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REMARKS ON TOTALISATION OF SERIES

BY

K. KRZYŻEWSKI (WARSAW)

This paper contains some remarks concerning totalisation of series introduced by A. Denjoy in [1] and [2]. I give the definition of totalisation of series directly based on totalisation of functions (the Denjoy integrals). Next (Theorem 1 and Theorem 2) I give the necessary and sufficient condition of totalisation of series which is an analogue to the descriptive definition of the Denjoy integrals. A similar condition was given by A. Denjoy in [1] and [2]. I shall use the notation and terminology of [3].

Let W be a denumerable subset of real numbers and $f(w) = a_w$ be a real function defined on W. We shall say that $\{a_w\}$ *is a sequence defined on the set of indices* W. Further, let $\Pi = \{P_w\}$, $w \in W$, be a family of closed disjoint intervals ordered by the relation:

$$(P_{w_1} < P_{w_2}) \equiv \Big(\prod_{x_1 \in P_{w_1}} \prod_{x_2 \in P_{w_2}} (x_1 < x_2)\Big).$$

We shall write $\Pi \sim W$ if for each pair w_1 and w_2 belonging to W the condition $w_1 < w_2$ is equivalent to $P_{w_1} < P_{w_2}$.

Let us now define a function $f_{W,\Pi}$ on the set of all real numbers R as follows:

$$f_{W,\Pi}(x) = \begin{cases} \dfrac{a_w}{|P_w|} & \text{for} \quad x \in P_w, \\ 0 & \text{for} \quad x \notin P_w. \end{cases} \tag{1}$$

A series $\sum\limits_{w \in W} a_w$ will be termed *D-convergent* (*D_*-convergent*) if there exists a family $\Pi \sim W$ such that the respective function $f_{W,\Pi}$ given by (1) is D-integrable (D_*-integrable) on R. We shall prove (Theorem 1) that if a series $\sum\limits_{w \in W} a_w$ is D-convergent (D_*-convergent), then $f_{W,\Pi}$ is D-integrable (D_*-integrable) on R for each family $\Pi \sim W$ and the definite D-integral (the definite D_*-integral) of $f_{W,\Pi}$ over R is independent on the family H.

The number $(D)\int_R f_{W,\Pi}(x)\,dx$ (the number $(D_*)\int_R f_{W,\Pi}(x)\,dx$) will be termed the *sum of the series* $\sum_{w\in W} a_w$ and it will be written

$$(D)\sum_{w\in W} a_w = (D)\int_R f_{W,\Pi}(x)\,dx \qquad \Big((D_*)\sum_{w\in W} a_w = (D_*)\int_R f_{W,\Pi}(x)\,dx\Big).$$

First we shall prove some lemmas.

LEMMA 1. *If a function F is VBG (VBG_*) on a set E (on an interval (a,b)) and φ is a non-decreasing function on a set E_1 (on an interval (a_1,b_1)) such that $\varphi[E_1]\subset E$ ($\varphi[(a_1,b_1)]\subset(a,b)$), then the function $F(\varphi)$ is VBG (VBG_*) on the set E_1 (on the interval (a_1,b_1)).*

The proof of this lemma is obvious.

LEMMA 2. *Let F be a function ACG on a measurable set E and let $E_1\subset E$. The conditions:*

(a) $F'_{ap}(x)=0$ *almost everywhere on* E_1,

(b) $|F[E_1]|=0$,

are equivalent.

