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Solution of linear systems of differential equations by the use of the method of successive approximations

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I. A scrutiny of linear parametric networks leads to the necessity of solving systems of differential-integral equations which can formally be presented in the form

$$(1)\qquad \begin{aligned} &A_{11}(t)i_1(t)+A_{12}(t)i_2(t)+\ldots+A_{1n}(t)i_n(t)=f_1(t)\,,\\ &A_{21}(t)i_1(t)+A_{22}(t)i_2(t)+\ldots+A_{2n}(t)i_n(t)=f_2(t)\,,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &A_{n1}(t)i_1(t)+A_{n2}(t)i_2(t)+\ldots+A_{nn}(t)i_n(t)=f_n(t) \end{aligned}$$

where $f_k(t)$, $k=1,2,\ldots,n$, are known continuous functions for $t\geqslant 0$; $i_k(t)$ are unkown functions; $A_{lk}(t)$ are operators determined by

$$(2)\qquad A_{lk}(t)=L_{lk}(t)\cdot\frac{d}{dt}+R_{lk}(t)+S_{lk}(t)\int(\;)\,d\tau\,;$$

$L_{lk}(t)$, $R_{lk}(t)$, $S_{lk}(t)$, $l,k=1,2,\ldots,n$, are known continuous functions for $t\geqslant 0$.

From now on we will carry on considerations assuming that operators $A_{lk}(t)$, $l,k=1,2,\ldots,n$, are defined by the formula

$$(2')\qquad A_{lk}(t)=L_{lk}(t)\frac{d}{dt}+R_{lk}(t)+S_{lk}(t)\int_0^t(\;)\,d\tau\,;$$

on account of

$$S_{lk}(t)\int i_k(t)\,dt=S_{lk}(t)\int_0^t i_k(\tau)\,d\tau+q_kS_{lk}(t)\,,\qquad q_k=\mathrm{const}\,,$$

one can pass from considering system (1) when the operators $A_{lk}(t)$ are defined by formula (2) to considering system (1) when the operators are of form (2′). Functions $f_k(t)$ then include the expressions $q_kS_{lk}(t)$.

Writing

$$A = \begin{bmatrix} A_{11}(t) & A_{12}(t) & \dots & A_{1n}(t) \\ A_{21}(t) & A_{22}(t) & \dots & A_{2n}(t) \\ \cdot & \cdot & \cdot & \cdot \\ A_{n1}(t) & A_{n2}(t) & \dots & A_{nn}(t) \end{bmatrix}, \quad i = \begin{bmatrix} i_1(t) \\ i_2(t) \\ \cdots \\ i_n(t) \end{bmatrix}, \quad f = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \cdots \\ f_n(t) \end{bmatrix}$$

system (1) can be written more briefly in the form

$$Ai = f. \tag{3}$$

Prof. dr Józef Lenkowski in his work [1]—where the results of the present paper were adopted—used an iteration process to solve system (3), which, with the use of the notation here applied can be expressed as follows:

Solutions of the system of equations (1) satisfying an initial condition $i(0) = c$ can be found according to the following scheme: We write equation (3) in the form

$$\bar{A}i + (A - \bar{A})i = f, \tag{4}$$

where

$$\bar{A} = \begin{bmatrix} \bar{A}_{11} & \dots & \bar{A}_{1n} \\ \bar{A}_{21} & \dots & \bar{A}_{2n} \\ \cdot & \cdot & \cdot \\ \bar{A}_{n1} & \dots & \bar{A}_{nn} \end{bmatrix}, \quad \bar{A}_{lk} = \bar{L}_{lk}\frac{d}{dt} + \bar{R}_{lk} + \bar{S}_{lk}\int_0^t (\ )\,d\tau,$$

$\bar{L}_{lk}, \bar{R}_{lk}, \bar{S}_{lk}$, $l, k = 1, 2, \dots, n$, are integers connected with functions $L_{lk}(t), R_{lk}(t), S_{lk}(t)$ (but they may be chosen arbitrarily). When these functions are periodical with periods T_{lk}, $l, k = 1, 2, \dots, n$, then we assume for instance

$$\bar{L}_{lk} = \frac{1}{T_{lk}}\int_0^{T_{lk}} L_{lk}(\tau)\,d\tau, \quad \bar{R}_{lk} = \frac{1}{T_{lk}}\int_0^{T_{lk}} R_{lk}(\tau)\,d\tau, \quad \bar{S}_{lk} = \frac{1}{T_{lk}}\int_0^{T_{lk}} S_{lk}(\tau)\,d\tau. \tag{5}$$

We assume that a zero approximation of solution of equation (3) with the initial condition $i(0) = c$ is a vector-function $i_0(t)$, being the solution of the equation

$$\bar{A}i = f \tag{$\bar{3}$}$$

with the initial condition $i_0(0) = c$.

