$a = -1$ equality (9) is impossible. Hence $a_1 = 0$. So $a_1 = a_2 = \ldots = a_{n-1} = 0$ for the extremal function and we end the proof similarly to the proof of theorem 2.

Note 3. The case here studied is particulary interesting as it concerns the classes of meromorphic starlike schlicht functions and meromorphic spiral schlicht functions.

Note 4. The theorems here proved were known for regular starlike schlicht functions and for starlike meromorphic schlicht functions satisfying the additional assumption $a_1 = 0$ [1].

References

[1] J. Clunie, *On meromorphic schlicht functions*, Journ. London Math. Soc. 34 (1959), p. 115-116.

[2] L. Špaček, *Příspěvek k teorii funkcí prostych*, Časopis Pěst. Mat. 62 (1933), p. 12-19.

[3] J. Zamorski, *Remarks on a class of analytic functions*, Bull. Acad. Polon. Sci., Sér. des Sci. Math., Astr. et Phys. 8 (1960), p. 377-380.

INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES

Reçu par la Rédaction le 15. 6. 1960

ANNALES
POLONICI MATHEMATICI
X (1961)

Note on abstract differential inequalities and Chaplighin method

by W. Mlak (Kraków)

We are interested in this paper in an abstract treatment of the Chaplighin method [1], [9], [10], [11] for the equation

$$\frac{dx}{dt} = Ax(t) + f\big(t, x(t)\big). \tag{1}$$

A is an infinitesimal generator of a semi-group of linear bounded operators of class (C_0) in the Banach space E. The essential moment in the Chaplighin method is the fact that $f(t, x)$ is convex in x. The second feature of that method is the use differential inequalities. The purpose of the present paper is the investigation of the Chaplighin method by using methods which are closely related to the Hille-Yosida semigroups theory (see [4]). We make use of some theorems concerning ordinary differential and integral inequalities. In section 1 we give a brief outline of the notation and definitions. We also discuss some geometric properties of positive cones. Section 2 presents some results concerning abstract linear differential inequalities. In sections 3 and 4 we examine almost linear differential inequalities. Sections 5, 6 and 7 are devoted to the main object of this paper. Three principal questions are considered. The first one is the question of existence of the Chaplighin sequence on a common interval. Next we discuss the problem of uniform boundedness and convergence of the Chaplighin sequence. We then use some assumptions imposed on the relationship between the partial ordering and metric properties. Following R. Kalaba [6] we introduce the concept of Newtonian sequences. Finally we present some results which concern the estimation of the norm of the difference between the exact solution of (1) and the approximate one. The last section deals with the uniform boundedness of Newtonian sequences.

1. Preliminaries. Let E be a real Banach space. The norm of $x \in E$ is denoted by $|x|$. The norm of bounded linear operators is also denoted by simple bars. The function $f(t, x)$ is defined on $\langle 0, a\rangle \times E$

and takes on values in E. Let V be the infinitesimal generator of a semi-group linear bounded operators in E. The semi-group generated by V is denoted by $T(t:V)$. In what follows we consider semi-groups of class (C_0), i.e. such that $\lim_{h\to 0+} T(h:V)x = x$ for $x \in E$. As far as possible the semi-groups in question are supposed to be semi-groups of contraction operators, i.e. with norm not greater than 1. These assumptions are introduced for the sake of simplicity.

The closed set $S \subset E$, $S \neq E$ is a cone if it satisfies the following two conditions (see [4], [13]):

(2) if $x \in S$, $y \in S$ then $x+y \in S$,

(3) if $\lambda \geqslant 0$ and $x \in S$ then $\lambda x \in S$.

An element x is said to be *positive* if $x \in S$. The partial ordering may be defined in terms of S with $x \leqslant y$ being equivalent to $y-x \in S$. The functional ξ is said to be *positive* if $\xi x \geqslant 0$ for $x \in S$. S may be completely characterised by means of positive linear and continuous functionals. We start with the following theorem (see for instance [2], p. 417):

If K_1 and K_2 are disjoint closed convex subsets of E and K_1 is compact, then there exist constants a, β and a continuous linear functional ξ on E such that

$$\xi x \leqslant a < \beta \leqslant \xi y$$

for $x \in K_1$, $y \in K_2$.

Suppose now that $x_0 \bar{\in} S$. We now put $K_1 = \{x_0\}$ and $K_2 = S$. Hence, there are a $\xi \in E^*$ and constants a, β such that $\xi x_0 \leqslant a < \beta \leqslant \xi y$ for $y \in S$. We will prove that $\xi y \geqslant 0$ for each $y \in S$. If we suppose the contrary, then for a certain $y_0 \in S$ $\xi y_0 < 0$. On the other hand, $n y_0 \in S$ for $n = 1, 2, \ldots$ and consequently $\xi x_0 \leqslant a < \beta < \xi n y_0 < 0$. Obviously $\lim_{n\to+\infty} \xi n y_0 = -\infty$. This contradicts the fact that β is finite. We have just proved that the dual cone $S^* = \underset{\xi}{E}\{\xi x \geqslant 0 \text{ for } x \in S,\ \xi \in E^*\}$ contains non-trivial elements. Moreover, it follows from the above reasoning that the following property holds:

If $\xi x \geqslant 0$ for $\xi \in S^*$ then $x \in S$.

This property is fundamental for our purposes. Let V be a linear operator. The domain of V is denoted by $D[V]$. V is said to be *positive* if $Vx \geqslant \theta$ for $x \in S \cap D[V]$. We then write $V \geqslant \theta$. The formula $V_1 \leqslant V_2$ means that $V_1 - V_2 \leqslant \theta$. A semi-group $T(\tau:U)$ is called *positive* if $T(\tau:U)$ are positive operators for $0 \leqslant \tau$.

Throughout the present paper with the exception of sections 7 and 8 we assume that A is an infinitesimal generator of a positive semi-group. For the terminology and other notation conventions we refer in the sequel to [4].

2. Linear differential inequalities. The point of departure is the following lemma (see [18], th. 1):

LEMMA 1. *Let $U(t,s)$ be a family of positive, linear and bounded operators defined for $0 \leqslant s \leqslant t \leqslant a$. Suppose that $U(t,s)$ is strongly continuous in (t,s) and $U(t,t) = I$. We assume that the strong derivative $\partial U(t,s)x/\partial s$ exists for $x \in D[A(s)]$ and equals $-U(t,s)A(s)x$. Let $x(t), y(t)$ be strongly continuous and let $x(t)$ be strongly differentiable to $x'(t)$ in $\langle 0, a\rangle$. Suppose that*

$$x'(t) \leqslant A(t)x(t) + y(t) \quad \textit{for} \quad t \in \langle 0, a\rangle .$$

Then

$$x(t) \leqslant U(t,s)x(s) + \int_s^t U(t,\tau)y(\tau)\,d\tau \quad \textit{for} \quad 0 \leqslant s \leqslant t \leqslant a .$$

It is a simple matter to verify that $T(t-s) = U(t,s)$ where $T(t)$ is a positive semi-group of class (C_0) satisfies the properties assumed in lemma 1: $A = A(t) = \text{const}$ is its infinitesimal generator.