Proof. Since the set E is expressed as the sum of a sequence of bounded and closed sets B_n and a set of measure zero such that F is AC on each B_n, we may clearly assume that the set E is closed and bounded and F is AC on it. In this case, let a and b denote the bounds of the set E and F_1 be the function which coincides with F on the set E and is linear in the intervals contiguous to E. The function F_1 is evidently AC on the interval $[a,b]$. Since F and F_1 coincide on E, we have $F'_{ap}(x)=F'_1(x)$ at almost all points of this set. Let us now assume that the condition (a) is satisfied. It is easy to see that then $F'_1(x)=0$ at each point x of $G\subset E_1$ such that $|E_1-G|=0$. Therefore on account of a well-known theorem we have $|F_1[G]|=0$. Since F fulfils the condition (N), we easily obtain (b). Conversely, let the condition (b) be satisfied. In view of the definition of F_1, we have at once $|F_1[E_1]|=0$. Now, by [4], Theorem 38.2, p. 213, we obtain $F'_1(x)=0$ almost everywhere on E_1 and therefore $F'_{ap}(x)=0$ almost everywhere on E_1. This completes the proof.

THEOREM 1. *Let $\{P^i_w\}=\Pi_i\sim W$ for $i=1,2$. Then D-integration (D_*-integration) of the function f_{W,Π_1} on R is equivalent to D-integration (D_*-integration) of f_{W,Π_2} on R and the respective integrals are equal.*

Proof. Let f_{W,Π_1} be D-integrable (D_*-integrable) on R and F_1 denote an indefinite D-integral (D_*-integral) of f_{W,Π_1}. Further, let $M(\Pi_i)$ and $m(\Pi_i)$ denote, for $i=1,2$, the supremum and infimum of the set $\sum_{w\in W} P^i_w$. There exists a non-decreasing function φ on the interval $(m(\Pi_2), M(\Pi_2))$ which satisfies the following conditions:

(i) φ is linear in P^2_w and $\varphi[P^2_w]=P^1_w$ for each $w\in W$,

(ii) $\lim\limits_{t\to m(\Pi_2)+0}\varphi(t)=m(\Pi_1)$ and $\lim\limits_{t\to M(\Pi_2)-0}\varphi(t)=M(\Pi_1)$.

We shall prove that $F_2=F_1(\varphi)$ is indefinite D-integral (D_*-integral) of f_{W,Π_2} on the interval $\big(m(\Pi_2), M(\Pi_2)\big)$.

First we shall prove that F_2 is continuous. To show this it is enough to prove that F_2 is continuous at t_0 such that $\varphi(t_0-0)<\varphi(t_0+0)$. But then the interval $\big(\varphi(t_0-0), \varphi(t_0+0)\big)$ and the set $\sum_{w\in W} P^1_w$ are disjoint, therefore F_1 is constant on the interval $[\varphi(t_0-0),\varphi(t_0+0)]$ and so the continuity of F_2 at t_0 is obvious. We shall now show that F_2 fulfils the condition (N). It is easy to see that it is enough to prove that F_2 fulfils the condition (N) on $A_2=\big(m(\Pi_2), M(\Pi_2)\big)-\sum_{w\in W} P^2_w$. On account of Lemma 2, we have $|F_1[A_1]|=0$ since $F'_{1ap}(x)=0$ almost everywhere on $A_1=\big(m(\Pi_1), M(\Pi_1)\big)-\sum_{w\in W}\operatorname{int}(P^1_w)$. We also have $\varphi[A_2]\subset A_1$, and therefore it follows that $|F_2[A_2]|=0$. This completes the proof that F_2 fulfils the condition (N) on $\big(m(\Pi_2), M(\Pi_2)\big)$. In view of our Lemma 1 and Theorem (6.8) of [3], p. 228 (Theorem (8.8) of [3], p. 233), it easily follows that F_2 is ACG (ACG_*) on $\big(m(\Pi_2), M(\Pi_2)\big)$.

We shall now prove that

(iii) $F'_{2ap}(t)=f_{W,\Pi_2}(t)$ almost everywhere on $\big(m(\Pi_2), M(\Pi_2)\big)$.