As the $m+1$-th "correction" we take a vector-function $i_{m+1}(t)$ which satisfies the equation

$$\bar{A}i + (A - \bar{A})i_m = 0 \tag{$\bar{4}$}$$

and the initial condition $i_{m+1}(0) = 0$, $m = 0, 1, 2, \dots$

A solution of equation (3) satisfying the given initial condition $i(0) = c$ is obtained in the form

$$i(t) = \sum_{m=0}^{\infty} i_m(t). \tag{6}$$

The iteration process thus formulated has been worked out by applying to the system of equations (1) some modification of Picard's method of successive approximations, given by S. A. Schelkunoff [2] for a simple linear system of two differential equations; next it was adopted by L. A. Zadeh [3] for linear differential equations of the n-th order.

Sufficient conditions of convergence of an iteration sequence in the modified Picard method of successive approximation were given in paper [4].

In 1937 prof T. Ważewski [5] dealt already with a modification of Picard's method for nonlinear systems of differential equations and he formulated a general theorem concerning the convergence of an appropriately formed iteration sequence. The theorem includes neither the case discussed in paper [4] nor the case considered in this work.

The aim of this paper is to give a sufficient condition for uniform convergence of series (6) to solve equation (3). The following theorem includes this condition:

THEOREM 1. *A sufficient condition of uniform convergence of series* (6) *in the interval* $\langle 0, a\rangle$ *—where a is an arbitrary positive real number— for the solution of equation* (3) *satisfying the initial condition* $i(0) = c$ *is the following:*

1° *there exists a derivative of the matrix* $L(t)$ *bounded in the interval* $\langle 0, a\rangle$,

$$L(t) = \begin{bmatrix} L_{11}(t) & \cdot & L_{1n}(t) \\ \cdot & \cdot & \cdot \\ L_{n1}(t) & \dots & L_{nn}(t) \end{bmatrix},$$

2° $\det \bar{L} \neq 0$,

$$\bar{L} = \begin{bmatrix} \bar{L}_{11} & \dots & \bar{L}_{1n} \\ \cdot & \cdot & \cdot \\ \bar{L}_{n1} & \dots & \bar{L}_{nn} \end{bmatrix},$$

3° *the condition*

$$\max_{0 \leqslant t \leqslant a} \|\bar{L}^{-1}(\bar{L} - L)\| < 1$$

must be satisfied.

CONCLUSION 1. *If the assumptions of theorem* 1 *are satisfied for any* $a > 0$, *series* (6) *is nearly uniformly convergent in the interval* $\langle 0, \infty)$ *for the solution of system* (1) *satisfying the initial condition* $i(0) = c$.

It can easily be seen that if series (6) and one formed from it by differentiating term by term are uniformly convergent in the interval $\langle 0, a\rangle$, then the sum of the series (6) is the solution of the system of equations (1) satisfying the initial condition $i(0) = c$.

II. Now we will deal with a linear system of differential equations and some method of successive approximations to which—as will be shown later on—the iteration process described above is introduced. Let us note some fundamental facts concerning a system of differential equations of the form

$$x' = A(t)x + f(t)\,, \tag{7}$$

where the matrix $A(t)$ and the vector-function $f(t)$ are continuous in the interval $\langle 0, a\rangle$. Let us denote by $X(t)$ the normed fundamental matrix of system (7). The matrix $X(t)$ satisfies the differential matrix equation

$$X' = A(t)X \tag{8}$$

and the initial condition $X(0) = I$, I—unitary matrix. Put

$$X(t, \xi) = X(t)X^{-1}(\xi)\,,$$

where ξ is a parameter. The matrix $X(t, \xi)$ with an arbitrarily fixed ξ with respect to t satisfies equation (8) and the initial condition $X(\xi, \xi) = I$. The solution of system (7) satisfying the initial condition $x(0) = c$ is expressed by the formula (cf. [6])

$$x(t) = X(t)\cdot c + \int_0^t X(t, \xi)f(\xi)\,d\xi\,. \tag{9}$$

If $A(t) = \text{const}$ then

$$\begin{gathered} X(t) = e^{At}, \quad X(t, \xi) = X(t-\xi) = e^{A(t-\xi)}, \\ x(t) = e^{At}\cdot c + \int_0^t e^{A(t-\xi)} f((\xi)\,d\xi\,. \end{gathered} \tag{10}$$