An important role in our investigations is played by the following lemma:

LEMMA 2. *If $\beta(t)$ is summable and if $x(t)$ is strongly absolutely continuous and strongly differentiable almost everywhere to $x'(t)$, then, if*

$$\frac{\partial\left[\exp\left(\int_s^t \beta(\tau)\,d\tau\right) T(t-s)x(s)\right]}{\partial s} \leqslant y(s)$$

for almost all $s \in (0, t)$, we have

$$\exp\left(\int_{s_2}^t \beta(\tau)\,d\tau\right) T(s_2) - x(s_2) \leqslant \exp\left(\int_{s_1}^t \beta(\tau)\,d\tau\right) T(t-s_1)x(s_1) + \int_{s_1}^{s_2} y(\tau)\,d\tau ,$$

for $s_1 < s_2 \leqslant \tau$; $T(t)$ is a positive semigroup of class (C_0) and $y(t)$ is supposed to be Bochner integrable in $\langle 0, a\rangle$.

Lemma 2 may be proved by using methods similar to that used in the proof of theorem 5 of [18].

We first prove the following theorem:

THEOREM 1. *Let A be the infinitesimal generator of a contraction positive semi-group. Let $B(t)$ be a strongly continuous operator-valued function. We assume that there is a β such that $B(t) + \beta I \geqslant \theta$ for $t \in \langle 0, a\rangle$. Let the strongly differentiable function $x(t)$ satisfy in $\langle 0, a\rangle$ the inequality*

(4) $$x'(t) \leqslant [A + B(t)]x(t) .$$

Suppose that

(5) $$x(0) \leqslant \theta .$$

Then $x(t) \leqslant \theta$ in $\langle 0, a\rangle$.

Proof. Let $z(t) = e^{\beta t}x(t)$. By (4) we then have

$$z'(t) \leqslant Az(t) + [B(t) + \beta I]z(t) . \tag{6}$$

It follows from lemma 1 and from (5) and (6) that

$$z(t) \leqslant \int_0^t T(t-\tau:A)[B(\tau)+\beta I]z(\tau)\,d\tau .$$

The operator $T(t-\tau:A)[B(\tau)+\beta I]$ is positive. We conclude therefore that the sequence defined by formulas

$$z_0(t) = z(t) , \qquad z_{n+1}(t) = \int_0^t T(t-\tau:A)[B(\tau)+\beta I]z_n(\tau)\,d\tau$$

is an increasing one: $z_n(t) \leqslant z_{n+1}(t)$. But $z_n(t)$ is a sequence of successive approximations for the equation

$$u(t) = \int_0^t T(t-\tau:A)[B(\tau)+\beta I]u(\tau)\,d\tau .$$

Hence $z_0(t) \leqslant \lim_{n\to\infty} z_n(t) = \theta$, q. e. d.

The assumption that $B(t)+\beta I \geqslant \theta$ may be weakened as follows:

(7) for each $t \in \langle 0, a\rangle$ there is a real $\beta(t)$ such that $B(t)+\beta(t)I \geqslant \theta$.

Our next result describes the case of $x(t)$ absolutely continuous.

THEOREM 2. *Let $B(t)$ be a bounded strongly Bochner integrable operator-valued function and let $B(t)$ satisfy (7) with summable $\beta(t)$. Let the function $x(t)$ be strongly absolutely continuous in $\langle 0, a\rangle$. We assume that the inequality*

$$x'(t) \leqslant [A + B(t)]x(t)$$

holds almost everywhere in $\langle 0, a\rangle$. Suppose that $x(0) \leqslant \theta$. Then $x(t) \leqslant 0$ in $\langle 0, a\rangle$.

Proof. The operator $V(t, s) = \exp\left(-\int_s^t \beta(\tau)\,d\tau\right)T(t-s:A)$ is positive. Hence

$$V(t, s)x'(s) \leqslant [V(t, s)A + V(t, s)B(s)]x(s) .$$

It is easy to verify that

$$\frac{\partial\big(V(t, s)x(s)\big)}{\partial s} = -V(t, s)[A-\beta(s)I]x(s) + V(t, s)x'(s) .$$

The above relations imply that

$$\frac{\partial[V(t, s)x(s)]}{\partial s} \leqslant V(t, s)[B(s)+\beta(s)I]x(s) .$$

Using lemma 2 we then obtain

$$x(t) \leqslant \int_0^t V(t, \tau)[B(\tau)+\beta(\tau)I]x(\tau)\,d\tau .$$

By (7) $V(t, s)[B(s)+\beta(s)I] \geqslant \theta$. The relation $x(t) \leqslant \theta$ can now be proved by using the method of successive approximations, just as in the proof of theorem 1.

The case where nothing is known about the regularity of $\beta(t)$ is much more complicated. It is then natural to consider some strong assumptions concerning the behaviour of $B(t)$. In order to do so we introduce the following condition discussed by Phillips in [16]:

(8) $B(t)$ is strongly continuously differentiable in t in the whole interval $\langle 0, a\rangle$.

In what follows the lemma below is very useful (see [4], th. 13.4.2):

LEMMA 3. *Let A be an infinitesimal generator of a positive semi-group $T(t:A)$ of class (C_0) and let B a linear bounded operator. It is supposed that $B+\beta I \geqslant 0$ for a certain real β. Then $A+B$ is the infinitesimal generator of a positive semi-group of class (C_0).*

Suppose now that (7) is satisfied. Then the semi-group $T\big(t:A+B(s)\big)$ is positive. Observe that (8) implies that the family of closed operators $C(t) = A+B(t)$ satisfies the conditions introduced by Kato in [7]: $D[A+B(t)]$ does not depend on t and $H(t, s) = \big[\lambda I - \big(A+B(t)\big)\big]$. $R\big(\lambda, A+B(s)\big)$ is strongly continuously differentiable in t for fixed s for a certain λ sufficiently large. It follows from the results of [7] that the strong limit

$$\lim_{\substack{\max|t_{i+1}-t_i|\to 0\\ s=t_1<\ldots<t_n=t}} \prod_{i=0}^{n-1} T\big(t_{i+1}-t_i:A+B(t_i)\big) = U(t, s) \tag{9}$$

exists provided that (8) is satisfied. Moreover $U(t, t) = I$. One easily verifies that $|U(t, s)| \leqslant \exp\left(\int_s^t |B(\tau)|\,d\tau\right)$. The operator $U(t, s)$ is positive and by formula (12) of [8]

$$\frac{\partial U(t, s)x}{\partial s} = -U(t, s)[A+B(s)]x$$

for $x \in D[A]$. Using lemma 1 we can summarize our discussion in the following theorem:

THEOREM 3. *Let A be the infinitesimal generator of a positive semigroup of class (C_0). Suppose that $B(t)$ satisfies* (7) *and* (8). *Let $x(t)$ satisfy the following inequalities:*

$$x'(t) \leqslant [A+B(t)]x(t) \quad \textit{for} \quad t \in \langle 0, a\rangle ,$$
$$x(0) \leqslant \theta .$$

Then $x(t) \leqslant \theta$ in $\langle 0, a\rangle$.