Since $F'_1(x)=f_{W,\Pi_1}(x)$ at each point x of $\sum_{w\in W}\operatorname{int}(P^1_w)$ and in view of (i), we see that (iii) is valid almost everywhere on $\sum_{w\in W} P^2_w$. On account of Lemma 2, since $|F_2[A_2]|=0$ and since F_2 is ACG on A_2, we obtain that (iii) is also valid almost everywhere on A_2. In the case of D_*-integral, since F_2 is almost everywhere derivable in the ordinary sense (as the function ACG_*), we may replace F'_{2ap} by F'_2 in (iii). Further, in view of (ii), we have

$$\lim_{t\to m(\Pi_2)+0} F_2(t)=\lim_{x\to m(\Pi_1)+0} F_1(x) \quad\text{and}\quad \lim_{t\to M(\Pi_2)-0} F_2(t)=\lim_{x\to M(\Pi_1)-0} F_1(x).$$

Therefore f_{W,Π_2} is D-integrable (D_*-integrable) on $\big(m(\Pi_2), M(\Pi_2)\big)$ and

$$(D)\int_{m(\Pi_2)}^{M(\Pi_2)} f_{W,\Pi_2}(t)\,dt=(D)\int_{m(\Pi_1)}^{M(\Pi_1)} f_{W,\Pi_1}(x)\,dx$$

$$\Big((D_*)\int_{m(\Pi_2)}^{M(\Pi_2)} f_{W,\Pi_2}(t)\,dt=(D_*)\int_{m(\Pi_1)}^{M(\Pi_1)} f_{W,\Pi_1}(x)\,dx\Big).$$

It is easy to see that this completes the proof of Theorem 1.

It is evident that if a series $\sum_{w\in W} a_w$ is D_*-convergent, then it is also D-convergent but as like as for D_*-integral and D-integral the converse is false. To show this it is enough to give an example of a function which is D-integrable on R but is not D_*-integrable such that it is constant on each I_n of a sequence $\{I_n\}$ of closed disjoint intervals and vanishes beyond the $\sum_{n=1}^{\infty} I_n$.

Example. Let $\{I_n\}$ be the sequence of intervals contiguous to Cantor's set C. There exists a one-to-one correspondence between the intervals I_n and the rational numbers of open interval $(0,1)$ so that, denoting by $r(I_n)$ the number which corresponds to the interval I_n, the relation $I_n < I_k$ is equivalent to $r(I_n) < r(I_k)$. Let us now put $a_n = 1/q_n$, where $r(I_n) = p_n/q_n$ and p_n and q_n are relatively prime natural numbers. It is easy to see that (a) $\lim\limits_{n\to\infty} a_n = 0$. We shall show that (b) for every portion P of C, $\sum_{k=1}^{\infty} a_{n_k} = +\infty$, where $\{I_{n_k}\}$ is the sequence of all different intervals contiguous to $\overline{P}$. For this purpose, it is enough to prove the following proposition: if p and q are arbitrary natural numbers and $\{r(I_{n_k})\}$ is the sequence of all different rational numbers of an open interval the ends of which are equal to $r(I_p)$ and $r(I_q)$, then $\sum_{k=1}^{\infty} a_{n_k} = +\infty$. To see the last observe that $\sum_{n=1}^{\infty} 1/p_n = +\infty$, where $\{p_n\}$ is the sequence of all different prime numbers. Let us now define a function f as follows:

$$f(x) = \begin{cases} \dfrac{a_n}{|I_n'|} & \text{for} \quad x \in I_n', \\ 0 & \text{for} \quad x \in R - \sum\limits_{n=1}^{\infty} (I_n' + I_n''), \\ -\dfrac{a_n}{|I_n''|} & \text{for} \quad x \in I_n'', \end{cases}$$

where $\{I_n'\}$ and $\{I_n''\}$ are arbitrary sequences of closed disjoint intervals such that $I_n' < I_n''$ and $I_n' + I_n'' \subset I_n$ for each n. The function f is evidently D-integrable on I_n and $(D)\int\limits_{I_n} f(x)\,dx = 0$ as well as $O(D; f; I_n) = a_n$. Now it suffices to apply Theorem (5.1) of [3], p. 257, to see that f is D-integrable on $[0,1]$ and therefore also on R. But f is not D_*-integrable on R, since it follows at once from (b) and from the second part of Theorem (1.4) of [3], p. 244, applied to the closed set C that f is not D_*-integrable on $[0,1]$.

