quad k = 1, 2, \ldots, m\,, \tag{2.6}$$

can be considered as the analytic elements of the algebraic function $\mathcal{A}$.

Suppose the curve in the real x, y, z space

$$L:\quad x = x(t)\,, \quad y = y(t)\,, \quad z = z(t)\,, \quad 0 \leqslant t \leqslant 1\,, \tag{2.7}$$

where

$$x(0) = x_0\,, \quad y(0) = y_0\,, \quad z(0) = z_0\,,$$

contains no points of the set S^*. Therefore L contains no singular points of $\mathcal{A}$ [4, p. 25]. Then the elements (2.6) can be continued along the curve L. This means that for every k of (2.6) there is a chain of analytic elements of (See [12], [10]),

$$\big\{\zeta_k(x, y, z, t); \big(x(t), y(t), z(t)\big)\big\}\,, \quad 0 \leqslant t \leqslant 1\,, \tag{2.8}$$

so that

$$Q\big[u\big(x, y, z, \zeta_k(x, y, z, t)\big), \zeta_k(x, y, z)\big] = 0 \tag{2.9}$$

in a sufficiently small neighborhood of $\big(x(t), y(t), z(t)\big)$, and

$$\zeta_k(x, y, z, 0) = \zeta_{k0}(x, y, z)\,, \quad k = 1, 2, \ldots, m\,. \tag{2.10}$$

We shall now show that the harmonic element $\{F_0; p_0\}$ can be continued along the arbitrary curve L described in (2.7).

For this purpose we form the family of harmonic elements

$$\mathcal{F}(t) = \big\{F(x, y, z, t); \big(x(t), y(t), z(t)\big)\big\}\,, \quad 0 \leqslant t \leqslant 1\,, \tag{2.11}$$

where

$$F(x, y, z, t) = \sum_{k=1}^{m} \frac{P\big[u\big(x, y, z, \zeta_k(x, y, z, t)\big), \zeta_k(x, y, z, t)\big]}{Q'_\zeta\big[u\big(x, y, z, \zeta_k(x, y, z, t)\big), \zeta_k(x, y, z, t)\big]}\,. \tag{2.12}$$

For a fixed k and for every t, $0 \leqslant t \leqslant 1$, there corresponds a number $\varepsilon > 0$ such that for $|h| < \varepsilon$ and $0 \leqslant t+h \leqslant 1$, the element

$$\big\{\zeta_k(x, y, z, t+h); \big(x(t+h), y(t+h), z(t+h)\big)\big\} \tag{2.13}$$

is a direct continuation of the element

$$\big\{\zeta_k(x, y, z); \big(x(t), y(t), z(t)\big)\big\}\,. \tag{2.14}$$

This means that

$$\{\zeta_k(x, y, z, t+h) = \zeta_k(x, y, z, t)\} \tag{2.15}$$

for every (x, y, z) in a sufficiently small neighborhood of

$$x(t+h), y(t+h), z(t+h).$$

Thus, according to the definition (2.12), in the same neighborhood

$$\begin{aligned} F(x, y, z, t+h) &= \sum_{k=1}^{m} \frac{P[u(x, y, z, \zeta_k(x, y, z, t+h)), \zeta_k(x, y, z, t+h)]}{Q'_\zeta[u(x, y, z, \zeta_k(x, y, z, t+h)), \zeta_k(x, y, z, t+h)]} \\ &= \sum_{k=1}^{m} \frac{P[u(x, y, z, \zeta_k(x, y, z, t)), \zeta_k(x, y, z, t)]}{Q'_\zeta[u(x, y, z, \zeta_k(x, y, z, t)), \zeta_k(x, y, z, t)]} = F(x, y, z, t). \end{aligned} \tag{2.16}$$

Hence, for the same h the harmonic element

$$\{F(x, y, z, t+h); (x(t+h), y(t+h), z(t+h))\} \tag{2.17}$$

is a direct continuation of the element

$$\{F(x, y, z, t); (x(t), z(t))\}. \tag{2.18}$$

At this time we note that the element $\{F_0; p_0\}$ is equal to the initial element $\{F(x, y, z, 0); (x(0), y(0), z(0))\}$ of (2.11).