Next the following determination of norm is used:

$$\|A\| = \max_i \Big[\sum_{k=1}^{n} |a_{ik}|\Big]\,, \qquad \|x\| = \max_{(i)} |x_i|\,.$$

It can easily be verified that the norm thus defined has the following properties:

$$\begin{gathered} \|A\| \geqslant 0\,, \quad \|A+B\| \leqslant \|A\| + \|B\|, \quad \|AB\| \leqslant \|A\|\cdot\|B\|\,, \\ \|Ay\| \leqslant \|A\|\cdot\|y\|\,, \quad \Big\|\int_0^t A(\xi)\,d\xi\Big\| \leqslant \int_0^t \|A(\xi)\|\,d\xi\,. \end{gathered} \tag{11}$$

Consider a system of differential equations of the form

$$x' = A(t)x + B(t)x' + f(t) \tag{12}$$

where matrices $A(t)$, $B(t)$ and the vector-function $f(t)$ are continuous in the interval $\langle 0, a\rangle$.

Let $\bar{A}(t)$ be an arbitrarily chosen matrix which is continuous in the interval $\langle 0, a\rangle$. The matrix $A(t)$ is presented as

$$A(t) = \bar{A}(t) + \bar{\bar{A}}(t)\,,$$

where

$$\bar{\bar{A}}(t) = A(t) - \bar{A}(t)\,.$$

The normed fundamental matrix of the system

$$x' = \bar{A}(t)x \tag{13}$$

is denoted by $\bar{X}(t)$. Put $\bar{X}(t, \xi) = \bar{X}(t)\bar{X}^{-1}(\xi)$. According to formula (9) we say that the system of equations (12) with the initial condition $x(0) = c$ is equivalent to a system of differential-integral equations of the form

$$x = \bar{X}(t)\cdot c + \int_0^t \bar{X}(t, \xi)f(\xi)\,d\xi + \int_0^t \bar{X}(t, \xi)\bar{\bar{A}}(\xi)x(\xi)\,d\xi + \int_0^t \bar{X}(t, \xi)B(\xi)x'(\xi)\,d\xi\,. \tag{14}$$

Put

$$x_0 = \bar{X}(t)\cdot c + \int_0^t \bar{X}(t, \xi)f(\xi)\,d\xi\,. \tag{15}$$

With respect to (9) one can see that x_0 satisfies the equation

$$x' = \bar{A}(t)x + f(t) \tag{16}$$

and the initial condition $x_0(0) = c$. System (14) is then of the form

$$x = x_0 + \int_0^t \bar{X}(t, \xi)\bar{\bar{A}}(\xi)x(\xi)\,d\xi + \int_0^t \bar{X}(t, \xi)B(\xi)x'(\xi)\,d\xi\,. \tag{17}$$

Assuming that the matrix $B(t)$ has a bounded derivative in the interval $\langle 0, a\rangle$ we get

$$\begin{aligned} &\int_0^t \bar{X}(t, \xi)B(\xi)x'(\xi)\,d\xi \\ &\quad = \bar{X}(t, \xi)B(\xi)x(\xi)\Big|_0^t - \int_0^t \frac{\partial}{\partial\xi}[\bar{X}(t, \xi)B(\xi)]x(\xi)\,d\xi \\ &\quad = B(t)x(t) - \bar{X}(t)\cdot B(0)\cdot c - \int_0^t \frac{\partial}{\partial\xi}[\bar{X}(t, \xi)B(\xi)]x(\xi)\,d\xi\,. \end{aligned} \tag{18}$$

According to (18) formula (17) is of the form

$$x = x_0 + B(t)x + \int_0^t \bar{\bar{D}}(t,\xi)x(\xi)d\xi + \bar{X}(t)b, \tag{19}$$

where

$$\bar{\bar{D}}(t,\xi) = \bar{X}(t,\xi)\bar{\bar{A}}(\xi) - \frac{\partial}{\partial\xi}[\bar{X}(t,\xi)B(\xi)], \qquad b = -B(0)c.$$

It can be seen that the systems of equations (14) and (19) are equivalent under these assumptions.

