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SOME REMARKS ON SYMMETRIC RELATIONS

BY

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Let X be a set, R — a symmetric relation defined in X and α — an ordinal number. We consider the following conditions:

(s_α) *If* $Y \subset X$ *and* $\aleph_\alpha \leqslant \overline{\overline{Y}}$, *there exist* $y_1, y_2 \in Y$ *such that* $y_1 \neq y_2$ *and* $y_1 R y_2$.

(s_α^*) *If* $Y \subset X$ *and* $\aleph_\alpha \leqslant \overline{\overline{Y}}$, *there exists a* $Z \subset Y$ *such that* $\aleph_\alpha \leqslant \overline{\overline{Z}}$ *and*

$$\aleph_\alpha \leqslant \overline{\overline{\{p : p \in Z \text{ and } zRp\}}}$$

for every $z \in Z$.

(k_α^*) *If* $Y \subset X$ *and* $\aleph_\alpha \leqslant \overline{\overline{Y}}$, *there exists a* $Z \subset Y$ *such that* $\aleph_\alpha \leqslant \overline{\overline{Z}}$ *and*

$$\overline{\overline{\{p : p \in Z \text{ and } z \operatorname{non} Rp\}}} < \aleph_\alpha$$

for every $z \in Z$.

(k_α) *If* $Y \subset X$ *and* $\aleph_\alpha \leqslant \overline{\overline{Y}}$, *there exists a* $Z \subset Y$ *such that* $\aleph_\alpha \leqslant \overline{\overline{Z}}$ *and the conditions* $z_1, z_2 \in Z$, $z_1 \neq z_2$ *imply that* $z_1 R z_2$.

Therefore (s_1) can be called the condition of Souslin and (k_1) — the condition of Knaster (cf. [2] and [4]). They have an application to the ordered sets theory (see [2]), where the relation R is defined in a family of sets and $x_1 R x_2$ means that $x_1 \cap x_2 \neq 0$.

It is obvious that each of these four conditions is stronger than the preceding one, i. e.

$$(k_\alpha) \to (k_\alpha^*) \to (s_\alpha^*) \to (s_\alpha)$$

for every α.

The implication

$$(s_0) \to (k_0)$$

has been proved by Sierpiński [3]. It follows that for $\alpha = 0$ all our four conditions are equivalent. It has also been proved by Sierpiński [3] that $(s_1) \to (k_1)$ is false. Thus we can write

$$(s_1) \not\to (k_1).$$

It has been proved by Dushnik and Miller [1] that

$$(s_0) \to (k_\alpha)$$

for every α, and by Knaster [2] [(1)] that

$$(k_1^*) \to (k_1) \quad \text{and} \quad (s_1) \to (s_1^*).$$

The last implications will be generalized in the present paper: we shall prove [(2)] that

$$(k_{\alpha+1}^*) \to (k_{\alpha+1}) \quad \text{and} \quad (s_{\alpha+1}) \to (s_{\alpha+1}^*)$$

for every α.

The following diagram contains all these results:

$$\begin{tikzcd}
(s_0) \arrow[r] & (k_\alpha) \arrow[r] & (k_\alpha^*) \arrow[r] \arrow[l, bend left, "\text{for non-limit } \alpha"] & (s_\alpha^*) \arrow[r] \arrow[l, bend left, "\text{for } \alpha = 1", "/" marking] & (s_\alpha) \arrow[l, bend left, "\text{for non-limit } \alpha"]
\end{tikzcd}$$

Proof that $(k_{\alpha+1}^*) \to (k_{\alpha+1})$. Let $Y \subset X$ and $\aleph_{\alpha+1} \leqslant \overline{\overline{Y}}$. It follows from $(k_{\alpha+1}^*)$ that there exists a $Z \subset Y$ such that

$$\aleph_{\alpha+1} \leqslant \overline{\overline{Z}} \tag{1}$$

and for every $z \in Z$ the set

$$P(z) = \{p : p \in Z \text{ and } z \, non R p\} \cup \{z\} \tag{2}$$

has a power $< \aleph_{\alpha+1} + 1 = \aleph_{\alpha+1}$, i. e.

$$\overline{\overline{P(z)}} \leqslant \aleph_\alpha \quad \text{for} \quad z \in Z. \tag{3}$$

We shall define for every ordinal number $\beta < \omega_{\alpha+1}$ an element z^β such that

$$z_\beta \in Z - \bigcup_{\iota<\beta} P(z_\iota) \quad \text{for} \quad \beta < \omega_{\alpha+1}. \tag{4}$$

Indeed, let z_1 be an arbitrary element of the set Z. Having defined the elements z_ι for $\iota < \beta$ we shall define z_β. For this purpose it is enough

(1) In [2] the relation R of special kind is considered, but it is not essential (cf. [4]).