THEOREM 2. *If a series $\sum\limits_{w\in W} a_w$ is D-convergent (D_*-convergent), then there exists a real function S on R which satisfies the following conditions:*

(1) $\lim\limits_{t\to-\infty} S(t)$ *and* $\lim\limits_{t\to+\infty} S(t)$ *exist and are finite,*

(2) *S is continuous at each point $t \notin W$ and continuous on the left at each point $w \in W$, moreover $S(w+0) - S(w) = a_w$ at each point $w \in W$,*

(3) *S is VBG (VBG$_*$) on R and $|S[R]| = 0$,*

(4) $\lim\limits_{t\to+\infty} S(t) - \lim\limits_{t\to-\infty} S(t) = (D)\sum\limits_{w\in W} a_w$ $\left(\lim\limits_{t\to+\infty} S(t) - \lim\limits_{t\to-\infty} S(t) = (D_*)\sum\limits_{w\in W} a_w\right)$.

Proof. It is easy to see that it is enough to prove our theorem in the special case in which the set of indices W is everywhere dense in the set of all real numbers. We shall therefore assume that the set W has this property. On account of our supposition, we have $\Pi \sim W$, where $\Pi = \{[c_w, d_w]\}$, $w \in W$, is the family of all intervals contiguous to Cantor's set C. Let us now define a function φ as follows:

$$\varphi(t) = \sup_{w \leqslant t} c_w.$$

The function φ satisfies the following conditions:

(a) φ is increasing and $\varphi[R] = (0,1) - \sum\limits_{w\in W} (c_w, d_w]$,

(b) $\lim\limits_{t\to-\infty} \varphi(t) = 0$, $\lim\limits_{t\to+\infty} \varphi(t) = 1$,

(c) φ is continuous at each point $t \notin W$ and $\varphi(w+0) = d_w$, $\varphi(w-0) = \varphi(w) = c_w$ for each $w \in W$.

Let F be an indefinite D-integral (D_*-integral) of $f_{W,\Pi}$ on R. We shall show that the function $S = F(\varphi)$ is the required one. In fact, it is easy to see that this function satisfies the conditions (1), (2), (4). On account of Lemma 1, S also is VBG (VBG$_*$) on R. Since evidently $|F[C]| = 0$ and $\varphi[R] \subset C$, it follows that S also satisfies the condition (3), and this completes the proof.

We shall now prove the converse of the preceding theorem:

THEOREM 3. *If for a sequence $\{a_w\}$ there exists a function S which satisfies the conditions (1), (2), (3) of the preceding theorem, then S is uniquely determined up to an additive constant and the series $\sum\limits_{w\in W} a_w$ is D-convergent (D_*-convergent) as well as the condition (4) is satisfied.*

Proof. We may make the assumption concerning the set of indices W similar as in the proof of Theorem 2. Further, let Π and φ mean the same as in the proof of Theorem 2 and φ^{-1} be the inverse function of φ. Let us now define a function F as follows:

$$F(x) = \begin{cases} \lim\limits_{t\to-\infty} S(t) & \text{for} \quad x \leqslant 0, \\ S(\varphi^{-1}(x)) & \text{for} \quad x \in (0,1) - \sum\limits_{w\in W} (c_w, d_w], \\ S(a_w + 0) & \text{for} \quad x = d_w, \text{ where } w \in W, \\ \text{linear in the interval } (c_w, d_w), \text{ where } w \in W, \\ \lim\limits_{t\to+\infty} S(t) & \text{for} \quad x \geqslant 1. \end{cases}$$

