We are not able, however, to prove the existence of a continuous solution defined in the whole interval $\langle a, b\rangle$. Also we cannot say anything about the number of solutions that are defined in the whole interval $\langle a, b\rangle$. It is our conjecture that under hypotheses (i)-(iii) equation (1) possesses exactly one solution defined in the whole interval $\langle a, b\rangle$, i.e. that each of the sets $\Delta(x)$ contains exactly one point. It can easily be shown that if the solution defined in the whole interval $\langle a, b\rangle$ is unique, then it is continuous and strictly increasing.

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Reçu par la Rédaction le 24. 5. 1960

ANNALES
POLONICI MATHEMATICI
X (1961)

On the continuous dependence of solutions of some functional equations on given functions. II

by J. Kordylewski and M. Kuczma (Kraków)

In the first part of this paper [2] we have proved (under suitable assumptions) continuous dependence on given functions for solutions of the functional equation

$$\varphi[f(x)]+\eta\varphi(x)=F(x), \tag{1}$$

where $\eta=\pm 1$. Presently we shall deal with a more general equation

$$\varphi(x)=H(x,\varphi[f(x)]), \tag{2}$$

where $\varphi(x)$ denotes the unknown function and $f(x)$ and $H(x,y)$ are given. Making use of the results obtained we shall prove a theorem about continuous dependence for solutions of equation (1) stronger than those proved in [2]. Although equation (1) is a particular case of equation (2), the hypotheses which we assume concerning equation (2) are not fulfilled in the case of equation (1). Thus the theorems proved in [2] do not follow from the results of the present paper.

II. Equation $\varphi(x)=H(x,\varphi[f(x)])$

§ 1. We assume the following hypotheses regarding the functions $f(x)$ and $H(x,y)$:

(i) The function $f(x)$ is defined, continuous and strictly increasing in an interval $\langle a, b\rangle$ and $f(x)>x$ for $x\in(a,b)$, $f(b)=b$.

(ii) The function $H(x,y)$ is continuous and has the continuous derivative $\partial H/\partial y\neq 0$ in a region Ω normal with respect to the x-axis.

(iii) $\Omega_x\neq 0$, $\Omega_x=\Gamma_{f(x)}$ for $x\in\langle a, b\rangle$, where Ω_x denotes the x-section of the region Ω:

$$\Omega_x=\{y:\ (x,y)\in\Omega\},$$

and Γ_x denotes the set of values of the function $H(x,y)$ for $y\in\Omega_x$ (1).

(1) In the case $f(a)\neq a$ it is enough to postulate only $\Omega_x\subset\Gamma_{f(x)}$, instead of the relation $\Omega_x=\Gamma_{f(x)}$.

(iv) There exists d such that $(b, d) \in \Omega$ and

$$H(b, d) = d;$$

moreover,

$$\left|\frac{\partial H}{\partial y}(b, d)\right| < 1. \tag{3}$$

Now we define the sequence of functions $h_\nu(x)$:

$$h_0(x) \overset{\mathrm{df}}{=} d, \qquad h_{\nu+1}(x) \overset{\mathrm{df}}{=} H(x, h_\nu[f(x)]), \qquad \nu = 0, 1, 2, \ldots$$

Lemma I. *Under hypotheses* (i)-(iv) *equation* (2) *possesses exactly one solution $\varphi(x)$ that is continuous in the interval $(a, b\rangle$ and fulfils the condition $\varphi(b) = d$. This solution is given by the formula*

$$\varphi(x) = \lim_{\nu\to\infty} h_\nu(x).$$

This lemma has been proved in [4].

In the sequel by the solution of equation (2) we shall understand this unique solution furnished by the above lemma.

We shall consider also the sequence of equations

$$\varphi(x) = H_n(x, \varphi[f_n(x)]), \qquad n = 1, 2, 3, \ldots \tag{4}$$

We shall assume that

(v) The functions $f_n(x)$ are defined, continuous and strictly increasing in the interval $\langle a, b\rangle$ and $f_n(x) > x$ for $x \in (a, b)$, $f_n(b) = b$ $(n = 1, 2, \ldots)$.

(vi) The functions $H_n(x, y)$ are continuous and have the continuous derivatives $\partial H_n/\partial y \neq 0$ in the region Ω.

(vii) $\Omega_x = \Gamma_{n, f(x)}$ for $x \in \langle a, b\rangle$, $n = 1, 2, 3, \ldots$, where $\Gamma_{n,x}$ denotes the set of values of the function $H_n(x, y)$ for $y \in \Omega_x$ (²).

