Let

$$X = \sum_{n=1}^{\infty} \{x : |f(x) - f_n(x)| \geqslant 2^{-n}\}.$$

Evidently $X \in \boldsymbol{I}$. If $x \in \mathfrak{X} - X$ then $|f_n(x) - f_{n-1}(x)| < 3 \cdot 2^{-n}$, whence $x \in H_{n,1}$ and $g_{n,1}(x) = f_n(x)$.

By induction with respect to m, $g_{n,m}(x) = f_n(x)$ for $x \in \mathfrak{X} - X$ and for each n and $m = 1, 2, \ldots, n$. It follows for $x \in X'$ that

$$g(x) = \lim_{n\to\infty} g_{n,n}(x) = \lim_{n\to\infty} f_n(x) = f(x).$$

In consequence

$$g(x) = f(x) \quad \text{a. e.}$$

REFERENCES

[1] C. Kuratowski, *Topologie I*, quatrième édition, Warszawa 1958.

[2] G. Lederer, *A problem on Baire classes*, Fundamenta Mathematicae 48 (1960), p. 85-89.

[3] E. Marczewski and T. Traczyk, *On developable sets and almost-limit points*, Colloquium Mathematicum 8 (1960), p. 55-66.

Reçu par la Rédaction le 21. 12. 1959

COLLOQUIUM MATHEMATICUM

VOL. VIII 1961 FASC. 1

CONCERNING DISTANCES OF SETS AND DISTANCES OF FUNCTIONS

BY

S. GŁADYSZ, E. MARCZEWSKI AND C. RYLL-NARDZEWSKI (WROCŁAW)

Let $(X, \boldsymbol{M}, \mu)$ be a σ-finite σ-measure space. H. Steinhaus has introduced the following distance of measurable sets of finite measure:

$$(1) \qquad \sigma_\mu(A, B) = \begin{cases} \dfrac{\mu(A \dot{-} B)}{\mu(A+B)} & \text{if} \quad \mu(A+B) > 0, \\ 0 & \text{if} \quad \mu(A) = \mu(B) = 0, \end{cases}$$

where $A \dot{-} B$ denotes the symmetric difference of A and B, i. e. the set $(A+B) - AB$. The distance (1) is discussed in [1] and [2]; it has been applied by biologists.

In connection with a question raised recently by J. B. Faliński from the Botanical Institute of the Polish Academy of Sciences, J. Perkal observes that there exist finite sequences of sets arbitrarily near each other (in the sense of the distance σ_μ) and yet having an empty intersection. Namely, it suffices to consider the sequence $S_1, \ldots, S_n$ of all $(n-1)$-element subsets of a fixed n-element set X and to adopt the number of elements of $A \subset X$ as the measure $\mu_0(A)$. Then

$$\sigma_{\mu_0}(S_i, S_j) = 2/n \quad \text{for} \quad i \neq j \text{ and } S_1 \cdot \ldots \cdot S_n = 0.$$

We provide here the answer to a problem of J. Perkal by proving that, *for every σ-measure μ, if $\sigma_\mu(A_i, A_j) < 2/n$, then $A_1 \cdot \ldots \cdot A_n \neq 0$* (see 1.2(ii)). The preceding example of Perkal proves that our inequalities in section 1.2 may be considered as the strongest ones.

The second part of this paper contains analogous considerations concerning the distance of functions.

1. SETS

1.1. Definitions. For any sets $A_1, \ldots, A_n$ we put

$$D(A_1, \ldots, A_n) = (A_1 + \ldots + A_n) - A_1 \cdot \ldots \cdot A_n.$$

This is a generalization of the symmetric difference: $D(A_1, A_2) = A_1 \dot{-} A_2$.

Further, let us set for measurable sets of finite measure

$$\sigma_\mu(A_1, \ldots, A_n) = \begin{cases} \dfrac{\mu(D(A_1, \ldots, A_n))}{\mu(A_1 + \ldots + A_n)} & \text{if} \quad \mu(A_1 + \ldots + A_n) > 0, \\ 0 & \text{if} \quad \mu(A_1) = \ldots = \mu(A_n) = 0. \end{cases}$$

This is a generalization (for every finite number of sets) of the distance σ_μ of two sets), defined by formula (1).