This shows that (2.11) is a chain of harmonic elements and can be continued along the arbitrary curve L described above. Therefore, the harmonic function $\mathcal{F}$ determined by the harmonic element $\{F_0; p_0\}$ (see (2.1)) has no singular points in the x, y, z space $R - S^*$.

Remark 1. In the last theorem, we assumed that the center (x_0, y_0, z_0) of the initial harmonic element did not belong to both sets D^* and S^*. But the second assumption that the point (x_0, y_0, z_0) does not belong to S^* can be removed. Indeed, if we have a point $(\hat{x}_0, \hat{y}_0, \hat{z}_0)$ which only does not belong to D^*, there is, of course, also a harmonic function defined by the integral

$$\hat{F}_0(x, y, z) = \frac{1}{2\pi i}\int_{|\zeta|=1} \frac{P(u, \zeta)}{Q(u, \zeta)}\, d\zeta \tag{2.19}$$

in a neighborhood of $(\hat{x}_0, \hat{y}_0, \hat{z}_0)$. Since we assume that the discriminant does not vanish identically, then in this neighborhood lie infinitely many points (x_0, y_0, z_0) which do not belong to S^*. Then every harmonic element

$$\{F(x, y, z); (x_0, y_0, z_0)\} \tag{2.20}$$

where

$$F(x, y, z) = \hat{F}_0(x, y, z)$$

in a neighborhood of (x_0, y_0, z_0) defines the same multi-valued harmonic function as the element

$$\{\hat{F}_0(x, y, z); (\hat{x}_0, \hat{y}_0, \hat{z}_0)\}.$$

Consequently, the set of singular points of the multi-valued harmonic function generated by

$$\{\hat{F}_0(x, y, z); (\hat{x}_0, \hat{y}_0, \hat{z}_0)\}$$

is identical to the analogous function generated by

$$\{F_0(x, y, z); (x_0, y_0, z_0)\}.$$

But for the second function we can already apply our theorem.

Note that the initial elements of the multi-valued harmonic functions considered in this part are defined by the one valued functions considered in part 1. Thus we can say that the class of multi-valued functions is generated by the corresponding one valued function considered in part 1.

Remark 2. From our theorem and the previous remark immediately follows that for the classes of multi-valued harmonic functions generated by the one valued harmonic functions of examples 1, 1′ and 2, 2′, the corresponding locus of singular points are the lines given by the equations (1.44) and (1.54), (1.56) and (1.55), (1.57).

In the case of Example 2, it is possible to evaluate the integral F in a closed form and thus exhibit the actual singular curves of the harmonic function of which F is an element (see [2], for Example 1)

$$F = \frac{1}{2\pi i}\int_{|\zeta|=1} \frac{d\zeta}{au^2\zeta + bu\zeta + c}, \qquad u = \tfrac{1}{2}(iy+z)\zeta + x + \tfrac{1}{2}(iy-z)\frac{1}{\zeta}, \tag{2.21}$$

$$F = \frac{1}{2\pi i}\int_{|\zeta|=1} \frac{d\zeta}{a\left(\frac{1}{2}(iy+z)\right)^2(\zeta-\zeta^{(1)})(\zeta-\zeta^{(2)})(\zeta-\zeta^{(3)})(\zeta-\zeta^{(4)})}, \tag{2.22}$$

where

$$\zeta^{(\nu)}, \quad \nu = 1, 2, 3, 4, \tag{2.23}$$

are the four roots, assumed distinct, of the denominator.

The integral (2.21) can now be evaluated by the residue theorem.

The value of the integral will then depend upon which roots are within the contour. The roots (2.23) are functions of the coefficients a, b, c and the real variables x, y, z. If all the roots or none of the roots are within the contour, then the value of the integral will be zero.

If for every point (x, y, z) the integral is identically zero because every root is in magnitude greater than unity or because every root is in magnitude less than unity, then we can obtain a harmonic function not identically zero by considering another contour which contains some but not all of the roots in its interior. This can be done since the roots are distinct.