We solve system (19) by the use of the successive approximations method putting

$$\begin{aligned} &x_0 = x_0, \\ &x_m = x_0 + B(t)x_{m-1} + \int_0^t \bar{\bar{D}}(t,\xi)x_{m-1}(\xi)d\xi + \bar{X}(t)b, \qquad m = 1, 2, \ldots \end{aligned} \tag{20}$$

In order to test the convergence of a sequence $\{x_m\}$ we consider—as usual—the series

$$\sum_{m=1}^{\infty}(x_m - x_{m-1}). \tag{21}$$

Introducing letter symbols

$$G = \max_{0\leqslant t\leqslant a}\|x_1 - x_0\|, \qquad M = \max_{\substack{0\leqslant t\leqslant a\\ 0\leqslant \xi\leqslant t}}\|\bar{\bar{D}}(t,\xi)\|, \qquad K = \max_{0\leqslant t\leqslant a}\|B(t)\| \tag{22}$$

we get the estimate

$$\|x_{m+1} - x_m\| \leqslant G\Big[\sum_{s=0}^{m}\binom{m}{s}K^{m-s}\frac{(Mt)^s}{s!}\Big]. \tag{23}$$

Indeed, assume that the estimate is right for some $m \geqslant 0$ which is a natural number; then according to (11) we have

$$\begin{aligned} \|x_{m+2} - x_{m+1}\| &\leqslant \|B(t)\|\cdot\|x_{m+1} - x_m\| + \int_0^t \|\bar{\bar{D}}(t,\xi)\|\cdot\|x_{m+1} - x_m\|d\xi \\ &\leqslant G\Big[\sum_{s=0}^{m}\binom{m}{s}K^{m-s+1}\frac{(Mt)^s}{s!}\Big] + G\Big[\sum_{s=0}^{m}\binom{m}{s}K^{m-s}\frac{(Mt)^{s+1}}{(s+1)!}\Big] \\ &= G\Big[\sum_{s=0}^{m+1}\binom{m+1}{s}K^{m-s+1}\frac{(Mt)^s}{s!}\Big], \end{aligned}$$

which, together with the application of the mathematical induction rule, proves formula (23) for any natural number m.

The series

$$G\Big[\sum_{m=0}^{\infty}\sum_{s=0}^{m}\binom{m}{s}K^{m-s}\frac{(Mt)^s}{s!}\Big] \tag{24}$$

is a majorant for series (21) for any $t \in \langle 0, a\rangle$. The necessary and sufficient condition of the uniform convergence of series (24) in the interval $\langle 0, a\rangle$ is that $K < 1$. If this condition is satisfied than one can easily verify (cf. [4]) that the sum of series (24) is

$$\frac{G}{1-K}\exp\Big(\frac{Mt}{1-K}\Big). \tag{25}$$

We conclude that a sufficient condition of uniform convergence of series (21) is that the inequality $K < 1$ be satisfied. These considerations allow us to note the following theorem:

THEOREM 2. *If* 1° *a matrix* $B(t)$ *has a bounded derivative in the interval* $\langle 0, a\rangle$, 2° $\max_{0\leqslant t\leqslant a}\|B(t)\| < 1$, *then a sequence* $\{x_m\}$ *defined by formula* (20) *uniformly converges in the interval* $\langle 0, a\rangle$ *to the solution of the system of equations* (19), *and thus to the solution of system* (12) *with the initial condition* $x(0) = c$.

CONCLUSION 2. *If the assumptions of theorem* 2 *are satisfied for any* $a > 0$, *then the sequence* $\{x_m\}$ *is nearly uniformly convergent in the interval* $\langle 0, \infty)$.

Let us notice next that

$$\begin{aligned} &x_0(0) = c, \\ &x_m(0) = x_0(0) + B(0)x_{m-1}(0) + \bar{X}(0)b = c + B(0)x_{m-1}(0) - B(0)c, \quad m = 1, 2, \ldots \end{aligned}$$

whence we obtain

$$x_m(0) = c, \qquad m = 0, 1, \ldots$$

According to (18) we have

$$\begin{aligned} &\int_0^t \bar{X}(t,\xi)B(\xi)x_m'(\xi)d\xi \\ &\qquad = B(t)x_m(t) - \bar{X}(t)B(0)c - \int_0^t \frac{\partial}{\partial\xi}[\bar{X}(t,\xi)B(\xi)]x_m(\xi)d\xi. \end{aligned} \tag{26}$$