Let us notice, incidentally, that $\sigma_\mu(A_1, \ldots, A_n) \leqslant \sigma_\mu(A_1, \ldots, A_n, A_{n+1})$.

1.2. Inequalities for sets. We will prove that

(i) *For any $A_1, \ldots, A_n \in M$ of finite measure we have*

$$\sigma_\mu(A_1, \ldots, A_n) \leqslant \frac{1}{n-1} \sum_{i<j} \sigma_\mu(A_i, A_j)$$

or, in other words, under the assumption $\mu(A_1 + \ldots + A_n) > 0$,

$$\frac{\mu((A_1 + \ldots + A_n) - A_1 \cdot \ldots \cdot A_n)}{\mu(A_1 + \ldots + A_n)} \leqslant \frac{1}{n-1} \sum_{i<j} \frac{\mu(A_i \dot{-} A_j)}{\mu(A_i + A_j)}.$$

Proof. It is easy to verify that if $x \in D(A_1, \ldots, A_n)$ then there exist at least $n-1$ pairs $i<j$ such that $x \in D(A_i, A_j)$. Hence we obtain the following inequality for characteristic functions:

$$\chi_{A_1 + \ldots + A_n}(x) - \chi_{A_1 \cdot \ldots \cdot A_n}(x) \leqslant \frac{1}{n-1} \sum_{i<j} \left(\chi_{A_i + A_j}(x) - \chi_{A_i \cdot A_j}(x)\right)$$

or, in other words,

$$\mu(D(A_1, \ldots, A_n)) \leqslant \frac{1}{n-1} \sum_{i<j} \mu(D(A_i, A_j)),$$

which implies (i).

The definition of σ_μ implies:

(ii) $\sigma_\mu(A_1, \ldots, A_n) < 1$ *if and only if either* $\mu(A_1 \cdot \ldots \cdot A_n) > 0$ *or* $\mu(A_1) = \ldots = \mu(A_n) = 0$.

By the aid of (ii) we obtain the following corollary of (i) concerning only the mutual distances of pairs of sets A_i, A_j:

(iii) *If* $\sum_{i<j} \sigma_\mu(A_i, A_j) < n-1$ *(e. g. if* $\sigma_\mu(A_i, A_j) < 2/n$ *for* $1 \leqslant i < j \leqslant n$*), then* $\mu(A_1 \cdot \ldots \cdot A_n) > 0$.

2. FUNCTIONS

2.1. Definitions. If f is a real function defined on X and μ-measurable, then C_f denotes (as in [1]) the set of points (x, y) lying between the graph of f and the X-axis:

$$C_f = \{(x, y) : x \in X, \text{ and } 0 \leqslant y \leqslant f(x) \text{ or } f(x) \leqslant y \leqslant 0\}.$$

If $a_1, \ldots, a_n$ are real numbers, then $\delta(a_1, \ldots, a_n)$ denotes the length of the smallest closed interval containing $a_1, \ldots, a_n$, or, in other words,

$$\delta(a_1, \ldots, a_n) = \max_j a_j - \min_j a_j. \tag{2}$$

Consequently,

$$\delta(a, b) = |a-b|, \qquad \delta(0, a, b) = \max(|a|, |b|, |a-b|). \tag{3}$$

For any real μ-integrable functions $f_1, \ldots, f_n$ we put

$$\sigma_\mu(f_1, \ldots, f_n) = \frac{\int \delta(f_1(x), \ldots, f_n(x))\, d\mu(x)}{\int \delta(0, f_1(x), \ldots, f_n(x))\, d\mu(x)},$$

where the integrals (here and in all the following formulas) are extended over the whole X. This definition obviously requires completion: if $f_1, \ldots, f_n$ vanish a. e. (almost everywhere), we put $\sigma_n(f_1, \ldots, f_n) = 0$.

It follows from (3) that $\sigma_\mu(f_1, \ldots, f_\mu)$ is a generalization of the distance σ_μ of two functions, defined in [1] (p. 325).

