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A PROOF OF A THEOREM OF ŻELAZKO ON L^p-ALGEBRAS

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Let G be a locally compact Abelian topological group. For each $p \geqslant 1$, we define the space $L^p(G)$ as the space of all measurable complex-valued functions f on G such that $|f|^p$ is integrable with respect to the Haar measure m on G. Obviously, $L^p(G)$ is a Banach space under the norm

$$\|f\|_p = \Big(\int_G |f(x)|^p m(dx)\Big)^{1/p}.$$

In the sequel we shall denote by fg the convolution of functions f and g, i. e.

$$(fg)(x) = \int_G f(y) g(xy^{-1}) m(dx) \qquad (x \epsilon G).$$

In this note we shall give a simple proof of the following theorem, proved by W. Żelazko in paper [3]:

If, for a number $p > 1$, $L^p(G)$ is a topological ring under the convolution multiplication, then G is a compact group.

Proof. Let R be the extension of $L_p(G)$ to a topological ring with a unit element (see [1], p. 158). The norm in R, which is an extension of the norm in $L^p(G)$, will henceforth be denoted by $\| \ \|_p$. It is well known that the norm

$$(1) \qquad \|f\| = \sup_{g \epsilon R, \|g\|_p = 1} \|fg\|_p$$

makes R a normed ring (see [1], p. 168).

First we shall prove that the ring $L^p(G)$ admits a non-trivial continuous homomorphism into the complex field. To prove this it is sufficient to show that $L^p(G)$ contains an element which does not belong to the radical of R.

Let V be a symmetric compact neighbourhood of the unit element in G and let χ be the characteristic function of the set V^2. Obviously, $\chi \in L^p(G)$. Moreover, using the inequality

$$\chi^{n+1}(x) = \int_G \chi^n(y)\chi(xy^{-1})m(dy) \geqslant \int_V \chi^n(y)\chi(xy^{-1})m(dy)$$
$$= \int_V \chi^n(y)m(dy) \qquad (x \in V),$$

we can prove by induction the inequality $\chi^{n+1}(x) \geqslant m^n(V)$ $(x \in V)$. Hence and from (1) we get the inequality

$$\|\chi^n\| \geqslant c\|\chi^{n+1}\|_p \geqslant c\Big(\int_V |\chi^{n+1}(x)|^p m(dx)\Big)^{1/p} \geqslant cm^n(V)m^{1/p}(V),$$

where $c = \|\chi\|_p^{-1}$. Thus $\lim\limits_{n\to\infty}\|\chi^n\|^{1/n} > 0$ and, consequently, the function χ does not belong to the radical of R.

Let h be a non-trivial continuous homomorphism of $L^p(G)$ into the complex field. By a well-known theorem there exists a function $\varphi \in L^q(G)$, where $1/p+1/q = 1$, such that

$$h(f) = \int_G f(x)\varphi(x)m(dx)$$

for any $f \in L^p(G)$ (for non σ-finite measures see [2]). Moreover,

$$\int_G |\varphi(x)|^q m(dx) > 0. \tag{2}$$

From the equality

$$\int_G\int_G f(x)g(y)\varphi(xy)m(dx)m(dy) = h(fg) = h(f)h(g)$$
$$= \int_G\int_G f(x)g(y)\varphi(x)\varphi(y)m(dx)m(dy)$$

we get for almost every pair $x, y \in G$ the following one: $\varphi(xy) = \varphi(x)\varphi(y)$. Hence for almost every $y \in G$ we have the equality

$$\int_G |\varphi(x)|^q m(dx) = \int_G |\varphi(xy)|^q m(dx) = |\varphi(y)|^q \int_G |\varphi(x)|^q m(dx),$$

which, according to (2), implies the equality $|\varphi(y)| = 1$ almost everywhere. We have proved that the function identically equal to 1 belongs to $L^q(G)$. Thus $m(G)$ is finite and, consequently, the group G is compact.

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