From relations (20) and (26) we infer that the following relation is satisfied:

$$x_{m+1} = x_0 + \int_0^t \bar{X}(t,\xi)\bar{\bar{A}}(\xi)x_m(\xi)d\xi + \int_0^t \bar{X}(t,\xi)B(\xi)x_m'(\xi)d\xi, \qquad m = 0, 1, \ldots \tag{27}$$

From relation (27) according to (17) we conclude that vector-functions $x_{m+1}(t)$, $m = 0, 1, \ldots$, satisfy a system of differential equations

$$x' = \bar{A}(t)x + f(t) + \bar{\bar{A}}(t)x_m + B(t)x'_m \tag{28}$$

and the initial condition $x_{m+1}(0) = c$.

Now put

$$\tilde{x}_0 = x_0, \quad \tilde{x}_{m+1} = x_{m+1} - x_m, \quad m = 0, 1, \ldots \tag{29}$$

Conclusion 3. *If the assumptions of theorem 2 are satisfied, then the sum of the series*

$$\sum_{m=0}^{\infty} \tilde{x}_m \tag{30}$$

is a solution of system (12) *satisfying the initial condition* $x(0) = c$.

From (28) and (29) it follows that the vector-functions $\tilde{x}_{m+1}$, $m = 0, 1, \ldots$, satisfy the system of equations

$$x' = \bar{A}(t)x + \bar{\bar{A}}(t)\tilde{x}_m + B(t)\tilde{x}'_m \tag{31}$$

and the initial condition $\tilde{x}_{m+1}(0) = 0$.

Notice. The simple example given in paper [4] shows that when assumption 2° of theorem 2 is not satisfied, then series (30) can be divergent, as it is in the example.

The conclusions inferred with respect to system (12) are used in part IV of the paper to prove theorem 1. This fact supports the necessity of considering system (12), whose form is little artificial.

III. The iteration process described in part II is a modification of Picard's method of successive approximations.

If we assume in the above considerations that

$$B(t) \equiv 0, \quad \bar{A}(t) \equiv 0,$$

then we obtain the common Picard method of successive approximations.

In this part it is assumed that $B(t) \equiv 0$. The iteration process mentioned above can be more useful than the Picard method.

In many cases with an appropriate choice of matrixes $\bar{A}(t)$ series (30) can converge more rapidly than an appropriate series formed by the use of the Picard method. This fact has a great practical importance.

In the Picard method the first approximation is mostly equal to the initial condition and therefore it does not refer to the form of the differential equation itself, determined by the matrix $A(t)$; it is mostly qualitatively different (its behaviour at $t \to \infty$) from the exact solution and owing to that it is the cause of the slow convergence of the sequence of successive approximations. The modified iteration process prevents such situations. It is to be remembered that the matrix $\bar{A}(t)$ must be chosen so that equation (16) be solvable, and therefore in practice it is generally assumed that $\bar{A}(t) = \text{const}$, e. g.:

$$\bar{A} = \frac{1}{a}\int_0^a A(t)\,dt \quad \text{or} \quad \bar{A} = \frac{1}{\omega}\int_0^\omega A(t)\,dt,$$

where the matrix $A(t)$ is periodical, ω being its period. Now we will carry out some estimations, which will explain the method under discussion.

Assumptions H_0.

1° Matrix $A(t)$ and a vector-function $f(t)$ are continuous and bounded in the interval $\langle 0, \infty)$.

2° $\bar{A}(t) = \text{const}$, $B(t) \equiv 0$, $t \in \langle 0, \infty)$.

3° Roots λ_i, $i = 1, 2, \ldots, n$, of the characteristic polynomial of the equation

$$x' = \bar{A}x \tag{32}$$

have their real parts negative.

When the assumptions H_0 are satisfied, then there exist such constants $C_1 > 0$ and $a > 0$ that

$$\|\bar{X}(t)\| \leqslant C_1 e^{-at}, \quad t \geqslant 0. \tag{33}$$

When $\lambda_i \neq \lambda_j$ for $i \neq j$ then $a = \min\limits_i |\mathrm{Re}\,\lambda_i|$; otherwise

$$0 < a < \min_i |\mathrm{Re}\,\lambda_i|.$$

Put

$$F = \sup_{0 \leqslant t < +\infty} \|f(t)\|, \quad M_1 = \sup_{0 \leqslant t < +\infty} \|\bar{\bar{A}}(t)\|. \tag{34}$$