(2) In the proofs the axiom of choice will be used.

to show that the set in formula (4) is not empty. Suppose it is empty, i. e. $Z = \bigcup_{\iota<\beta} P(z_\iota)$. We obtain

$$\overline{\overline{Z}} = \overline{\overline{\bigcup_{\iota<\beta} [P(z_\iota) - \bigcup_{\varkappa<\iota} P(z_\varkappa)]}} = \sum_{\iota<\beta} \overline{\overline{P(z_\iota) - \bigcup_{\varkappa<\iota} P(z_\varkappa)}} \leqslant \sum_{\iota<\beta} \overline{\overline{P(z_\iota)}}.$$

It follows by virtue of (3) that $\overline{\overline{Z}} \leqslant \aleph_\alpha \cdot \overline{\beta}$. But we have $\beta < \omega_{\alpha+1}$, which implies $\overline{\beta} < \overline{\omega}_{\alpha+1} = \aleph_{\alpha+1}$, i. e. $\overline{\beta} \leqslant \aleph_\alpha$. Thus we obtain $\overline{\overline{Z}} \leqslant \aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$ contrary to (1).

Therefore the points z_β such that (4) holds are defined. Let Z' be the set of all those points (where $\beta < \omega_{\alpha+1}$). It follows from (2) and (4) that $z_\beta \neq z_{\beta'}$ for $\beta \neq \beta'$. Hence $\overline{\overline{Z'}} = \overline{\omega}_{\alpha+1} = \aleph_{\alpha+1}$ and $Z' \subset Z \subset Y$ by virtue of (4).

Now let $z_1, z_2 \in Z'$ and $z_1 \neq z_2$. Thus we have $z_1 = z_{\beta_1}$, $z_2 = z_{\beta_2}$ and $\beta_1 \neq \beta_2$. We can assume that $\beta_1 < \beta_2$. Condition (4) implies that z_{β_2} $non \in P(z_{\beta_1})$, which by virtue of (2) means that $z_{\beta_1} R z_{\beta_2}$, i. e. $z_1 R z_2$. Therefore $(k_{\alpha+1})$ is proved.

Proof that $(s_{\alpha+1}) \to (s_{\alpha+1}^*)$. Suppose $(s_{\alpha+1}^*)$ is not true. It means that there exists a $Y \subset X$ such that

$$\aleph_{\alpha+1} \leqslant \overline{\overline{Y}} \tag{5}$$

and for every $Z \subset Y$ satisfying the inequality $\aleph_{\alpha+1} \leqslant \overline{\overline{Z}}$ an element $q(Z) \in Z$ exists such that the set

$$Q(Z) = \{p : p \in Z \text{ and } q(Z) R p\} \cup \{q(Z)\} \tag{6}$$

has a power $< \aleph_{\alpha+1} + 1 = \aleph_{\alpha+1}$, i. e.

$$\overline{\overline{Q(Z)}} \leqslant \aleph_\alpha. \tag{7}$$

Let $Z_1 = Y$. Thus $\aleph_{\alpha+1} \leqslant \overline{\overline{Z}}_1$ according to (5).

Now let $1 < \beta < \omega_{\alpha+1}$ and for every $\iota < \beta$ let the set Z_ι be defined os that $Z_\iota \subset Y$, $\aleph_{\alpha+1} \leqslant \overline{\overline{Z}}_\iota$ and the sets $Q(Z_\iota)$ are disjoint. We put

$$Z_\beta = Y - \bigcup_{\iota<\beta} Q(Z_\iota). \tag{8}$$

We shall prove that $\aleph_{\alpha+1} \leqslant \overline{\overline{Z}}_\beta$. Indeed, suppose the contrary, i. e. that

$$) \qquad \overline{\overline{Z}}_\beta \leqslant \aleph_\alpha.$$

From (8) we obtain $Y = Z_\beta \cup \bigcup_{\iota<\beta} Q(Z_\iota)$, where the sets in the sums on the right are disjoint according to (8) and to the hypothesis concerning the sets $Q(Z_\iota)$ for $\iota<\beta$. Thus

$$\overline{\overline{Y}} = \overline{\overline{Z}}_\beta + \sum_{\iota<\beta} \overline{\overline{Q(Z_\iota)}} \leqslant \aleph_\alpha + \aleph_\alpha \cdot \bar{\beta}$$

by virtue of (7) and (9). But $\beta < \omega_{\alpha+1}$ implies $\bar{\beta} \leqslant \aleph_\alpha$. Hence $\overline{\overline{Y}} \leqslant \aleph_\alpha + \aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$, contrary to (5).