The residue at each root is a harmonic function as can be easily verified by formal computation. The residue at each root is the harmonic function

$$(2.24_1)\qquad H_1 = \frac{1}{a(p_1-p_2)\sqrt{R^2+2(z+iy)p_1}},$$

$$(2.24_2)\qquad H_2 = \frac{1}{a(p_2-p_1)\sqrt{R^2+2(z+iy)p_1}},$$

$$(2.24_3)\qquad H_3 = \frac{1}{a(p_1-p_2)\sqrt{R^2+2(z+iy)p_2}},$$

$$(2.24_4)\qquad H_4 = \frac{1}{a(p_2-p_1)\sqrt{R^2+2(z+iy)p_2}},$$

where

$$(2.25)\qquad R^2 = x^2+y^2+z^2$$

and p_1, p_2 are the roots of

$$(2.26)\qquad ap^2+bp+c = 0$$

(see (1.32), (1.36)).

It is evident that the harmonic function generated by (2.21) is singular along the branch curve given by setting the appropriate denominator of (2.24) equal to zero. This locus is identical with part of the locus where the discriminant of the denominator of (1) vanishes (see (1.50), (1.51)).

The locus of possible singular points consists, in general, of two distinct curves. The actual singular locus is seen to consist of one of the curves completely, or both of the curves completely, or neither curve, depending on which roots are within the contour. If neither curve is the locus of singular points, we see that the harmonic function, which is the sum of the residues within the contour, is identically zero.

It is interesting to note that the integral

$$(2.27)\qquad F_2 = \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \frac{d\zeta}{a(u\zeta)^n+b(u\zeta)^{n-1}+\ldots+e},$$

n a positive integer, can be treated similarly to (2.21), and that the actual singular curve is of the same nature as the singular curve of (2.21).

The integral (2.27) can be written in the form

$$(2.28)\qquad F = \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \frac{d\zeta}{a\left(\frac{1}{2}(iy+z)\right)^n(\zeta-\zeta^{(1)})(\zeta-\zeta^{(2)})\ldots(\zeta-\zeta^{(2n)})},$$

where

$$(2.29)\qquad \zeta^{(\nu)},\quad \nu = 1, 2, \ldots, 2n,$$

are the roots of the denominator.

Assuming that the roots (2.29) are distinct, the integral (2.28) can be evaluated by the residue theorem. The value of the residue at one particular root is

$$(2.30)\qquad \mathcal{H} = \frac{1}{a\left(\frac{1}{2}(iy+z)\right)^n(\zeta^{(1)}-\zeta^{(2)})(\zeta^{(1)}-\zeta^{(3)})\ldots(\zeta^{(1)}-\zeta^{(2n-1)})(\zeta^{(1)}-\zeta^{(2n)})}$$
$$= \frac{1}{a(q_1-q_2)(q_1-q_3)\ldots(q_1-q_n)\sqrt{R^2+2(z+iy)q_1}},$$

where

$$(2.31)\qquad R^2 = x^2+y^2+z^2$$

and

$$(2.32)\qquad q_\nu,\quad \nu = 1, 2, \ldots, n$$

are the roots of

$$(2.33)\qquad aq^n+bq^{n-1}+\ldots+e = 0.$$

Thus, it is seen that the singularity locus of the harmonic function generated by (2.30) is similar in nature to the singularity locus of the harmonic function generated by (2.24).

3. Multi-valued harmonic functions generated by series development. We consider the Bergman integral operator

$$(3.1)\qquad F_0(x, y, z) = \frac{1}{2\pi i}\int\limits_{|\zeta|=1} f_0(u, \zeta)\zeta,\quad u = x+\tfrac{1}{2}(iy+z)\zeta+\tfrac{1}{2}(iy-z)\cdot 1/\zeta$$

where $f_0(u, \zeta)$ is an analytic function of u and ζ definem in the product domain of a complex neighborhood ω of 0 and a complex annulus σ containing the circle $|\zeta| = 1$. Then the function $f_0(u, \zeta)$ can be expressed in the product domain in the form

$$(3.2)\qquad f_0(u, \zeta) = \sum_{n=0}^{\infty}\sum_{j=-\infty}^{\infty} T_{nj}u^n\zeta^j.$$

Suppose further, that there is given an arbitrary polynomial $M(u, \zeta, 1/\zeta)$ which vanishes for $u = 0$ and all ζ. In addition, suppose that $A[p]/B[p]$ is an arbitrary rational function which is regular at $p = 0$. Then for u sufficiently close to 0 and for $|\zeta| = 1$, the composite function

$$(3.3)\qquad \frac{A[M(u, \zeta, 1/\zeta)]}{B[M(u, \zeta, 1/\zeta)]}$$

can be developed in a series of the form

$$(3.4) \qquad \sum_{n=0}^{\infty} \sum_{j=-\infty}^{\infty} s_{nj} u^n \zeta^j .$$

Thus the function (3.2) can be expressed in the form

$$(3.5) \qquad f_0(u, \zeta) = \sum_{n=0}^{\infty} \sum_{j=-\infty}^{\infty} s_{nj} u^n \zeta^j + \sum_{n=0}^{\infty} \sum_{j=-\infty}^{\infty} (T_{nj} - s_{nj}) u^n \zeta^j .$$

Suppose that the second sum is an analytic function for all u and for ζ belonging to an annulus containing the circle $|\zeta| = 1$.