3. Almost linear differential inequalities. This section deals with inequalities of the following form:

$$(10) \qquad x'(t) \leqslant Ax(t)+f\big(t, x(t)\big), \quad 0 \leqslant t \leqslant a ,$$

$$(11) \qquad y'(t) \geqslant Ay(t)+f\big(t, y(t)\big), \quad 0 \leqslant t \leqslant a .$$

These inequalities imply that

$$(12) \qquad [x(t)-y(t)]' \leqslant A[x(t)-y(t)]+\big[f\big(t, x(t)\big)-f\big(t, y(t)\big)\big] .$$

The aim of our investigations is the linearization of (12). The linearization makes it possible to use the results of the preceding section. We shall make use of the following elementary lemma:

LEMMA 4. *Let $f(t, x)$ be Frechet differentiable to $f_x(t, x)$. Suppose that $f_x(t, x)$ is strongly continuous in x. Then*

$$(13) \qquad f(t, x)-f(t, y) = \int_0^1 f_x\big((t, z(\tau)\big)(x-y)\,d\tau$$

where $z(\tau) = y+\tau(x-y)$.

The integral appearing in (13) is the Riemann integral of the function $f_x\big(t, z(\tau)\big)(x-y) \in E$. Owing to the continuity of $f_x(t, x)$ in x the following formula holds

$$\int_0^1 f_x\big(t, z(\tau)\big)(x-y)\,d\tau = \Big[\int_0^1 f_x\big(t, z(\tau)\big)\,d\tau\Big](x-y)$$

where the integral on the right-hand side is the Riemann integral (in the strong sense) of the operator-valued function $f_x\big(t, z(\tau)\big)$.

THEOREM 4. *Let $x(t), y(t)$ satisfy* (10) *and* (11) *respectively. Suppose that $x(0) \leqslant y(0)$. Let $f(t, x)$ satisfy the assumptions of lemma 4. It is supposed that $f_x(t, x)$ is strongly continuous in (t, x). Suppose that there is a real β such that $f_x(t, x)+\beta I \geqslant \theta$ for $(t, x) \in \langle 0, a\rangle \times E$. Under our assumptions the inequality $x(t) \leqslant y(t)$ holds for $t \in \langle 0, a\rangle$.*

Proof. We first define

$$B(t) = \int_0^1 f_x\big(t, y(t)+\tau[x(t)-y(t)]\big)\,d\tau .$$

$B(t)$ is strongly continuous and $B(t)+\beta I \geqslant \theta$. Inequality (12) and lemma 4 imply that difference $z(t) = x(t)-y(t)$ satisfies the inequality $z'(t) \leqslant [A+B(t)]z(t)$. The assertion of our theorem now follows from theorem 1.

Using theorem 2 and lemma 4 one easily proves the following theorem:

THEOREM 5. *Let $x(t)$ and $y(t)$ be strongly absolutely continuous and strongly differentiable almost everywhere in $\langle 0, a\rangle$. Suppose that* (10) *and* (11) *hold almost everywhere in $\langle 0, a\rangle$. It is supposed that $f(t, x)$ satisfies the assumptions of lemma 4. Furthermore, $f_x(t, x)$ is strongly continuous in (t, x) and $f_x\big(t, y(t)+\tau[x(t)-y(t)]\big)+\beta(t)I \geqslant \theta$ for $t \in \langle 0, a\rangle$ and $\tau \in \langle 0, 1\rangle$. We assume that $\beta(t)$ is summable in $\langle 0, a\rangle$ and $x(0) \leqslant y(0)$. Then $x(t) \leqslant y(t)$ in $\langle 0, a\rangle$.*

We now introduce the following condition:

(14) The function $f(t, x)$ is Frechet differentiable in x to $f_x(t, x)$. The differential $f_x(t, x)$ is strongly continuously differentiable in (t, x).

It should be observed that the following property holds: *if $x(t)$ is strongly continuously differentiable and if $f(t, x)$ satisfies* (14) *then $f_x\big(t, x(t)\big)$ is strongly continuously differentiable in t.* Combining this result with theorem 3 and lemma 4 we get the following theorem:

THEOREM 6. *Let $x(t), y(t)$ be strongly continuously differentiable on $\langle 0, a\rangle$. Suppose that* (10) *and* (11) *hold and $x(0) \leqslant y(0)$. We assume that $f(t, x)$ satisfies* (14). *It is supposed that for each $t \in \langle 0, a\rangle$ there is a real $\beta(t)$ such that $f_x\big(t, y(t)+\tau[x(t)-y(t)]\big)+\beta(t)I \geqslant \theta$ for $\tau \in \langle 0, 1\rangle$. Under our assumptions the inequality $x(t) \leqslant y(t)$ holds for $t \in \langle 0, a\rangle$.*

4. Differential inequalities with convex f. Suppose we are given the function $g(x)$ defined on E. The values of g belong to E. Let $g(x)$ be Frechet differentiable to $g_x(x)$. The function $g(x)$ is said to be *convex* if its Frechet differential $g_x(u)$ increases in u, i.e. $g_x(u_1)z < g_x(u_2)z$ for $u_1 \leqslant u_2$ and $z \geqslant \theta$.

LEMMA 5. *Suppose that $g(x)$ is convex and $u \leqslant v$. Then $g_x(u)(v-u)+g(u) \leqslant g(v)$.*

Proof. Owing to the properties of S and S^* mentioned in section 1 it is sufficient to prove that

$$\xi g_x(u)(v-u)+\xi g(u) \leqslant \xi g(v) \quad \text{for} \quad \xi \in S^* .$$

We first define $h(\lambda) = \xi g\big(\lambda v+(1-\lambda)u\big)$. For real λ $h'(\lambda) = \xi g_x\big(\lambda v+ +(1-\lambda)u\big)(v-u)$. Suppose now that $\lambda_1 < \lambda_2$. Then $\lambda_1 v+(1-\lambda_1)u \leqslant \lambda_2 v+(1-\lambda_2)u$ and consequently $h'(\lambda_1) \leqslant h'(\lambda_2)$. It is thus seen that the real-valued function $h(\lambda)$ is convex. We infer therefore that $h'(\lambda_1)(\lambda_2-\lambda_1)+h(\lambda_1) \leqslant h(\lambda_2)$. For $\lambda_1 = 0$ and $\lambda_2 = 1$ we therefore get $h'(0)+h(0) = \xi g_x(u)(v-u)+\xi g(u) \leqslant \xi g(v) = h(1)$, which was to be proved.

We shall make use of the following condition:

(15) The function $f(t,x)$ is Frechet differentiable to $f_x(t,x)$. There exists a continuous and non-negative function $\omega(t,u)$ $(t\in\langle 0,a\rangle, u\geqslant 0)$ such that $|f_x(t,x)-f_x(t,y)|\leqslant\omega(t,|x-y|)$.

The function $\omega(t,u)$ increases in u.