(viii) We have

$$f_n(x) \underset{\langle a,b\rangle}{\Rightarrow} f(x), \tag{5}$$

$$H_n(x, y) \underset{\Lambda}{\Rightarrow} H(x, y), \qquad \frac{\partial}{\partial y} H_n(x, y) \underset{\Lambda}{\Rightarrow} \frac{\partial}{\partial y} H(x, y) \tag{6}$$

for every compact set $\Lambda \subset \Omega$.

Lemma II. *Let us suppose that hypotheses* (i)-(viii) *are fulfilled. Then there exist for n sufficiently large d_n such that $(b, d_n) \in \Omega$ and*

$$H_n(b, d_n) = d_n, \tag{7}$$

$$\left|\frac{\partial H_n}{\partial y}(b, d_n)\right| < 1, \tag{8}$$

$$\lim_{n\to\infty} d_n = d. \tag{9}$$

(²) In the case $f(a) \neq a$ it is enough to postulate only $\Omega_x \subset \Gamma_{n, f(x)}$.

Proof. According to (3) one can choose positive numbers β, δ and $\vartheta < 1$ such that the rectangle

$$R: \begin{cases} b - \beta \leqslant x \leqslant b, \\ d - \delta \leqslant y \leqslant d + \delta \end{cases}$$

is contained in Ω and

$$|\partial H/\partial y| < \vartheta \quad \text{in} \quad R. \tag{10}$$

It follows from (6) that for n sufficiently large

$$|\partial H_n/\partial y| < \vartheta \quad \text{in} \quad R. \tag{11}$$

We have by (11) for $y \in \langle d-\delta, d+\delta\rangle$ and n sufficiently large

$$|H_n(b, y) - H_n(b, d)| < \vartheta|y - d|. \tag{12}$$

There exists N such that for $n > N$ we have (12) and

$$|H_n(b, d) - d| < (1 - \vartheta)\delta. \tag{13}$$

Consequently for $n > N$ and $y \in \langle d-\delta, d+\delta\rangle$ we have by (12) and (13)

$$|H_n(b, y) - d| \leqslant |H_n(b, y) - H_n(b, d)| + |H_n(b, d) - d| < \delta.$$

This means that the function $H_n(b, y)$ maps the interval $\langle d-\delta, d+\delta\rangle$ into itself and on account of the fixed-point theorem there is in $\langle d-\delta, d+\delta\rangle$ exactly one d_n such that relation (7) is fulfilled. Relation (8) follows from (11) and the inclusion $R \subset \Omega$. Since the rectangle R may be chosen arbitrarily small (of course, the index N depends on the choice of δ), relation (9) is also fulfilled. This completes the proof.

It follows from lemmas I and II that for n sufficiently large equation (4) possesses exactly one solution $\varphi_n(x)$ that is continuous in the interval $(a, b\rangle$ and fulfils the condition $\varphi_n(b) = d_n$. This solution is given by the formula

$$\varphi_n(x) = \lim_{\nu\to\infty} h_{n,\nu}(x),$$

where

$$h_{n,0}(x) = d_n, \qquad h_{n,\nu+1}(x) = H_n(x, h_{n,\nu}[f_n(x)]), \qquad \nu = 0, 1, 2, \ldots$$

In the sequel by the solution of equation (4) we shall understand the above-mentioned unique solution.

Now we shall prove the following

Theorem I. *Under hypotheses* (i)-(viii)

$$\varphi_n(x) \underset{\langle c,b\rangle}{\Rightarrow} \varphi(x) \quad \textit{for every} \quad c \in (a, b)\ (^3), \tag{14}$$

where $\varphi_n(x)$ and $\varphi(x)$ denote the solutions of equation (4) *and* (2) *respectively.*

(³) In the case $f(a) \neq a$, c may also equal a.

