References

[1] N. Bourbaki, *Topologie générale*, Act. sci. ind., Hermann et Cie., Paris.
[2] S. Eilenberg, *Ordered topological spaces*, Amer, J. Math. 63 (1941), pp. 39-45.

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On isomorphic free algebras

by

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1. Preliminaries and summary. Given a class $\mathfrak{A}$ of abstract algebras which have the same operations, let $\mathcal{A}_m$ denote the free $\mathfrak{A}$-algebra with m generators (cf. [1], p. viii). In this paper we consider the class of all free algebras $\mathcal{A}_m$ with finite m. Assuming that two of these algebras are isomorphic, i.e. $\mathcal{A}_k \simeq \mathcal{A}_l$, $k \neq l$, we investigate the distribution of pairs of isomorphic algebras in the sequence $A_1, A_2, \ldots$

THEOREM 1. *If k is the smallest integer such that $\mathcal{A}_k \simeq \mathcal{A}_m$ holds for some $m \neq k$, and $l = k+d$ is the smallest integer satisfying $\mathcal{A}_k \simeq \mathcal{A}_l$, $l \neq k$, then $\mathcal{A}_m \simeq \mathcal{A}_n$ holds for $m \neq n$ if and only if $m \equiv n \pmod d$, and $m, n \geqslant k$.*

Theorem 1 implies that, for any fixed free $\mathfrak{A}$-algebra $\mathcal{A}_m$, the indices j for which $\mathcal{A}_j$ is isomorphic to $\mathcal{A}_m$ form an arithmetic progression. Let in particular $\{\mathcal{A}\}$ be the class consisting of a single abstract algebra $\mathcal{A}$ and suppose that $\mathcal{A}$ has a finite basis (set of independent (¹) generators). Then the above consequence of Theorem 1 yields a theorem of E. Marczewski (cf. also [2], Theorem 5) which says that the finite ranks (cardinals of bases) of $\mathcal{A}$ form an arithmetic progression. The transition from our results to this theorem follows by observing that r is a rank of $\mathcal{A}$ if and only if $\mathcal{A}$ is isomorphic to the free $\{\mathcal{A}\}$-algebra $\mathcal{A}_r$.

THEOREM 2. *Given any integers $0 < k < l$, there exists a class of algebras $\mathfrak{A}_{(k,l)}$ satisfying the assumptions of Theorem 1, i.e. with the property that k and l are the smallest integers such that $\mathcal{A}_k \simeq \mathcal{A}_l$.*

For proving Theorem 2 we use the class $\mathfrak{A}_{(k,l)}$ of all algebras having the following $k+l$ operations

$$\varphi_i(x_1, \ldots, x_k), \quad \omega_j(x_1, \ldots, x_l), \quad i = 1, \ldots, l; \; j = 1, \ldots, k, \tag{1}$$

which satisfy the axioms (²)

$$\varphi_i(\omega_1(x_1, \ldots, x_l), \ldots, \omega_k(x_1, \ldots, x_l)) = x_i; \quad i = 1, \ldots, l, \tag{2}$$

$$\omega_j(\varphi_1(x_1, \ldots, x_k), \ldots, \varphi_l(x_1, \ldots, x_k)) = x_k; \quad j = 1, \ldots, k. \tag{3}$$

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(¹) In the sense of E. Marczewski's definition (see [5]; cf. also Def. 5 below).

(²) Axioms of this kind were considered in [4].

$\mathfrak{A}_{(k,l)}$ is a variety in the sense of P. Hall (cf. [3], p. 66, Def. 2). In the course of the proof of Theorem 2 we show (Lemma 5 below) that a free $\mathfrak{A}_{(k,l)}$-algebra $\mathcal{A}_m$ is isomorphic to a free $\mathfrak{A}_{(k,l)}$-algebra $\mathcal{A}_n$ if and only if n is a rank of $\mathcal{A}_m$. Hence, by Theorem 1, the arithmetic progression $k+qd$ $(q = 0, 1, \ldots)$ is the set of all ranks of the algebra $\mathcal{A} = \mathcal{A}_k$. This gives a positive answer to a question raised by E. Marczewski (cf. [2]) whether every arithmetic progression is the set of all ranks of a certain algebra $\mathcal{A}$ (this question was partially answered by A. Goetz and C. Ryll-Nardzewski in [2]).