THEOREM 7. *Let the functions $x(t), y(t)$ satisfy* (10) *and* (11). *Suppose that $x(0)\leqslant y(0)$. We assume that $f(t,x)$ is continuous in (t,x) and convex in x for every $t\in\langle 0,a\rangle$. It is supposed that the operator-valued function $f_x(t,x(t))=B(t)$ is strongly continuously differentiable in t on $\langle 0,a\rangle$ and $B(t)$ satisfies* (7). *Let $f(t,x)$ satisfy* (15) *and suppose that there is a function $\varphi(t)$ such that $|x(t)-y(t)|\leqslant\varphi(t)$ and*

$$(*)\quad \int_0^t \exp\Big(\int_\tau^t |B(s)|\,ds\Big)\big[\omega\big(\tau,\varphi(\tau)\big)\varphi(\tau)+\omega\big(\tau,|x(\tau)-y(\tau)|\big)\varphi(\tau)\big]\,d\tau\leqslant\varphi(t)$$

for $t\in\langle 0,a\rangle$. Our assumptions imply that $x(t)\leqslant y(t)$ on $\langle 0,a\rangle$.

Proof. It follows from our assumptions and from the results of section 2 that there exists an operator-valued function $U(t,s)\geqslant\theta$ which satisfies the assumptions of lemma 1 with $A(s)=A+f_x\big(s,x(s)\big)$. We therefore infer that (10) and (11) imply

$$(16)\quad x(t)\leqslant y(t)+\int_0^t U(t,\tau)\big\{f\big(\tau,x(\tau)\big)-f\big(\tau,y(\tau)\big)-f_x\big(\tau,x(\tau)\big)[x(\tau)-y(\tau)]\big\}\,d\tau.$$

We define a sequence

$$x_0(t)=x(t),$$

$$x_{n+1}(t)=y(t)+\int_0^t U(t,\tau)\big[f\big(\tau,x_n(\tau)\big)-f\big(\tau,y(\tau)\big)-f_x\big(\tau,x(\tau)\big)\big(x_n(\tau)-y(\tau)\big)\big]\,d\tau.$$

Suppose that $x_{i-1}(t)\leqslant x_i(t)$ for $t\in\langle 0,a\rangle$ and $i=1,2,\ldots,n$. The function $f(t,x)$ is convex in x. Applying lemma 5 we have

$$f_x\big(\tau,x_{n-1}(\tau)\big)\big(x_n(\tau)-x_{n-1}(\tau)\big)+f\big(\tau,x_{n-1}(\tau)\big)\leqslant f\big(\tau,x_n(\tau)\big).$$

But $f_x\big(\tau,x_0(\tau)\big)\leqslant f_x\big(\tau,x_{n-1}(\tau)\big)$ and consequently

$$f\big(\tau,x_{n-1}(\tau)\big)-f\big(\tau,y(\tau)\big)-f_x\big(\tau,x_0(\tau)\big)\big(x_{n-1}(\tau)-y(\tau)\big)$$
$$\leqslant f\big(\tau,x_n(\tau)\big)-f\big(\tau,y(\tau)\big)-f_x\big(\tau,x_0(\tau)\big)\big(x_{n-1}(\tau)-y(\tau)\big).$$

This last inequality together with the fact that $U(t,\tau)$ are positive implies that $x_n(t)\leqslant x_{n+1}(t)$. We have thus proved that the sequence $\{x_n(t)\}$ is

increasing. Applying arguments similar to that used in [19] (*p.* 114-115) one shows that (15) implies the following inequality:

$$(16^\circ)\quad \big|f\big(\tau,x_n(\tau)\big)-f\big(\tau,y(\tau)\big)-f_x\big(\tau,x_0(\tau)\big)\big(x_n(\tau)-y(\tau)\big)\big|$$
$$\leqslant\omega\big(\tau,|x_n(\tau)-y(\tau)|\big)\,|x_n(\tau)-y(\tau)|+\omega\big(\tau,|x(\tau)-y(\tau)|\big)\,|x_n(\tau)-y(\tau)|.$$

We shall now prove that

$$(17)\quad |x_\nu(t)-y(t)|\leqslant\varphi(t)\quad\text{for}\quad \nu=0,1,2,\ldots\quad\text{and}\quad t\in\langle 0,a\rangle.$$

Obviously (17) holds for $\nu=0$. Suppose that $|x_n(t)-y(t)|\leqslant\varphi(t)$. Then, by (16°)

$$(18)\quad |x_{n+1}(t)-y(t)|\leqslant\int_0^t\exp\Big(\int_\tau^t|B(s)|\,ds\Big)\big[\omega\big(\tau,|x_n(\tau)-y(\tau)|\big)\,|x_n(\tau)-y(\tau)|+$$
$$+\omega\big(\tau,|x(\tau)-y(\tau)|\big)\,|x_n(\tau)-y(\tau)|\big]\,d\tau.$$

But $\omega(t,u)$ increases in u. By (18) and the assumed property of $\varphi(t)$ we get

$$|x_{n+1}(t)-y(t)|$$
$$\leqslant\int_0^t\exp\Big(\int_\tau^t|B(s)|\,ds\Big)\big[\omega\big(\tau,\varphi(\tau)\big)\varphi(\tau)+\omega\big(\tau,|x(\tau)-y(\tau)|\big)\varphi(\tau)\big]\,d\tau\leqslant\varphi(t).$$

We infer therefore that (17) holds for each ν.

We now define $R=\max\limits_{\langle 0,a\rangle}\omega\big(t,\max\limits_{\langle 0,a\rangle}\varphi(t)\big)$. From (16°) we get

$$(19)\quad |x_{n+1}(t)-y(t)|\leqslant 2RM\int_0^t|x_n(\tau)-y(\tau)|\,d\tau$$

where

$$M=\exp\Big(\int_0^a|B(s)|\,ds\Big).$$

It follows from (19) that

$$|x_{n+1}(t)-y(t)|\leqslant(2RM)^{n+1}\varrho\frac{t^{n+1}}{(n+1)!},\qquad 0\leqslant t\leqslant a$$

where $\varrho=\max\limits_{0\leqslant t\leqslant a}\varphi(t)$. This implies that $\lim\limits_{n\to\infty}x_n(t)=y(t)$ in $\langle 0,a\rangle$. On the other hand, the sequence $x_n(t)$ is increasing and S is closed in the norm topology. We conclude therefore that $x(t)=x_0(t)\leqslant x_n(t)\leqslant\lim\limits_{n\to\infty}x_n(t)=y(t)$, q. e. d.

Remark. For small a we can construct $\varphi(t)$ satisfying $(*)$ as follows: let $|x(0)-y(0)|<\eta$. Then $|x(t)-y(t)|<\eta$ for t sufficiently small, say for $t\in\langle 0,\beta\rangle$. Then an arbitrary solution $\varphi(t)$ of the equation

$$u' = \exp\Big(\int_0^\beta |B(s)|\,ds\Big)\,\omega(t,u)u+\omega\big(t,|x(t)-y(t)|\big)u$$

such that $\varphi(0)=\eta$ satisfies $(*)$ and $|x(t)-y(t)|<\varphi(t)$ for t sufficiently small.