According to (27) and (10)

$$\tilde{x}_m = \int_0^t \bar{X}(t-\xi)\bar{\bar{A}}(\xi)x_{m-1}(\xi)\,d\xi, \quad m = 1, 2, \ldots \tag{35}$$

From relations (15), (29), (33), (34), (35) we easily obtain

$$\|\tilde{x}_m\| \leqslant (C_1 M_1)^m e^{-at}\left\{\frac{C_2 t^m}{m!} + \frac{C_1 F}{a^{m+1}}\left[e^{at} - \sum_{i=0}^{m} \frac{(at)^i}{i!}\right]\right\} \tag{36}$$

or

$$\|\tilde{x}_m\| \leqslant (C_1 M_1)^m e^{-at}\frac{C_2 t^m}{m!} + \frac{F}{M_1}\left(\frac{C_1 M_1}{a}\right)^{m+1}, \tag{37}$$

where

$$C_2 = C_1\|c\|\,.$$

If, moreover the inequality

(38) $$\frac{C_1M_1}{a} < 1$$

is satisfied, then the series

(39) $$\sum_{m=0}^{\infty}\left[(C_1M_1)^m e^{-at}\frac{C_2t^m}{m!} + \frac{F}{M_1}\left(\frac{C_1M_1}{a}\right)^{m+1}\right]$$

is the convergent majorant for series (30). The sum of series (39) is

(40) $$C_2e^{-\beta t} + \frac{C_1F}{\beta}\,, \qquad \beta = C_1M_1 - a > 0\,.$$

Thus under these assumptions we have the estimate

(41) $$\|x\| \leqslant C_2e^{-\beta t} + \frac{C_1F}{\beta}\,, \qquad t \geqslant 0\,,$$

which allows us to formulate the theorem:

THEOREM 3. *If inequality* (38) *is satisfied where the constants a, C_1, M_1 are determined by relations* (33) *and* (34) *respectively, then the asymptotic stability of the zero solution of the equation*

$$x' = \bar{A}x$$

causes the asymptotic stability of the zero solution of the equation

$$x' = A(t)x\,.$$

Theorems of this type can be found in papers [7] and [8], but here the meaning of the respective constants a, C_1M_1 is determined exactly and consequently the theorem has greater practical importance.

If the assumptions H_0 are satisfied and the solution of equation (12) is bounded for $t \geqslant 0$, then one can easily obtain the following error estimate:

(42) $$\left\|x - \sum_{i=0}^{m}\tilde{x}_i\right\| \leqslant K\left(\frac{C_1M_1}{a}\right)^{m+1} e^{-at}\left[e^{at} - \sum_{i=0}^{m}\frac{(at)^i}{i!}\right], \qquad t \geqslant 0$$

where

$$K = \sup_{0\leqslant t<+\infty}\|x(t)\|\,.$$

From inequality (38) and relation (41) it follows that

$$K \leqslant C_2 + \frac{C_1F}{\beta}\,.$$

Then we can note:

CONCLUSION 4. *If the assumptions* H_0 *and inequality* (38) *are satisfied, then from* (42) *we obtain*

(43) $$\left\|x - \sum_{i=0}^{m}\tilde{x}_i\right\| \leqslant \left(C_2 + \frac{C_1F}{\beta}\right)\left(\frac{C_1M_1}{a}\right)^{m+1}, \qquad t \geqslant 0\,.$$

It ought to be stressed that estimate (43) is uniform for the whole straight $t \geqslant 0$. Estimates of this type are not obtained by the use of the normal Picard method. The estimates (42) and (43) allow us to maintain that in cases when the assumptions H_0 and (38) are satisfied the modified iteration process is more rapidly convergent than the normal Picard iteration process.

IV. Proof of theorem 1. Equation (3) can be written in the form

(44) $$L(t)i' + R(t)i + S(t)\int_0^t i(\tau)\,d\tau = f(t)\,,$$

where

$$L(t) = \begin{bmatrix} L_{11}(t) & \dots & L_{1n}(t) \\ \cdot & \cdot\;\cdot\;\cdot\;\cdot & \cdot \\ L_{n1}(t) & \dots & L_{nn}(t)\end{bmatrix}, \qquad R(t) = \begin{bmatrix} R_{11}(t) & \dots & R_{1n}(t) \\ \cdot & \cdot\;\cdot\;\cdot\;\cdot & \cdot \\ R_{n1}(t) & \dots & R_{nn}(t)\end{bmatrix},$$

$$S(t) = \begin{bmatrix} S_{11}(t) & \dots & S_{1n}(t) \\ \cdot & \cdot\;\cdot\;\cdot\;\cdot & \cdot \\ S_{n1}(t) & \dots & S_{nn}(t)\end{bmatrix}.$$