Proof. We put

$$r_\nu(x) \overset{\mathrm{df}}{=} h_{\nu+1}(x) - h_\nu(x), \qquad r_{n,\nu}(x) \overset{\mathrm{df}}{=} h_{n,\nu+1}(x) - h_{n,\nu}(x), \qquad \nu = 0, 1, 2, \ldots$$

Thus we have

$$\varphi(x) = d + \sum_{\nu=0}^{\infty} r_\nu(x), \qquad \varphi_n(x) = d_n + \sum_{\nu=0}^{\infty} r_{n,\nu}(x).$$

Hence

$$(15) \qquad |\varphi_n(x) - \varphi(x)| \leqslant |d_n - d| + \sum_{\nu=0}^{\infty} |r_{n,\nu}(x) - r_\nu(x)|.$$

In [4] it has been proved that there exist $x_0 \in (a, b)$ and $\vartheta < 1$ such that for $x \in \langle x_0, b\rangle$

$$(16) \qquad |r_{\nu+1}(x)| < \vartheta |r_\nu[f(x)]|; \qquad \nu = 0, 1, 2, \ldots$$

From the uniform convergence (5), (6) it follows that this x_0 may be chosen in such a manner that also

$$(17) \qquad |r_{n,\nu+1}(x)| < \vartheta |r_{n,\nu}[f_n(x)]| \quad \text{for} \quad x \in \langle x_0, b\rangle, \quad \nu = 0, 1, 2, \ldots$$

(at least for n sufficiently large; but since in the assertion of the present theorem only a limit effect occurs, we may always restrict ourselves to n sufficiently large).

Now let us take an arbitrary $\varepsilon > 0$. We can find $x_1 \in (x_0, b)$ and an index N_1 such that for $x \in \langle x_1, b\rangle$ and $n > N_1$

$$|r_0(x)| < \frac{\varepsilon}{4}(1-\vartheta)$$

and

$$|r_{n,0}(x)| < \frac{\varepsilon}{4}(1-\vartheta).$$

Next we can find indices ν_0 and $N_2 \geqslant N_1$ such that for $\nu \geqslant \nu_0$, $n > N_2$ and $x \in \langle x_0, b\rangle$, $f^\nu(x) \in \langle x_1, b\rangle$ and $f_n^\nu(x) \in \langle x_1, b\rangle$ (4). According to (16) and (17) we have for $x \in \langle x_0, b\rangle$

$$|r_{\nu_0}(x)| < \vartheta^{\nu_0} |r_0[f^{\nu_0}(x)]| < \frac{\varepsilon}{4}(1-\vartheta)$$

and

$$|r_{n,\nu_0}(x)| < \vartheta^{\nu_0} |r_{n,0}[f_n^{\nu_0}(x)]| < \frac{\varepsilon}{4}(1-\vartheta).$$

(4) $f^\nu(x)$, $f_n^\nu(x)$ denote the ν-th iterations of the functions $f(x)$, $f_n(x)$. As has been proved in [3], for each $x \in (a, b)$ the sequences $f^\nu(x)$, $f_n^\nu(x)$ are increasing and converge to b.

Hence also

$$\sup_{\langle x, b\rangle} |r_{\nu_0}(t)| < \frac{\varepsilon}{4}(1-\vartheta)$$

and

$$\sup_{\langle x, b\rangle} |r_{n,\nu_0}(t)| < \frac{\varepsilon}{4}(1-\vartheta)$$

for $x \in \langle x_0, b\rangle$ and $n > N_2$. Thus we have, according to (16) and (17), for $x \in \langle x_0, b\rangle$ and $n > N_2$

$$(18) \qquad \sum_{\nu=\nu_0}^{\infty} |r_{n,\nu}(x)| \leqslant \sum_{\nu=\nu_0}^{\infty} |\vartheta^{\nu-\nu_0} r_{n,\nu_0}[f_n^{\nu-\nu_0}(x)]| \leqslant \sup_{\langle x, b\rangle} |r_{n,\nu_0}(t)| \sum_{\nu=\nu_0}^{\infty} \vartheta^{\nu-\nu_0} < \varepsilon/4$$

and

$$(19) \qquad \sum_{\nu=\nu_0}^{\infty} |r_\nu(x)| \leqslant \sum_{\nu=\nu_0}^{\infty} |\vartheta^{\nu-\nu_0} r_{\nu_0}[f^{\nu-\nu_0}(x)]| \leqslant \sup_{\langle x, b\rangle} |r_{\nu_0}(t)| \sum_{\nu=\nu_0}^{\infty} \vartheta^{\nu-\nu_0} < \varepsilon/4.$$

Moreover, since (on account of (6) and (9)) $r_{n,\nu}(x) \underset{\langle x_0, b\rangle}{\Rightarrow} r_\nu(x)$ and $d_n \to d$, we can find an index $N \geqslant N_2$ such that for $n > N$ and $x \in \langle x_0, b\rangle$

$$(20) \qquad |d_n - d| < \varepsilon/4$$

and

$$(21) \qquad \sum_{\nu=0}^{\nu_0-1} |r_{n,\nu}(x) - r_\nu(x)| < \varepsilon/4.$$