5. Chaplighin sequences. This section concerns the Chaplighin method. We begin with the following lemma (see [16], th. 6.3):

LEMMA 6. *Let A be the infinitesimal generator of a semi-group of class (C_0). Suppose that $x(t)$ is strongly continuously differentiable to $x'(t)$. Let $f(t,x)$ satisfy* (14) *and let $f_t(t,x)$ be continuous. Then there exists a unique continuously differentiable solution $y(t)$ in $\langle 0,a\rangle$ of the initial value problem*

$$y' = Ay+f_x\big(t,x(t)\big)\big(y-x(t)\big)+f\big(t,x(t)\big)\,,\qquad y(0)=y_0\in D[A]\,.$$

Let $x(t)$ be continuously differentiable: $x(t)$ is said to be *admissible* if there exists a function $\gamma(t)$ such that $\gamma(t)I\leqslant f_x\big(t,x(t)\big)$ for $t\in\langle 0,a\rangle$. We first prove the following

THEOREM 8. *Suppose that the function $x(t)$ is admissible and*

(20) $$x'(t)\leqslant Ax(t)+f\big(t,x(t)\big)\,,\qquad 0\leqslant t\leqslant a\,.$$

Let the function $f(t,x)$ be convex in x in the sense of the definition of section 4. We assume that $f(t,x)$ satisfies condition (14). *Then the solution $y(t)$ of the problem*

(21) $$y' = Ay+f_x\big(t,x(t)\big)\big(y-x(t)\big)+f\big(t,x(t)\big)\,,\qquad y(0)=x(0)$$

exists in $\langle 0,a\rangle$ and is an admissible function. Moreover, the following conditions hold:

(22) $$x(t)\leqslant y(t)\qquad for\qquad t\in\langle 0,a\rangle\,,$$

(23) $$y'(t)\leqslant Ay(t)+f\big(t,y(t)\big)\qquad for\qquad t\in\langle 0,a\rangle\,.$$

Proof. The existence of $y(t)$ satisfying (21) follows from lemma 6. By (20) and (21) we have

$$z'(t)\leqslant Az(t)+f_x\big(t,x(t)\big)z(t)\,,\qquad z(0)=0$$

where $z(t)=x(t)-y(t)$. Inequality (22) follows now from theorem 3 and from the fact that $x(t)$ is admissible. (22) being proved, we infer by the monotonicity of f_x that

$$\gamma(t)u\leqslant f_x\big(t,x(t)\big)u\leqslant f_x\big(t,y(t)\big)u\qquad \text{for}\qquad u\geqslant 0\,.$$

Hence $y(t)$ is an admissible function. Applying lemma 5 and (22) we get $f_x\big(t,x(t)\big)\big(y(t)-x(t)\big)+f\big(t,x(t)\big)\leqslant f\big(t,y(t)\big)$. This last inequality together with (21) implies (23).

Observe that the strong differentiability of $f_x(t,x)$ is needed only to ensure the existence of the solution of (21). This supposed property of $f_x(t,x)$ is a very strong assumption. However, our investigations may serve as a model provided that the existence problem for (21) is already solved.

Suppose now that $x(t)$ satisfies the assumptions of theorem 8. Then there exists a unique solution $y(t)$ of (21) corresponding to $x(t)$. We have thus to do with a transformation law which assignes to $x(t)$ a uniquely determined function $y(t)$. Denote this transformation by C. We then have $y(\cdot)=Cx(\cdot)$. It is easy to verify that $y(t)$ also satisfies the aasumptions of theorem (8). Hence the sequence

(24) $$x_0(t)=x(t)\,,\qquad x_{n+1}(t)=Cx_n(t)$$

is well defined. Using theorem 8 we conclude that

(25) $$x_n(t)\leqslant x_{n+1}(t)\,,\qquad x_n(0)=x_{n+1}(0)\in D[A]\,,$$

(26) $$x'_{n+1}(t)\leqslant Ax_{n+1}(t)+f_x\big((t,x_n(t)\big)\big(x_{n+1}(t)-x_n(t)\big)+f\big(t,x_n(t)\big)\,,$$

(27) $$x'_{n+1}(t)\leqslant Ax_{n+1}(t)+f\big(t,x_{n+1}(t)\big)\,.$$

The sequence $\{x_n(t)\}$ is said to be the *Chaplighin sequence* if it satisfies (25), (26) and (27). Hence, the assumptions of theorem 8 may be treated as an example of conditions which are sufficient for the existence of a Chaplighin sequence in a common interval $\langle 0,a\rangle$. Up to this point we have not been interested in the convergence of Chaplighin sequences. In certain special functional spaces the monotonicity of a Chaplighin sequence together with some additional conditions imply its convergence to the solution of (1). The question of convergence of a Chaplighin sequence is closely related to the uniform boundedness (in the sense of norm or in the sense of partial ordering) of that sequence.

We give here an illustration of how the theorems of the previous sections may be applied to the estimation of Chaplighin sequences. In what follows we shall make use of the following condition:

(28) $$\text{If}\qquad \theta\leqslant x\leqslant y\qquad\text{then}\qquad |x|\leqslant|y|\,.$$

In other words, the norm is an increasing function for positive elements. Property (28) holds for $C(\Omega)$ and $L(\Omega)$ spaces with $x(\cdot)\geqslant\theta$ being equivalent to $x(q)\geqslant 0$ for $q\in\Omega$.

Suppose we are given a function $y(t)$ which satisfies in $\langle 0,a\rangle$ the inequality

(29) $$y'(t)\geqslant Ay(t)+f\big(t,y(t)\big)\,.$$

Let $x_n(t)$ be a Chaplighin sequence. If for each n the functions $y(t)$ and $x_n(t)$ satisfy the assumptions of any of theorems 4, 5, 6, 7 then $x_n(t) \leqslant x_{n+1}(t) \leqslant y(t)$. Now we use (28) and thus conclude that $|x_n(t)| \leqslant |y(t)| + |x_0(t) - y(t)|$. It is thus seen that the problem of the uniform boundedness of a Chaplighin sequence is reduced to the question whether theorems of sect. 3 and 4 may be applied. A new problem now arises. This is the problem of existence of $y(t)$ satisfying (29). In many instances $y(t)$ may be a solution of (1). In general, however, as in the classical case, the existence of such a $y(t)$ must be postulated.

In order to illustrate the above discussion we formulate one of the possible theorems.