We will prove another formula for σ_μ for a special case:

(i) *If $f_1, \ldots, f_n$ are non-negative and μ-integrable, and do not all vanish a. e., then*

$$\sigma_\mu(f_1, \ldots, f_n) = 1 - \frac{\int \min_j f_j(x)\, d\mu(x)}{\int \max_j f_j(x)\, d\mu(x)}.$$

This is an easy consequence of the definition of $\sigma_\mu(f_1, \ldots, f_n)$, of (2) and of the following equality, valid for non-negative numbers a_j:

$$\delta(0, a_1, \ldots, a_n) = \max_j a_j.$$

Let us remark that

(ii) *For real functions $f_1, \ldots, f_n$ which are μ-integrable and do not all vanish a. e., $\sigma_\mu(f_1, \ldots, f_n) < 1$ if and only if the functions $f_1, \ldots, f_n$ are all positive or all negative on a set of positive measure.*

This easily follows from the following equivalence: $\delta(a_1, \ldots, a_n) < \delta(0, a_1, \ldots, a_n)$ if and only if the numbers $a_1, \ldots, a_n$ are all positive or all negative.

2.2. Functions and sets. We will prove some relations between σ_μ for sets and σ_μ for functions.

(i) $\sigma_\mu(A_1, \ldots, A_n) = \sigma_\mu(\chi_{A_1}, \ldots, \chi_{A_n})$, *where A_j are μ-measurable sets of finite measure.*

This is an easy consequence of the definition of σ_μ for sets, of 2.1 (i) and of the following equalities:

$$\mu(A_1 \cdot \ldots \cdot A_n) = \int \chi_{A_1 \cdot \ldots \cdot A_n} d\mu = \int \min_j \chi_{A_j}(x) d\mu(x),$$

$$\mu(A_1 + \ldots + A_n) = \int \chi_{A_1 + \ldots + A_n} d\mu = \int \max_j \chi_{A_j}(x) d\mu(x).$$

(ii) $\sigma_\mu(f_1, \ldots, f_n) = \sigma_\nu(C_{f_1}, \ldots, C_{f_n})$, *where f_j are μ-integrable real unctions, and ν denotes the direct product of μ and the Lebesgue measure $|\;|$.*

In order to prove this, it suffices to note that intersecting the sets $C_{f_1} + \ldots + C_{f_n}$ and $D(C_{f_1}, \ldots, C_{f_n})$ with the vertical line $x = x_0$ we obtain

$$|\{y : (x_0, y) \in C_{f_1} + \ldots + C_{f_n}\}| = \delta\big(0, f_1(x_0), \ldots, f_n(x_0)\big),$$

$$|\{y : (x_0, y) \in D(C_{f_1}, \ldots, C_{f_n})\}| = \delta\big(f_1(x_0), \ldots, f_n(x_0)\big),$$

which is easy to verify by considering first the case when the numbers $f_1(x_0), \ldots, f_n(x_0)$ have the same sign, and next the other case.

The preceding equalities give, by the theorem of Fubini,

$$\nu\big(D(C_{f_1}, \ldots, C_{f_n})\big) = \int |\{y : (x, y) \in D(C_{f_1}, \ldots, C_{f_n})\}| \, d\mu(x),$$

$$\nu(C_{f_1} + \ldots + C_{f_n}) = \int |\{y : (x, y) \in C_{f_1} + \ldots + C_{f_n}\}| \, d\mu(x),$$

which implies equality (ii).

2.3. Inequalities for functions. Propositions 1.2 (i) and 2.2 (ii) imply the following inequality for μ-integrable functions:

(i) $$\sigma_\mu(f_1, \ldots, f_n) \leqslant \frac{1}{n-1} \sum_{i<j} \sigma_\mu(f_i, f_j).$$

Hence, by 2.1 (ii),

(ii) *If $\sum_{i<j} \sigma_\mu(f_i, f_j) < n-1$ (e. g. if $\sigma_\mu(f_i, f_j) < 2/n$ for $1 \leqslant i \leqslant j \leqslant n$) and not all f_j vanish a. e., then the functions $f_1, \ldots, f_n$ are all positive or all negative on a set of positive measure.*

REFERENCES

[1] E. Marczewski and H. Steinhaus, *On a certain distance of sets and the corresponding distance of functions*, Colloquium Mathematicum 6 (1958), p. 321-327.

[2] E. Marczewski i H. Steinhaus, *O odległości systematycznej biotopów*, Zastosowania Matematyki 4 (1959), p. 195-203.

Reçu par la Rédaction le 8. 3. 1960