We have by (15)

$$|\varphi_n(x) - \varphi(x)| \leqslant |d_n - d| + \sum_{\nu=0}^{\nu_0-1} |r_{n,\nu}(x) - r_\nu(x)| + \sum_{\nu=\nu_0}^{\infty} |r_{n,\nu}(x)| + \sum_{\nu=\nu_0}^{\infty} |r_\nu(x)|,$$

whence, by (20), (21), (18) and (19), for $x \in \langle x_0, b\rangle$ and $n > N$

$$|\varphi_n(x) - \varphi(x)| < \varepsilon,$$

which proves that

$$(22) \qquad \varphi_n(x) \underset{\langle x_0, b\rangle}{\Rightarrow} \varphi(x).$$

Now let us take an arbitrary $c \in (a, b)$ and let us put

$$c_\mu \overset{\mathrm{df}}{=} f^\mu(c-\gamma), \qquad \gamma > 0, \qquad c - \gamma > a, \qquad \mu = 0, 1, 2, \ldots$$

We can find M such that $c_M \in (x_0, b)$. We further take $\gamma_1 > 0$ (which may be chosen arbitrarily small) such that $c_{M-1} + \gamma_1 < b$. Of course

$$f_n(c_{M-1} + \gamma_1) \to f(c_{M-1} + \gamma_1) > f(c_{M-1}) = c_M.$$

Consequently for n sufficiently large $f_n(c_{M-1} + \gamma_1) \in (x_0, b)$ and on account of the monotonity of the functions $f_n(x)$

$$f_n(x) \in \langle x_0, b\rangle$$

for $x \in \langle c_{M-1}+\gamma_1, b\rangle$ and for n sufficiently large. Hence we have, according to (2), (4) and (22)

$$\varphi_n(x) \underset{\langle c_{M-1}+\gamma_1, b\rangle}{\Rightarrow} \varphi(x).$$

Now we take $\gamma_2 > 0$ such that $c_{M-2}+\gamma_2 < b$. Since

$$f_n(c_{M-2}+\gamma_2) \to f(c_{M-2}+\gamma_2) > c_{M-1},$$

we get for n sufficiently large

$$f_n(c_{M-2}+\gamma_2) \in \langle c_{M-1}+\gamma_1, b\rangle$$

and

$$f_n(x) \in \langle c_{M-1}+\gamma_1, b\rangle \quad \text{for} \quad x \in \langle c_{M-2}+\gamma_2, b\rangle,$$

provided we choose γ_1 so small that $f(c_{M-2}+\gamma_2) > c_{M-1}+\gamma_1$. Consequently

$$\varphi_n(x) \underset{\langle c_{M-2}+\gamma_2, b\rangle}{\Rightarrow} \varphi(x).$$

Of course, also γ_2 may be chosen arbitrarily small.

After M steps of such a procedure we obtain

$$\varphi_n(x) \underset{\langle c-\gamma+\gamma_M, b\rangle}{\Rightarrow} \varphi(x),$$

where γ_M may be arbitrarily small. Consequently choosing $\gamma_M \leqslant \gamma$ we get hence relation (14).

If $f(a) \neq a$ and we choose $c = a$, we cannot apply the considerations described above. In that case, however, we may continue the functions $f(x)$ and $f_n(x)$ to the left from the point a in such a manner that in an interval $\langle a-\alpha, b\rangle$ hypotheses (i) and (v) will be fulfilled. Then we can apply the above considerations and finally we obtain the desired result also in that case. This completes the proof of the theorem.

§ 2. Now we shall apply theorem I in order to obtain a result concerning equation (1). Besides equation (1) we shall consider also the sequence of equations

$$\varphi[f_n(x)]+\eta\varphi(x) = F_n(x), \quad \eta^2 = 1. \tag{23}$$

We assume that:

(ix) The functions $f(x)$ and $f_n(x)$ $(n = 1, 2, 3, \ldots)$ are of the class C^1 in the interval $\langle a, b\rangle$, $f(x) > x$ and $f_n(x) > x$ in (a, b), $f(b) = f_n(b) = b$; moreover, $f'(x) > 0$ and $f_n'(x) > 0$ in $\langle a, b\rangle$, $f'(b) < 1$, $f_n'(b) < 1$.

(x) The functions $F(x)$ and $F_n(x)$ $(n = 1, 2, 3, \ldots)$ are of the class C^1 in the interval $\langle a, b\rangle$, $F(b) = 0$, $F_n(b) = 0$ [5].