We shall now consider in the x, y, z space the multi-valued harmonic function $\mathcal{F}_0$ generated by the harmonic element $\{F_0; (0, 0, 0)\}$ where F_0 is given by (3.1) in a sufficiently small neighborhood of $(0, 0, 0)$ and an analogous function $\mathcal{F}_1$ generated by the element $\{F_1; (0, 0, 0)\}$ where

$$(3.6) \qquad \begin{gathered} F_1(x, y, z) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \frac{A[M(u, \zeta, 1/\zeta)]}{B[M(u, \zeta, 1/\zeta)]} d\zeta , \\ u = x + \tfrac{1}{2}(iy + z)\zeta + \tfrac{1}{2}(iy - z) \cdot 1/\zeta . \end{gathered}$$

Then we can obtain the following:

THEOREM 3. *The only possible singular points of the multi-valued function $\mathcal{F}_0$ must be identical with the singular points of the function $\mathcal{F}_1$.*

Proof. The function $F_0(x, y, z)$ of (3.1) can be expressed, according to (3.4), (3.5), in the form

$$(3.7) \qquad F_0(x, y, z) = F_1(x, y, z) + \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \sum_{n=0}^{\infty} \sum_{j=-\infty}^{\infty} (T_{nj} - s_{nj}) u^n \zeta^j d\zeta .$$

The second term in (3.7) is obviously a harmonic function defined in the full x, y, z space.

If we continue the element $\{F_0; (0, 0, 0)\}$, we obtain in this way a multi-valued harmonic function $\mathcal{F}_0$ whose initial elements equal the sum of the initial elements of $\mathcal{F}_1$ and the second term of (3.7).

Thus, all regular points of $\mathcal{F}_1$ will be regular points of $\mathcal{F}_0$ and conversely. This ends the proof.

4. Remark. Bergman shows that every harmonic function $F(x, y, z)$ of three real variables defined in a neighborhood of the origin can be represented in the form

$$(4.1) \qquad F(x, y, z) = \frac{1}{2\pi i} \int f(u, \zeta) d\zeta , \quad u = x + \tfrac{1}{2}(iy + z)\zeta + \tfrac{1}{2}(iy - z) \cdot 1/\zeta ,$$

where

$$(4.2) \qquad f(u, \zeta) = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_{nk} u^n \zeta^k$$

is the so called *B-associate* or *normalized B-associate* [1, p. 468]. In other words, there is a one-to-one correspondence between the mentioned harmonic functions and the B-associates in corresponding linear spaces. Thus, having the harmonic function $F(x, y, z)$, we can obtain, at first, the corresponding function (4.2); and secondly, we cas try to apply theorem 3 to obtain the singularities of the multi-valued harmonic function generated by $F(x, y, z)$.

More exactly, we can try to divide the series development of (4.2) into two parts

$$(4.3) \qquad \sum_{n=0}^{\infty} \sum_{j=-\infty}^{\infty} s_{nj} u^n \zeta^j + \sum_{n=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} V_{nj} u^n \zeta^j$$

in which the first part of (4.3) represents for u in a neighborhood of the origin and for ζ in an annulus containing the unit circle, the development of a rational function $A[p]/B[p]$ of the type described in theorem 3. To distinguish the first part of (4.3) having this property, one can use various known criteria ([5], [11], [13]; compare also [2], [6]).

Final remarks. We gave here only the necessary conditions characterizing the locii of singular points of the integral operators. But it is clear that these locii do not consist of only singular points. Thus, it could be interesting to obtain simple necessary and sufficient conditions characterizing the locii of singular points. Also, the investigation of the nature of the singularities presents an interesting problem in this domain. These problems will be investigated in a future paper.

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