Put

(45) $$q(t) = \int_0^t i(\tau)\,d\tau\,.$$

By introducing a new unknown vector-function $q(t)$ the differential-integral equation (44) can be transformed into an equivalent differential equation of the second order of the form

(46) $$L(t)q'' + R(t)q' + S(t)q = f(t)\,.$$

Let $\bar{L}$, $\bar{R}$, $\bar{S}$ be arbitrarily chosen constant matrices, $\det\bar{L} \neq 0$. Put

$$L(t) = \bar{L} - \bar{\bar{L}}(t)\,, \qquad R(t) = \bar{R} - \bar{\bar{R}}(t)\,, \qquad S(t) = \bar{S} - \bar{\bar{S}}(t)$$

where

$$\bar{\bar{L}}(t) = \bar{L} - L(t)\,, \qquad \bar{\bar{R}}(t) = \bar{R} - R(t)\,, \qquad \bar{\bar{S}}(t) = \bar{S} - S(t)\,.$$

Equation (46) is of the form

(47) $$q'' + \bar{L}^{-1}\bar{R}q' + \bar{L}^{-1}\bar{S}q = \bar{L}^{-1}\bar{\bar{R}}(t)q' + \bar{L}^{-1}\bar{\bar{S}}(t)q + \bar{L}^{-1}f(t) + \bar{L}^{-1}\bar{\bar{L}}(t)q''\,.$$

Introducing new unknowns

$$x = q\,, \qquad y = q'$$

and putting

$$\bar{\mathcal{R}} = -\bar{L}^{-1}\bar{R}, \quad \bar{\bar{\mathcal{R}}} = \bar{L}^{-1}\bar{\bar{R}}(t), \quad \bar{\mathcal{S}} = -\bar{L}^{-1}\bar{S}, \quad \bar{\bar{\mathcal{S}}} = \bar{L}^{-1}\bar{\bar{S}}(t),$$
$$\mathcal{L} = \bar{L}^{-1}\bar{\bar{L}}(t)$$

we obtain a system

$$(48) \qquad \begin{aligned} y' &= \bar{\mathcal{R}}y + \bar{\mathcal{S}}x + \bar{L}^{-1}f(t) + \bar{\bar{\mathcal{R}}}y + \bar{\bar{\mathcal{S}}}x + \mathcal{L}y', \\ x' &= y. \end{aligned}$$

We write system (48) in the form

$$(49) \qquad z' = \bar{D}z + \bar{\bar{D}}z + g + \mathcal{E}z',$$

where

$$z = \begin{bmatrix} y \\ x \end{bmatrix}, \quad \bar{D} = \begin{bmatrix} \bar{\mathcal{R}} & \bar{\mathcal{S}} \\ I & 0 \end{bmatrix}, \quad \bar{\bar{D}} = \begin{bmatrix} \bar{\bar{\mathcal{R}}} & \bar{\bar{\mathcal{S}}} \\ 0 & 0 \end{bmatrix}, \quad \mathcal{E} = \begin{bmatrix} \mathcal{L} & 0 \\ 0 & 0 \end{bmatrix}, \quad g = \begin{bmatrix} \bar{L}^{-1}f \\ 0 \end{bmatrix}.$$

It can be seen that equation (44) with the initial condition $i(0) = c$ is equivalent to equations (46) and (47) with the initial conditions $q(0) = 0$, $q'(0) = c$ and to equation (49) with the initial condition $z(0) = c^* = \begin{bmatrix} c \\ 0 \end{bmatrix}$. For equation (49) we can use the iteration process described in part II.