2. Proof of Theorem 1. The following lemma and the corollaries are straightforward generalizations of results contained in [2] (Theorems 2 and 3).

LEMMA 1. *Two free $\mathfrak{A}$-algebras $\mathcal{A}_m$ and $\mathcal{A}_n$ are isomorphic if and only if there are algebraic operations*

$$(4)\qquad \psi_i(x_1, \ldots, x_n), \quad \theta_j(x_1, \ldots, x_m); \quad i = 1, \ldots, m; \quad j = 1, \ldots, n$$

such that the equalities

$$(5)\qquad \psi_i\big(\theta_1(x_1, \ldots, x_m), \ldots, \theta_n(x_1, \ldots, x_m)\big) = x_i; \qquad i = 1, \ldots, m,$$

$$(6)\qquad \theta_j\big(\psi_1(x_1, \ldots, x_n), \ldots, \psi_m(x_1, \ldots, x_n)\big) = x_j; \qquad j = 1, \ldots, n,$$

hold identically in every algebra $\mathcal{A} \in \mathfrak{A}$.

Proof of Lemma 1. Suppose that τ is an isomorphism of $\mathcal{A}_m$ onto $\mathcal{A}_n$. Denoting by $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$ the free generators of $\mathcal{A}_m$ and $\mathcal{A}_n$ respectively, we have that there are operations (4) for which

$$\tau a_i = \psi_i(b_1, \ldots, b_n), \qquad \tau^{-1} b_j = \theta_j(a_1, \ldots, a_m).$$

Thus

$$a_i = \tau^{-1}\tau a_i = \psi_i(\tau^{-1}b_1, \ldots, \tau^{-1}b_n) = \psi_i\big(\theta_1(a_1, \ldots, a_m), \ldots, \theta_n(a_1, \ldots, a_m)\big),$$

$$b_j = \tau\tau^{-1}b_j = \theta_j(\tau a_1, \ldots, \tau a_m) = \theta_j\big(\psi_1(b_1, \ldots, b_n), \ldots, \psi_m(b_1, \ldots, b_n)\big).$$

Since $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$ generate free $\mathfrak{A}$-algebras, we infer that (5) and (6) hold identically in every algebra $\mathcal{A} \in \mathfrak{A}$.

Now assume that there are operations (4) such that (5) and (6) hold identically in every $\mathcal{A} \in \mathfrak{A}$. Denoting, as before, the free generators of $\mathcal{A}_n$ by $b_1, \ldots, b_n$ consider the elements

$$c_i = \psi_i(b_1, \ldots, b_n); \qquad i = 1, \ldots, m.$$

We have, by (6) that c_i generate $\mathcal{A}_n$. The lemma will be proved if we show that they generate freely $\mathcal{A}_n$, i.e. that an arbitrary mapping λ_1 of $c_1, \ldots, c_m$ into an algebra $\mathcal{A} \in \mathfrak{A}$ can be extended to a homomorphism λ of $\mathcal{A}_n$ into $\mathcal{A}$. Suppose that $\lambda_1 c_i = d_i$, where d_i belong to $\mathcal{A}$. Consider the mapping λ defined on $b_1, \ldots, b_n$ by $\lambda b_j = \theta_j(d_1, \ldots, d_m)$. This mapping can certainly be extended to a homomorphism λ since $b_1, \ldots, b_n$ are freely generating $\mathcal{A}_n$. But then, by (5),

$$\lambda c_i = \psi_i(\lambda b_1, \ldots, \lambda b_n) = \psi_i\big(\theta_1(d_1, \ldots, d_m), \ldots, \theta_n(d_1, \ldots, d_m)\big) = d_i$$

which shows that λ is the required extension of λ_1. This completes the proof of Lemma 1.