THEOREM 9. *Let $x_n(t)$ be a Chaplighin sequence. Suppose that $f_x(t, x)$ is strongly continuous in (t, x) and $f_x(t, x) + \beta I \geqslant 0$ for a certain real β and $(t, x) \in \langle 0, a\rangle \times E$. Assume that $y(t)$ satisfies* (29). *It is supposed that* (28) *holds. Then $|x_n(t)| \leqslant |y(t)| + |x_0(t) - y(t)|$ for $n = 0, 1, 2, \ldots$ and $t \in \langle 0, a\rangle$.*

6. Convergence of Chaplighin sequences. In this section we examine the relationship of the monotonicity of a Chaplighin sequence and its convergence. Suppose that the partial ordering possesses the following property:

$$(30)\qquad \text{If } x_n \leqslant y_n \leqslant z_n \text{ and } \lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = y_0 \text{ then } \lim_{n\to\infty} y_n = y_0\ (^1)\,.$$

We start with the following lemma (Krein):

LEMMA 7. *Let condition* (30) *be satisfied. If $u_n \leqslant u_{n+1}$ and u_n converges weakly to u_0 then u_n converges strongly to u_0.*

Proof. It is a classical result of S. Mazur (see [12]) that the weak convergence of u_n implies that there exists a sequence $v_n = \sum_{i=1}^{n} c_i^{(n)} u_i$ such that $\sum_{i=1}^{n} c_i^{(n)} = 1$, $c_i^{(n)} \geqslant 0$ and $v_n \to u_0$ strongly. On the other hand,

$$(31)\qquad v_k \leqslant u_k\,.$$

Observe now that $u_{n+k} \to u_0$ weakly for fixed k. Applying the above mentioned result of Mazur we conclude that there exists a sequence $v_n^{(k)} = \sum_{i=1}^{n} c_{in}^{(k)} u_{i+k}$ such that $v_n^{(k)} \to u_0$ strongly and $c_{in}^{(k)} \geqslant 0$ and $\sum_{i=1}^{n} c_{in}^{(k)} = 1$. Then, for each k we may choose n_k in such a way that $n_k < n_{k+1}$ and $v_{n_k}^{(k)} \to u_0$ strongly. But $u_k \leqslant v_n^{(k)}$ for each n and consequently $u_k \leqslant v_{n_k}^{(k)}$. It now follows from (31) that $v_k \leqslant u_k \leqslant v_{n_k}^{(k)}$. Using (30) we infer that $u_k \to u_0$ strongly, q.e.d.

(1) The limits here are in the strong sense.

It is thus seen that there is no difference between strongly and weakly convergent Chaplighin sequences provided that condition (30) is satisfied. Lemma 7 may be treated as a generalization of Dini's theorem on monotonic sequences of continuous functions on compact spaces. Indeed, in spaces $C(\Omega)$ of real-valued continuous functions defined on a compact Hausdorff space Ω weak convergence of a sequence is equivalent to uniform boundedness and pointwise convergence. Strong convergence is uniform. In such spaces the $(\cdot)$ pointwise convergence of a Chaplighin sequence follows from its monotonicity. The $(\cdot)$ continuity of the limit function is implied by the fact that it satisfies a suitable integral equation. A more general situation may be characterised by the following conclusion, which summarizes our discussion:

If (30) *is satisfied, then every weakly convergent Chaplighin sequence is strongly convergent.*

This trivial conclusion generalizes in some sense the well-known property of Chaplighin sequences in $C(\Omega)$ spaces.

It is not true in the case of $C(\Omega)$ spaces that if $x_n \leqslant x_{n+1}$ and $x_n \leqslant y$ for each n, then x_n converges weakly. It is known, however, (see [5], lemma 3.2) that if the norm is additive for positive elements, i.e.

$$(32)\qquad |x+y| = |x| + |y| \quad \text{for} \quad x, y \in S\,,$$

then the following property holds:

If $x_n \leqslant x_{n+1}$ and $x_n \leqslant y$ for certain y then x_n converges strongly to a certain limit.

In order to establish strong convergence of a Chaplighin sequence $x_n(t)$ in spaces satisfying (32) it suffices therefore to find a function $y(t)$ such that $x_n(t) \leqslant y(t)$, $t \in \langle 0, a\rangle$, $n = 0, 1, 2, \ldots$ This can be done by using any of the theorems of sections 3 and 4.

7. Convergence to the solution. After a brief outline of some typical assumptions which ensure the convergence of the Chaplighin sequence $x_n(t)$, we consider the question whether $x_n(t)$ converges to the solution of (1). The function $x_n(t)$ satisfies the integral equation

$$(33)\qquad x_n(t) = T(t\colon A)x_n(0) + \int_0^t T(t-\tau\colon A)\big[f_x\big(\tau, x_{n-1}(\tau)\big)\big(x_n(\tau) - x_{n-1}(\tau)\big) + f\big((\tau, x_{n-1}(\tau)\big)\big]\,d\tau$$

provided that f, f_x and $x_n'(t)$ are strongly continuous. If $x_n(t) \to x(t)$ strongly and $\{x_n(t)\}$ is uniformly bounded in $\langle 0, a\rangle$ then, by the Lebesgue theorem for Bochner integral we infer by (33) that

$$(34)\qquad x(t) = T(t\colon A)x(0) + \int_0^t T(t-\tau\colon A) f\big(\tau, x(\tau)\big)\,d\tau\,.$$

In some cases (see for instance [3], th. 12.2) the solution $x(t)$ of (34) is a unique solution of differential equation (1) with a prescribed initial value $x(0)$. In what follows we shall discuss the convergence of a Chaplighin sequence to the solution of (1) without making use of integral equations. Integral equations of the form (33) will be discussed in connection with the problem of uniform boundedness of Newtonian sequences in section 8.

It has already been remarked by Lusin [9] that the Chaplighin method is in some sense an extension of the well-known Newtons method of solving numerical equations to differential equations. Suppose now that $f_x(t, x)$ exists for $(t, x) \in \langle 0, a\rangle \times E$. It is of its own interest to discuss some metric properties of the sequences of solutions of approximate equations

$$(35)\qquad x'_{n+1}(t) = Ax_{n+1}(t)+f_x\big(t, x_n(t)\big)\big(x_{n+1}(t)-x_n(t)\big)+f\big(t, x_n(t)\big)$$

($x_n(0) = x_0 = \text{const}$). Such sequences will be referred to in the sequel as Newtonian sequences.

In what follows we do not introduce the relation of inequality. We are interested only in such properties of the sequence of solutions of (35) which can be expressed without the use of partial ordering. Some results for concrete forms of equation (1) in $C(\Omega)$ have been obtained by Kalaba in [6].

We now prove the following theorem:

THEOREM 10. *Let A be the infinitesimal generator of a semi-group of contraction operators of class (C_0). Let $x_n(t)$ be a Newtonian sequence and let $|x_n(t)| \leqslant M = \text{const} < +\infty$ for $n = 0, 1, 2, \ldots, t \in \langle 0, a\rangle$. Suppose that $f_x(t, x)$ satisfies (15) and $\sup|f_x(t, 0)| < +\infty$. Then $x_n(t)$ converges uniformly on $\langle 0, a\rangle$ to a certain limit. Moreover, if equation (1) has a (unique) solution $x(t)$ such that $x(0) = x_0$, then $|x_n(t)-x(t)|\to 0$ uniformly on $\langle 0, a\rangle$ and*

$$|x_n(t)-x(t)| \leqslant w_n(t)$$

where

$$w_1(t) \geqslant |x_1(t)-x(t)|$$

and

$$w_{n+1}(t) = \int_0^t \exp\big(K(t-s)\big)\,\omega\big(s, w_n(s)\big)\,w_n(s)\,ds$$

with $K = \sup|f_x(t, 0)|+\max\limits_{\langle 0,a\rangle}\omega(t, M)$.