[5] As to the essentiality of the supposition $F(b) = 0$, $F_n(b) = 0$, compare the final section of our paper [2].

Lemma III. *Under hypotheses* (ix)-(x) *equation* (1) *resp.* (23) *possesses exactly one solution* $\varphi(x)$ *resp.* $\varphi_n(x)$ *that is continuous in the interval* $(a, b\rangle$ *and fulfils the condition* $\varphi(b) = 0$ *resp.* $\varphi_n(b) = 0$. *These solutions are of the class* C^1 *in* $(a, b\rangle$.

The existence and the uniqueness of the continuous solution of (23) in $(a, b\rangle$ have been proved in [2] (cf. also [1] and [3]). The proof that this solution is of the class C^1 for the case $\eta = -1$ is to be found in [1]. In the case $\eta = +1$ the proof is analogical.

In the sequel by the solution of equation (1) resp. (23) we shall understand this unique solution furnished by lemma III.

Now let us assume that

(xi) $f_n(x) \underset{\langle a,b\rangle}{\Rightarrow} f(x)$, $f_n'(x) \underset{\langle a,b\rangle}{\Rightarrow} f'(x)$, $F_n(x) \underset{\langle a,b\rangle}{\Rightarrow} F(x)$, $F_n'(x) \underset{\langle a,b\rangle}{\Rightarrow} F'(x)$.

We shall prove

Theorem II. *Under hypotheses* (ix)-(xi)

$$\varphi_n(x) \underset{\langle c,b\rangle}{\Rightarrow} \varphi(x), \tag{24}$$

$$\varphi_n'(x) \underset{\langle c,b\rangle}{\Rightarrow} \varphi'(x) \tag{25}$$

for every $c \in (a, b)$ [6], *where* $\varphi(x)$ *and* $\varphi_n(x)$ *denote the solutions of equations* (1) *and* (23) *respectively.*

Proof. Convergence (24) has been proved in [2]. Thus it remains to prove relation (25).

The derivatives $\varphi'(x)$ and $\varphi_n'(x)$ of the functions $\varphi(x)$ and $\varphi_n(x)$ are continuous in the interval $(a, b\rangle$ and satisfy the functional equations

$$\eta\varphi'[f(x)]f'(x)+\varphi'(x) = \eta F'(x), \tag{26}$$

$$\eta\varphi_n'[f_n(x)]f_n'(x)+\varphi_n'(x) = \eta F_n'(x) \tag{27}$$

respectively. Putting

$$H(x, y) \overset{\text{df}}{=} \eta F'(x)-\eta f'(x)y, \qquad H_n(x, y) \overset{\text{df}}{=} \eta F_n'(x)-\eta f_n'(x)y,$$

we see that equations (26) and (27) are of the form (2) resp. (4). One can easily verify that hypotheses (ix)-(xi) imply hypotheses (i)-(viii) if we take as Ω the stripe

$$\begin{cases} a-\alpha < x < b+\alpha, \\ -\infty < y < \infty \end{cases}$$

(the functions $F'(x)$, $F_n'(x)$, $f'(x)$, $f_n'(x)$ can be continued onto the interval $(a-\alpha, b+\alpha)$). Consequently relation (25) follows immediately from theorem I.

[6] In the case $f(a) \neq a$, c may also equal a.

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Reçu par la Rédaction le 2. 6. 1960

ANNALES
POLONICI MATHEMATICI
X (1961)

Bemerkung zu meiner Arbeit: „Ein Problem der zweidimensionalen Minkowskischen Geometrie"

(Annales Polonici Mathematici 9 (1960), S. 39-48)

von L. TAMÁSSY (Debrecen)

Ich beweise in Nr. 4. meiner Arbeit, dass wenn die Eigenschaft (E) besteht, d. h. wenn für jedes Dreieck die drei Produkte: Seitenlänge mal Preudohöhenlänge in Minkowskischem Mass gemessen einander gleich sind, dann die Indikatrix ein zentrisches Ellipsoid ist. Im Buch von Busemann: *The geometry of geodesics* (New-Jork 1955) wird aber bewiesen (Seite 103), dass die Indikatrix eines n dimensionalen Minkowskischen Raumes ($n > 2$), in welchem die Transversalität symmetrisch ist, ein zentrisches Ellipsoid ist. Dies, und mein Hauptsatz in Nr. 2. geben aber sofort den erwähnten Satz in Nr. 4., ohne des dort erbrachten Beweises.

Reçu par la Rédaction le 25. 1. 1961