We then form a sequence $\{z_m\}$ using the relations

$$z'_0 = \bar{D}z_0 + g, \qquad z_0(0) = c^*,$$
$$z'_{m+1} = \bar{D}z_{m+1} + \bar{\bar{D}}z_m + \mathcal{E}z'_m, \qquad z_{m+1}(0) = 0, \qquad m = 0, 1, \ldots$$

According to theorem 2 and conclusion 3 we assert that a series of the form

$$(50) \qquad \sum_{m=0}^{\infty} z_m$$

uniformly converges in the interval $\langle 0, a \rangle$ to a solution of equation (49) satisfying the initial condition $z(0) = c^*$; if 1° matrix $\mathcal{E}(t)$ has a bounded derivative in the interval $\langle 0, a \rangle$, 2° $\max\limits_{0 \leqslant t \leqslant a} \|\mathcal{E}(t)\| < 1$. One can see that vector-functions x_0, y_0 such that

$$z_0 = \begin{bmatrix} y_0 \\ x_0 \end{bmatrix}$$

satisfy the system

$$\begin{aligned} y' &= \bar{\mathcal{R}}y + \bar{\mathcal{S}}x + g, \\ x' &= y \end{aligned}$$

and the initial conditions $x_0(0) = 0$, $y_0(0) = c$, and that the vector-function $q_0 = x_0$ satisfies the equation

$$\bar{L}q'' + \bar{R}q' + \bar{S}q = f$$

and the initial conditions $q(0) = 0$, $q'(0) = c$. Finally the vector-function $i_0 = q'_0$ satisfies the equation

$$\bar{L}i' + \bar{R}i + \bar{S}\int_0^t i(\tau)\,d\tau = f$$

and the initial condition $i(0) = c$, which means that it satisfies equation (3). Similarly vector-functions x_{m+1}, y_{m+1} such that

$$z_{m+1} = \begin{bmatrix} y_{m+1} \\ x_{m+1} \end{bmatrix}$$

satisfy the system

$$\begin{aligned} y' &= \bar{\mathcal{R}}y + \bar{\mathcal{S}}x + \bar{\bar{\mathcal{R}}}y_m + \bar{\bar{\mathcal{S}}}x_m + \mathcal{L}y'_m, \\ x' &= y \end{aligned}$$

and the initial conditions $x(0) = y(0) = 0$, and a vector-function $q_{m+1} = x_{m+1}$ satisfies the equation

$$\bar{L}q'' + \bar{R}q' + \bar{S}q = \bar{\bar{R}}q'_m + \bar{\bar{S}}q_m + \bar{\bar{L}}q''_m$$

and the initial conditions

$$q_{m+1}(0) = 0, \qquad q'_{m+1}(0) = 0;$$

finally a vector-function $i_{m+1} = q'_{m+1}$ satisfies the equation

$$\bar{L}i' + \bar{R}i + \bar{S}\int_0^t i(\tau)\,d\tau = \bar{\bar{R}}i_m + \bar{\bar{S}}\int_0^t i_m(\tau)\,d\tau + \bar{\bar{L}}i'_m, \qquad m = 0, 1, \ldots$$

and the initial condition $i_{m+1}(0) = 0$, which means that it satisfies equation (4).

One can see that a converse argument is also valid. The convergence of series (50) is equivalent to the convergence of series (6). Considering the fact that the existence of a bounded derivative of the matrix $\mathcal{E}(t)$ is equivalent to the existence of a bounded derivative of the matrix $L(t)$ and

$$\max_{0 \leqslant t \leqslant a} \|\mathcal{E}(t)\| = \max_{0 \leqslant t \leqslant a} \|\bar{L}^{-1}\bar{\bar{L}}\| = \max_{0 \leqslant t \leqslant a} \|\bar{L}^{-1}(\bar{L} - L)\|,$$

we conclude that theorem 1 is proved.

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References

[1] J. Lenkowski, *The application of a certain perturbational method to the analysis of time-varying circuits*, Bull. Acad. Polon. Sci., Ser. sci. techn. 7 (1959), p. 531-539.

[2] S. A. Schelkunoff, *Solution of linear and slightly nonlinear differential equations*, Quart. of. Appl. Math. 3 (1946), p. 348.

[3] L. A. Zadeh, *Frequency analysis of variable networks*, Proceedings of the I. R. E. 38 (1950), p. 296.

[4] M. Kwapisz, *About a modified Picard method of successive approximations for solution of ordinary linear differential equations of n-th order (in Polish)*, Zeszyty Naukowe Politechniki Gdańskiej, Łączność, 3 (1960), p. 31-46.

[5] T. Ważewski, *Sur la méthode des approximations successives*, Ann. Soc. Pol. Math. 16 (1937), p. 214.

[6] В. В. Немыцкий и В. В. Степанов, *Качественная теория дифференциальных уравнений*, Москва 1949.

[7] I. Г. Малкин, *Теория устойчивости движения*, Москва 1952.

[8] Р. Беллман, *Теория устойчивости решений дифференциальных уравнений*, Москва 1954.

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