Proof. The function $z_n(t) = x_{n+1}(t)-x_n(t)$ satisfies the equation

$$z'_n(t) = Az_n(t)+f_x\big(t, x_n(t)\big)z_n(t)+f\big(t, x_n(t)\big)-f_x\big(t, x_{n-1}(t)\big)z_{n-1}(t)-f\big(t, x_{n-1}(t)\big)\,.$$

Condition (15) implies that

$$\big|f\big(t, x_n(t)\big)-f_x\big(t, x_{n-1}(t)\big)z_{n-1}(t)-f\big(t, x_{n-1}(t)\big)\big| \leqslant \omega\big(t, |z_{n-1}(t)|\big)\,|z_{n-1}(t)|$$

and $\big|f_x\big(t, x_n(t)\big)\big| \leqslant K$. Making use of theorem 4 of [14] we obtain

$$|z_n(t)| \leqslant K\int_0^t |z_n(s)|\,ds+\int_0^t \omega\big(s, |z_{n-1}(s)|\big)\,|z_{n-1}(s)|\,ds\,.$$

The last inequality leads us to the following one:

$$(36)\qquad |z_n(t)| \leqslant R\int_0^t \big[|z_n(s)|+|z_{n-1}(s)|\big]\,ds$$

where $R = \max\big(K, \max\limits_{\langle 0,a\rangle}\omega(t, 2M)\big)$. By the theorem on integral inequalities (see for instance [15], th. 1) (36) implies that

$$|z_n(t)| \leqslant F\int_0^t |z_{n-1}(s)|\,ds\,,\qquad F = R\exp(Ra)\,.$$

On the other hand, $|z_1(t)| \leqslant 2M$. Hence

$$|z_n(t)| \leqslant 2M\,\frac{(Ft)^{n-1}}{(n-1)!}\,,\qquad n = 1, 2, \ldots$$

We infer therefore that $x_n(t)$ is uniformly convergent on $\langle 0, a\rangle$. Let $x(t)$ satisfy the equation

$$x'(t) = Ax(t)+f\big(t, x(t)\big)$$

and $x(0) = x_0$. The inequality

$$(37)\qquad |x_n(t)-x(t)| \leqslant w_n(t)$$

holds for $n = 1$. Let (37) be satisfied for n and write $\varphi_\nu(t) = |x_\nu(t)-x(t)|$. We have

$$\begin{aligned}[x_\nu(t)-x(t)] = A[x_\nu(t)-x(t)]+f_x\big(t, x_{\nu-1}(t)\big)[x_\nu(t)-x(t)]+\\ +f_x\big((t, x_{\nu-1}(t)\big)[x(t)-x_{\nu-1}(t)]+f\big(t, x_{\nu-1}(t)\big)-f\big(t, x(t)\big)\,.\end{aligned}$$

By theorem 4 of [14] and by (15) we get

$$\varphi_\nu(t) \leqslant \int_0^t \big[K\varphi_\nu(s)+\omega\big(s, \varphi_{\nu-1}(s)\big)\varphi_{\nu-1}(s)\big]\,ds\,.$$

Just as in the first part of the proof, the last inequality implies that

$$\varphi_\nu(t) \leqslant \int_0^t \exp\big(K(t-s)\big)\,\omega\big(s, \varphi_{\nu-1}(s)\big)\,\varphi_{\nu-1}(s)\,ds\,.$$

If $\nu = n+1$, then

$$\varphi_{n+1}(t) \leqslant \int_0^t \exp\big(K(t-s)\big)\,\omega\big(s, \varphi_n(s)\big)\varphi_n(s)\,ds$$
$$\leqslant \int_0^t \exp\big(K(t-s)\big)\,\omega\big(s, w_n(s)\big)w_n(s)\,ds = w_{n+1}(t)\,,$$

which was to be proved. The sequence $w_n(t)$ is a sequence of successive approximations for the equation

$$w(t) = \int_0^t \exp\big(K(t-s)\big)\,\omega\big(s, w(s)\big)w(s)\,ds\,.$$

Hence $w_n(t)$ converges on $\langle 0, a\rangle$ uniformly to 0 and this proves our assertion.

The estimation $|x_n(t)-x(t)| \leqslant w_n(t)$ generalizes a certain result of Lusin [10]. In [10] $\omega(t, u) = Qu$ where $Q = \sup|f_{xx}|$. Following Lusin we can easily prove that

$$|x_n(t)-x(t)| \leqslant \frac{2C}{2^n}$$

if $w_1(t) \leqslant 1/2Qa\exp(Qa)$. In general, the function $w_1(t)$ may be calculated by using methods developed in [3] and in [14].

8. The uniform boundedness of Newtonian sequences. An essential role in section 7 was played by the assumption that the Newtonian sequence is uniformly bounded on a certain interval. Some simple estimates for Newtonian sequences were given in [6] (see p. 534-535). There appeared the assumption that f_{xx} exists. We now establish, using a method similar to that developed in [14], the existence of a common interval $\langle 0, a\rangle$ on which the Newtonian sequence is uniformly bounded.

Suppose we are given a function $z(t) \in E$, $t \in \langle 0, a\rangle$. Assume that

$$|T(t{:}A)z(0)-z(t)| \leqslant \eta \quad \text{for} \quad t \in \langle 0, a\rangle$$

and suppose that $f(t, x)$ satisfies (15). It is assumed that $f(t, x)$, $f_x(t, x)$ are strongly continuous and $\big|f\big(t, z(t)\big)\big| \leqslant F(t)$, $\big|f_x\big(t, z(t)\big)\big| \leqslant G(t)$ in $\langle 0, a\rangle$. The functions $F(t)$ and $G(t)$ are continuous. In what follows $\varphi(t)$ is the right-hand maximum solution of the equation

$$u' = 3G(t)u + 3\omega(t, u)u + F(t) \tag{38}$$

and $\varphi(0) = \eta$. Let $\varphi(t)$ exist as a solution of (38) on the whole interval $\langle 0, a\rangle$. With the aid of a general theorem of T. Ważewski (see [17]) one easily proves the following lemma:

LEMMA 8. *Suppose that $\sigma(t)$ is the right-hand maximum solution of equation*

$$\sigma' = 2\omega\big(t, \varphi(t)\big)\varphi(t) + 2G(t)\varphi(t) + F(t) + \big[\omega\big(t, \varphi(t)\big) + G(t)\big]\sigma \tag{39}$$

and $\sigma(0) = \varphi(0)$. Then $\sigma(t) = \varphi(t)$ in $\langle 0, a\rangle$.

THEOREM 11. *Suppose that $|x(t)-z(t)| \leqslant \varphi(t)$ in $\langle 0, a\rangle$. Let $y(t)$ satisfy the integral equation*

$$y(t) = T(t{:}A)y(0) + \int_0^t T(t-s{:}A)\big[f_x\big(s, x(s)\big)\big(y(s)-x(s)\big) + f\big(s, x(s)\big)\big]ds$$

for $t \in \langle 0, a\rangle$. Suppose that $y(0) = z(0)$. Then $|y(t)-z(t)| \leqslant \varphi(t)$.