It is easy to see that F is continuous on R. Further, on account of Lemma 1, it is VBG on $(0,1)-\sum_{w\in W}(c_w, d_w]$ and therefore also VBG on R. In the case of S being VBG_*, it is necessary to use an additional argument to show that F is VBG_* on $(0,1)-\sum_{w\in W}[c_w, d_w]$, since F is evidently VBG_* on $(-\infty, 0]+\sum_{w\in W}[c_w, d_w]+[1, +\infty)$. But this follows at once from the definition of function VBG_*, in view of $O(F;[a,b]) = O\big(S;[\varphi^{-1}(a), \varphi^{-1}(b)]\big)$, where a and b belong to $(0,1)-\sum_{w\in W}[c_w, d_w]$ and $a<b$. Further, F fulfils the condition (N) on R. This follows from the second part of the condition (3). Now, on account of Theorem (6.8) of [3], p. 228 ([3], Theorem (8.8), p. 233), we easily deduce that F is ACG (ACG_*) on R. In view of the definition of F, we have $F'(x) = f_{W,H}(x)$ almost everywhere on R. In this way, since $\lim_{x\to-\infty} F(x)$ and $\lim_{x\to+\infty} F(x)$ exist and are finite, we have shown that $f_{W,H}$ is D-integrable (D_*-integrable) on R. Further, let us observe that the condition (4) is also satisfied. If S_1 and S_2 satisfy conditions (1), (2), (3), then the respective functions F_1 and F_2 are ACG on R and have the derivatives equal almost everywhere on R. Therefore, on account of Theorem (6.2) of [3], p. 225, it easily follows that F_1 and F_2 differ by a constant. The same clearly holds for S_1 and S_2. This completes the proof.

Remark. Let us observe that in Theorem 2 and in Theorem 3 the condition (3) may be replaced by the conditions:

(3′) *S is VBG (VBG_*) on R and fulfils the condition* (N),

(3″) $S'_{ap}(t) = 0$ ($S'(x) = 0$) *almost everywhere on* R.

This easily follows from Lemma 2.

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PROBLÈMES ET REMARQUES SUR LES CARRÉS DE CONVOLUTION

PAR

J.-P. KAHANE (MONTPELLIER)

Toutes les fonctions dont il s'agit dans la suite sont 2π-périodiques et sommables sur $[0, 2\pi]$. Le carré de convolution de f est

$$f*f(t) = \frac{1}{2\pi}\int_0^{2\pi} f(t-s)f(s)\,ds.$$

Ainsi, si $f \sim \sum_{-\infty}^{\infty} c_n e^{int}$, on a $f*f \sim \sum_{-\infty}^{\infty} c_n^2 e^{int}$.

S. Hartman a posé le problème suivant (un énoncé restreint a paru dans [1], voir aussi remarque [2] de C. Ryll-Nardzewski):

PROBLÈME 1. *Etant donné une classe de fonctions (par exemple L^p, C, $\mathrm{Lip}\,\alpha, \ldots$) déterminer s'il est vrai ou non que toute fonction de la classe soit le carré de convolution d'au moins une fonction sommable.*

Comme éléments de réponse, on a

1a) *C'est vrai pour* $\mathrm{Lip}\,\alpha$, $\alpha > 1/2$.

1b) *C'est faux pour* L^2.

En effet, 1a) est une conséquence immédiate d'un théorème de S. Bernstein selon lequel toute fonction de la classe $\mathrm{Lip}\,\alpha$, $\alpha > 1/2$, admet une série de Fourier absolument convergente ([4], p. 240). Et 1b) résulte d'un théorème de Banach sur les séries lacunaires ([4], p. 203): si $f \sim \sum_1^\infty a_k e^{in_k t}$ avec $n_{k+1}/n_k \geqslant 2$, on a $\sum_1^\infty |a_k|^2 < \infty$, c'est-à-dire que $f*f$ admet une série de Fourier absolument convergente; donc il est faux que toute $g \sim \sum_1^\infty b_k e^{in_k t}$ avec $\sum_1^\infty |b_k|^2 < \infty$ puisse s'écrire $f*f$.

On peut poser un problème un peu plus général.

PROBLÈME 2. *Etant donné deux classes de fonctions X et Y, déterminer s'il est vrai ou non que toute fonction de la classe X soit le carré de convolution d'au moins une fonction de la classe Y.*