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DIFFERENTIABILITY OF JUMP FUNCTIONS

BY

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A number of the many proofs of the existence almost everywhere of the derivative of a nondecreasing function on a real interval are simpler when the nondecreasing function is continuous. Since every nondecreasing function can be written as the sum of a continuous nondecreasing function and a nondecreasing jump function, there is some advantage in having an elementary proof of the fact that a nondecreasing jump function has, almost everywhere, a derivative equal to zero. Here a *nondecreasing jump function* means a countable sum of nondecreasing step functions, each with a single jump; that is, it is of the form $\sum_{k=1}^{\infty} f_k(x)$, where $f_k(x) = 0$ for $x < a_k$ and $f_k(x) = b_k$ for $x > a_k$, and $0 \leqslant f_k(a_k) \leqslant b_k$. The proof given here is elementary in the sense that it makes no use of the general theorem on differentiating monotonic functions, or of integration, or of density. For more details about the derivatives of jump functions see [1].

It is enough to show that $f^+(x) = 0$ almost everywhere: since $f^+(x) \geqslant f_+(x) \geqslant 0$ this shows that f has a zero right-hand derivative almost everywhere; then by considering $-f(-x)$ we see that f has a zero left-hand derivative almost everywhere. It is then enough to show that any set E_a on which $f^+(x) > 2/a > 0$ is of measure zero, since the set where $f^+(x) > 0$ is the union of the sets E_n, $n = 1, 2, \ldots$

Let the domain of f be a compact interval J. Suppose that E_a is not of measure zero; it is measurable because f^+ is measurable (see [3], p. 113 or, for a simpler proof when (as here) f is monotonic, [2], p. 194). Thus E_a is of positive measure μ. The variation of f over any interval is the sum of its jumps in that interval. If $\varepsilon > 0$, we can find a finite set $\{I\}$ of closed intervals, of arbitrarily small total length η, such that the variation of f outside $\bigcup I$ is at most ε. For, if we arrange the jumps of f in order of decreasing magnitude we can cover the point a_k where the kth jump occurs by an interval of length $2^{-k}\eta$. If $\eta < \mu/2$, the part of E_a outside $\bigcup I$ will be of measure at least $\mu/2$, and so will contain a closed set F of measure at least $\mu/4$. Since F is closed and the intervals I are closed (and finite in number), F is at a positive distance from $\bigcup I$. Therefore to each x in F we can assign a positive δ_x such that $f(x+\delta_x)-f(x) > \delta_x/a$ and the interval $(x-\delta_x, x+\delta_x)$ is disjoint from $\bigcup I$. Since f is nondecreasing we then have

$$\frac{f(x+\delta_x)-f(x-\delta_x)}{\delta_x} \geqslant \frac{f(x+\delta_x)-f(x)}{\delta_x} > 1/a. \tag{1}$$

A finite collection $\{G_k\}$ of the intervals $(x-\delta_x, x+\delta_x)$ cover F. By reducing the number of G_k, if necessary, we can arrange that no point of F is in more than two G_k. Let $2\delta_k$ be the length of G_k. Then

$$\operatorname{meas} F \leqslant 4\sum \delta_k. \tag{2}$$

By (1), the variation V_n of f over G_n is at least δ_n/a, so from (2) we have

$$\operatorname{meas} F \leqslant 4a\sum V_n.$$

But all the G_n are outside $\bigcup I$, they overlap at most in pairs, and the variation of f outside $\bigcup I$ is at most ε. Thus meas $F \leqslant 8\varepsilon a$. But meas $F \geqslant \mu/4$, and we obtain a contradiction if $\varepsilon < \mu/(32a)$. Thus E_a must be of measure 0, and the proof is complete.

REFERENCES

[1] J. S. Lipiński, *Sur la dérivée d'une fonction de sauts*, Colloquium Mathematicum 4 (1957), p. 197-205.

[2] E. J. McShane, *Integration*, Princeton 1944.

[3] S. Saks, *Theory of the integral*, Warszawa-Lwów 1937.

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MESURE ET DÉRIVÉE

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1. Problèmes et résultats. Le rapport entre les fonctions dérivées et la mesure de Lebesgue est bien connu. En particulier, si l'on se borne aux dérivées sommables au sens de Lebesgue sur tout intervalle fini, on peut les définir de la manière suivante.

Soit $f(x)$ une fonction sommable au sens de Lebesgue sur tout intervalle fini. Appelons *valeur moyenne* de la fonction $f(x)$ au point x_0 la limite

$$\lim_{h\to 0} h^{-1}\int_{x_0}^{x_0+h} f(t)\,dt,$$

évidemment si elle existe, et désignons-la par $S(f, x_0)$. Lorsque $S(f, x)$ existe en tout point et que, de plus, $S(f, x) = f(x)$, la fonction $f(x)$ est *une dérivée*.

L'intégrale qui intervient dans cette définition est liée à la mesure de Lebesgue. On peut se poser la question suivante: quelles fonctions obtiendra-t-on si l'on remplace, dans cette définition, la mesure de Lebesgue par une autre mesure μ, non-atomique et telle que tout intervalle admette une mesure μ positive, l'intégrale de Lebesgue étant remplacée en même temps par une μ-intégrale?

Nous ne considérerons dans la suite que des fonctions à valeurs finies. Admettons que la fonction $f(x)$ finie est μ-intégrable sur tout intervalle fini. Désignons par $S_\mu(f, a, b)$ la valeur moyenne, de poids μ, de la fonction $f(x)$ dans l'intervalle ouvert (a, b), c'est-à-dire le nombre

$$[\mu(a, b)]^{-1}\int_a^b f(t)\,d\mu(t)\cdot\operatorname{sign}(b-a).$$

On peut donc avoir $b < a$. Désignons encore par $S_\mu(f, x_0)$ la valeur moyenne, de poids μ, de la fonction $f(x)$ au point x_0, c'est-à-dire la