Proof. Obviously

$$y(t)-z(t) = T(t{:}A)z(0)-z(t) + \int_0^t T(t-s{:}A)\big[f_x\big(s, x(s)\big)\big(y(s)-z(s)\big) + f_x\big(s, x(s)\big)\big(z(s)-x(s)\big) + f\big(s, x(s)\big)\big]ds\,. \tag{40}$$

On the other hand, $|T(t-s{:}A)| \leqslant 1$ and

$$\big|f_x\big(s, x(s)\big)\big| \leqslant G(s) + \omega\big(s, \varphi(s)\big)\,,$$
$$\big|f\big(s, x(s)\big)\big| \leqslant F(s) + G(s)\varphi(s) + \omega\big(s, \varphi(s)\big)\varphi(s)\,.$$

It follows from (40) that

$$|y(t)-z(t)| \leqslant \eta + \int_0^t \big\{\big[\omega\big(s, \varphi(s)\big) + G(s)\big]|y(s)-z(s)| + 2\big[\omega\big(s, \varphi(s)\big) + G(s)\big]\varphi(s) + F(s)\big\}ds\,.$$

The last inequality and the theorem on integral inequalities (see [15], th. 1) imply that

$$|y(t)-z(t)| \leqslant \sigma(t)$$

where $\sigma(t)$ is the right-hand maximum solution of equation (39) for which $\sigma(0) = \eta$. By lemma 8 $\sigma(t) = \varphi(t)$. Hence $|y(t)-z(t)| \leqslant \varphi(t)$, q.e.d.

Suppose now that $f_x(t, x)$ exists for $(t, x) \in \langle 0, a\rangle \times E$ and let $\big|f_x\big(t, z(t)\big)\big| \leqslant G(t)$, $\big|f\big(t, z(t)\big)\big| \leqslant F(t)$ where $z(t) = T(t{:}A)u_0$, $u_0 \in D[A]$ and $F(t)$ and $G(t)$ are continuous in $\langle 0, a\rangle$. Denote by $\varphi(t)$ the right-hand maximum solution of (38) for which $\varphi(0) = 0$. We assume that $\varphi(t)$ exists as a solution in the whole interval $\langle 0, a\rangle$. In what follows we assume that (15) holds. As usual, A is the infinitesimal generator of a semigroup of contraction operators.

THEOREM 12. *Let $y(t)$ satisfy in $\langle 0, a\rangle$ the equation*

$$y'(t) = Ay(t) + f_x(t, x(t))(y(t) - x(t)) + f(t, x(t))$$

and assume that $y(0) = u_0$. Suppose that $|x(t) - z(t)| \leqslant \varphi(t)$. Then

$$|y(t) - z(t)| \leqslant \varphi(t) \quad \text{for} \quad t \in \langle 0, a\rangle .$$

Proof. Our assumptions imply that

$$\begin{aligned}&[y(t) - z(t)]' \\ &= A[y(t) - z(t)] + f_x(t, x(t))(y(t) - z(t)) + f_x(t, x(t))(z(t) - x(t)) + f(t, x(t))\end{aligned}$$

and

$$|f_x(t, x(t))(y(t) - z(t))| \leqslant [G(t) + \omega(t, \varphi(t))]|y(t) - x(t)| ,$$

$$|f_x(t, x(t))(z(t) - x(t)) + f(t, x(t))| \leqslant 2\omega(t, \varphi(t))\varphi(t) + 2G(t)\varphi(t) + F(t) .$$

Applying th. 4 of [14] we infer that $|y(t) - z(t)| \leqslant \sigma(t)$ where $\sigma(t)$ is the right-hand maximum solution of (39) for which $\sigma(0) = 0$. It is clear by lemma 8 that $\sigma(t) = \varphi(t)$. This completes the proof.

It should be remarked that in theorem 12 we do not assume that $f(t, x), f_x(t, x)$ are continuous.

Suppose that we want to find the solution $u(t)$ of (1) for which $u(0) = u_0 \in D[A]$. We now define $z(t) = T(t:A)u_0$. Then the interval $\langle 0, a\rangle$ and $\varphi(t)$ in theorem 12 are uniqeuly determined by $z(t), f(t, x)$ and $\omega(t, w)$. If we take $x_0(t) = z(t)$ as a first member of the Newtonian sequence $\{x_n(t)\}$ $(x_n(0) = u_0)$, then, by theorem 12 $|x_n(t) - x_0(t)| \leqslant \varphi(t)$ for $t \in \langle 0, a\rangle$ and $n = 1, 2, \ldots$ We may then estimate apriori the interval on which the Newtonian sequence is uniformly bounded.

References

[1] С. А. Чаплыгин, *Избранные труды по механике и математике*, Москва 1954.

[2] N. Dunford and J. T. Schwartz, *Linear operators (general theory, part I)*, Pure and Appl. Math. Ser., Vol. VIII (1958).

[3] C. Foias, G. Gussi et V. Poenaru, *Sur les solutions généralisées de certaines équations linéaires et quasi linéaires dans l'espace de Banach*, Rev. Math. Pures et Appl., Tome III, 2 (1958), p. 283-304, Ac. R. P. Roumaine.

[4] E. Hille and R. S. Phillips, *Functional analysis and semi-groups*, Amer. Math. Soc. Coll. Publ., Vol. XXXI (1957), Providence.

[5] S. Kakutani, *Concrete representation of abstract (L)-spaces and the mean ergodic theorem*, Ann. of Math. 42, 2 (1941), p. 523-537.

[6] R. Kalaba, *On non-linear differential equations, the maximum operation and monotone convergence*, Journ. Math. Mech. 8, 4 (1959), p. 519-574.

[7] T. Kato, *Integration of the equation of evolution in Banach space*, Journ. Math. Soc. Japan. 5 (1953), p. 208-234.

[8] М. А. Красносельский, С. Г. Крейн и П. Е. Соболевский, *О дифференциальных уравнениях с неограниченными операторами в банаховом пространстве*, Доклады Ак. Наук СССР, 111 (1956), p. 19-22.

[9] Н. Н. Лузин, *О методе приближенного интегрирования академика Чаплыгина*, Собрание сочинений, Том 3, Москва 1959, p. 146-167.

[10] — *О методе С. А. Чаплыгина о аналитической точки зрения*, ibid., p. 168-180.

[11] — *О методе приближенного интегрирования академика С. А. Чаплыгина*, ibid., p. 181-208.

[12] S. Mazur, *Über konvexe Mengen in linearen normierten Räumen*, Stud. Math. 4 (1933), p. 70-84.

[13] — et W. Orlicz, *Sur les espaces métriques linéaires I*, Stud. Math. 10 (1948), p. 184-208.

[14] W. Mlak, *Limitations and dependence on parameter of solutions of non-stationary differential operator equations*, Ann. Pol. Math. 6 (1959), p. 305-322.

[15] Z. Opial, *Sur un système d'inégalités intégrales*, Ann. Pol. Math. 3 (1957), p. 200-209.

[16] R. S. Phillips, *Perturbation theory for semi-groups of operators*, Trans. Amer. Math. Soc. 74, 2 (1953), p. 199-221.

[17] T. Ważewski (in preparation).

[18] W. Mlak, *Differential inequalities with unbounded operators in Banach spaces* (in print).

[19] Л. В. Канторович, *О методе Ньютона*, Труды Инст. Стеклова 28 (1949), p. 104-144.

INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES

Reçu par la Rédaction le 2. 4. 1960