COROLLARY 1. *If $\mathcal{A}_m \simeq \mathcal{A}_n$, then $\mathcal{A}_{m+r} \simeq \mathcal{A}_{n+r}$ holds for $r = 1, 2, \ldots$*

Proof. Assuming that $\mathcal{A}_m \simeq \mathcal{A}_n$ we have that there are operations (4) which satisfy (5) and (6). Denoting by e_j^q the identity operations $e_j^q(x_1, \ldots, x_q) = x_j$, $1 \leqslant j \leqslant q$, we define as follows the operations

$$\tilde{\psi}_i(x_1, \ldots, x_{n+r}), \quad \tilde{\theta}_j(x_1, \ldots, x_{m+r}); \quad i = 1, \ldots, m+r; \; j = 1, \ldots, n+r.$$

$$\tilde{\psi}_i = \begin{cases} e_i^{n+r}; & i = 1, \ldots, r, \\ \psi_{i-r}(e_{r+1}^{n+r}, \ldots, e_{r+n}^{n+r}); & i = r+1, \ldots, r+m; \end{cases}$$

$$\tilde{\theta}_j = \begin{cases} e_j^{m+r}; & j = 1, \ldots, r, \\ \theta_{j-r}(e_{r+1}^{m+r}, \ldots, e_{r+m}^{m+r}); & j = r+1, \ldots, r+n. \end{cases}$$

It is easily checked that $\tilde{\psi}_i$ and $\tilde{\theta}_j$ satisfy conditions analogous to (5) and (6) and hence $\mathcal{A}_{m+r} \simeq \mathcal{A}_{n+r}$ holds.

We note that our assumption $\mathcal{A}_k \simeq \mathcal{A}_{k+d}$ and Corollary 1 imply

COROLLARY 2. *$A_k \simeq A_{k+qd}$ holds for $q = 1, 2, \ldots$*

We are now in position to complete the proof of Theorem 1. We have, by the above Corollaries that the conditions $m \equiv n \pmod{d}$ and $m, n \geqslant k$ imply $\mathcal{A}_m \simeq \mathcal{A}_n$. We observe that $\mathcal{A}_m \simeq \mathcal{A}_n$ never holds when $k \leqslant m < n < k+d$. Indeed this would imply that $\mathcal{A}_{m+k+d-n} \simeq \mathcal{A}_{k+d} \simeq \mathcal{A}_k$, by Corollary 1 which contradicts the definition of k and d. Thus $\mathcal{A}_m \simeq \mathcal{A}_n$ implies $m \equiv n \pmod{d}$. This completes the proof of Theorem 1.

3. The free $\mathfrak{A}_{(k,l)}$-algebras. We denote by $\mathfrak{A}_{(k,l)}$ the class defined in section 1. The main result we wish to prove in this section is Lemma 3.

Given a set S and an integer n, the elements of the Cartesian product S^n will be called *n-sequences* in S. Let A_m be the set of elements of the free $\mathfrak{A}_{(k,l)}$-algebra $\mathcal{A}_m$. For any sequences $\sigma \in (A_m)^k$, $\varrho \in (A_m)^l$, we denote

$$(7)\qquad \varphi(\sigma) = \big(\varphi_1(\sigma), \ldots, \varphi_l(\sigma)\big), \qquad \omega(\varrho) = \big(\omega_1(\varrho), \ldots, \omega_k(\varrho)\big).$$

We denote by $a_1, \ldots, a_m$ the free generators of $\mathcal{A}_m$. We prove first

LEMMA 2. *There exists a decomposition $A_m = \bigcup_{i=0}^{\infty} {}^i\Delta$ in pairwise disjoint sets ${}^i\Delta$, where ${}^0\Delta = \{a_1, \ldots, a_m\}$ and each of the sets ${}^i\Delta$, $(i \geqslant 1)$, is a disjoint union of certain subsets having k elements or l elements each and these subsets can be ordered to form k-sequences and l-sequences in such a way that if we call these sequences normal, then, for $n = 0, 1, \ldots$,*

(a) *if ε is a not normal k-sequence, (l-sequence), in $\bigcup_{i=0}^{n} {}^{i}\Delta$ which has at least one term in ${}^{n}\Delta$, then $\varphi(\varepsilon)$, ($\omega(\varepsilon)$), is a normal sequence in ${}^{n+1}\Delta$,*

(b) *if π is a normal k-sequence, (l-sequence), in ${}^{n+1}\Delta$, then $\varphi(\pi)$, ($\omega(\pi)$), is a not normal sequence in $\bigcup_{i=0}^{n} {}^{i}\Delta$ which has at least one term in ${}^{n}\Delta$.*