From (6) we have $Q(Z_\beta) \subset Z_\beta$. That means according to (8) that the sets $Q(Z_\beta)$ and $Q(Z_\iota)$ are disjoint for every $\iota < \beta$.

Therefore the sets Z_β (where $\beta < \omega_{\alpha+1}$) are defined so that $Q(Z_\beta)$ are disjoint and $\aleph_{\alpha+1} \leqslant \overline{\overline{Z}}_\beta$ for $\beta < \omega_{\alpha+1}$. Put $z_\beta = q(Z_\beta)$ for $\beta < \omega_{\alpha+1}$.

Now let $\beta < \beta'$. Since $z_\beta \in Q(Z_\beta)$ and $z_{\beta'} \in Q(Z_{\beta'})$ by virtue of (6), we have $z_\beta \neq z_{\beta'}$ and $z_{\beta'}\, non \in Q(Z_\beta)$, that is $q(Z_\beta)\, non R z_{\beta'}$, i. e. $z_\beta\, non R z_{\beta'}$. Hence the set of points z_β (where $\beta < \omega_{\alpha+1}$) has a power $\aleph_{\alpha+1}$, is contained in Y, thus also in X, and $z_\beta\, non R z_{\beta'}$ for every two of its distinct elements z_β and $z_{\beta'}$. This contradicts $(s_{\alpha+1})$.

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CARTESIAN PRODUCTS AND CONTINUOUS IMAGES

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Studying the question whether the Cartesian product $A \times B$ of continua A and B is a continuous image of A provided that B is a continuous image of A, Siekluckі and Engelking have proved that the answer can be negative already for $A = B$, i. e. for topological squares (¹). Their examples are the following:

(i) $A = B = \mathcal{S}$, where $\mathcal{S}$ is the *condensed sinusoid*, i.e. the sum of the curve $\{(x, y): y = \sin 1/x,\ 0 < x \leqslant 1\}$ and the straight segment with end points $(0, -1)$ and $(0, 1)$,

(ii) $A = B = \mathcal{B}$, where $\mathcal{B}$ is the *countable brush*, i.e. the sum of the infinite sequence of straight segments with end points $(0, 1)$ and $(1/i, 0)$ for $i = 1, 2, \ldots$ and the straight segment with end points $(0, 1)$ and $(0, 0)$.

The purpose of this paper is to prove a more general theorem (²), which comprises the cases (i) and (ii) (see especially the corollary).

Let X, Y and Z be arbitrary compact spaces and let p be the projection of Cartesian product $X \times Y$ onto X. We denote by $L(X)$ the set of points of X at which the space X is locally connected and we put $N(X) = X - L(X)$.

(1) *If $f(X) = Y$ is a continuous mapping, $y \in Y$ and $f^{-1}(y) \subset \mathrm{Int}(V)$, then $y \in \mathrm{Int}(f(V))$.*

Proof. Suppose that $\lim y_n = y$ and $y_n \in Y - f(V)$. Then we have $f^{-1}(y_n) \subset X - f^{-1}f(V) \subset X - V$. Applying the compactness of X, let $x_n \in f^{-1}(y_n)$ and $\lim x_n = x'$. Then $f(x_n) = y_n$ and $x_n \in X - V$, that is $x' \in \overline{X - V} = X - \mathrm{Int}(V)$. Thus $x' \in X - f^{-1}(y)$ and hence $f(x') \neq y$, which contradicts the continuity of f.

(¹) See **P 290**, Colloquium Mathematicum 7 (1960), p. 110, and **P 290, R 1**, ibidem, p. 309.

(²) It is a result of a correspondence and discussion at the meeting on 16 December 1959 of the Wrocław Topological Seminar conducted by Professor B. Knaster.