Proof. We shall first construct a certain algebra $\mathscr{A}_{(m)} \in \mathfrak{A}_{(k,l)}$ which has the properties (a) and (b). Then we shall prove that $\mathscr{A}_{(m)} = \mathscr{A}_m$. We denote ${}^{0}\Delta = \{a_1, \ldots, a_m\}$. To construct $\mathscr{A}_{(m)}$ we need an infinite sequence of disjoint sets ${}^{1}\Delta, {}^{2}\Delta, \ldots$ with the following properties (we take the notation ${}^{n}A = \bigcup_{i=0}^{n} {}^{i}\Delta$)

(i) for each $n \geqslant 1$, ${}^{n}\Delta$ is a disjoint union of certain subsets having k elements or l elements each; these subsets are ordered and called *normal k-sequences* or *normal l-sequences* respectively;

(ii) for each $n \geqslant 0$, there is a 1:1 mapping Φ_n of the class of all these k-sequences in ${}^{n}A$ which are not normal and have at least one term in ${}^{n}\Delta$ onto the class of all normal l-sequences in ${}^{n+1}\Delta$;

(iii) for each $n \geqslant 0$, there is a 1:1 mapping Ω_n of the class of all not normal l-sequences in ${}^{n}A$ which have at least one term in ${}^{n}\Delta$ onto the class of all normal k-sequences in ${}^{n+1}\Delta$.

We construct the sets ${}^{i}\Delta$ by induction. The set ${}^{0}\Delta$ was already defined and there are, by definition, no normal sequences in ${}^{0}\Delta$. Assuming that we know the number of normal sequences in each of the sets ${}^{0}\Delta, \ldots, {}^{r}\Delta$, the conditions (i)-(iii) determine uniquely the cardinal of ${}^{r+1}\Delta$ (this cardinal is $kL_r + lK_r$ where L_r (K_r) is the number of all not normal l-sequences (k-sequences) in ${}^{r}A$ which have at least one term in ${}^{r}\Delta$). If ${}^{r+1}\Delta$ is any set of this required cardinal and disjoint to the sets ${}^{0}\Delta, \ldots, {}^{r}\Delta$, then there exists a decomposition of ${}^{r+1}\Delta$ into disjoint sets having k elements or l elements each and such that if we order these sets arbitrarily and call them normal k-sequences and normal l-sequences, then there are mappings Φ_r and Ω_r with the properties (ii) and (iii). Thus the sequence ${}^{0}\Delta, {}^{1}\Delta, \ldots$ can be constructed by induction.

We denote $A = \bigcup_{i=0}^{\infty} {}^{i}\Delta$. We consider the mappings $\Phi = \bigcup_{i=0}^{\infty} \Phi_i$, $\Omega = \bigcup_{i=0}^{\infty} \Omega_i$ (regarding mappings as sets of pairs, we have their union naturally defined). Let N_k (N_l) denote the class of all normal k-sequences (l-sequences) in A. It follows, by (ii) and (iii) that Φ and Ω are 1:1 mappings defined on $A^k - N_k$ and $A^l - N_l$ respectively and

$$\Phi(A^k - N_k) = N_l, \quad \Omega(A^l - N_l) = N_k. \tag{8}$$

We define now on A the operations (1) by defining the functions (7) on A^k and A^l respectively. We put

$$\varphi(\sigma) = \begin{cases} \Phi(\sigma) & \text{if} \quad \sigma \in A^k - N_k, \\ \Omega^{-1}(\sigma) & \text{if} \quad \sigma \in N_k, \end{cases}$$
$$\omega(\varrho) = \begin{cases} \Omega(\varrho) & \text{if} \quad \varrho \in A^l - N_l, \\ \Phi^{-1}(\varrho) & \text{if} \quad \varrho \in N_l. \end{cases} \tag{9}$$

It follows from (9), by (8), that $\varphi = \omega^{-1}$ and thus the equations (2) and (3) hold identically on A. Hence, denoting by $\mathscr{A}_{(m)}$ the algebra with the set of elements A and with the operations (1), we have $\mathscr{A}_{(m)} \in \mathfrak{A}_{(k,l)}$. Using (8), (9) and (i), (ii) and (iii) we see that $\mathscr{A}_{(m)}$ has the properties asserted in Lemma 2. To complete the proof of this lemma we show that $\mathscr{A}_{(m)}$ is the free algebra with the generators $a_1, \ldots, a_m$, i.e. that

(*) *every mapping λ_1 of $a_1, \ldots, a_m$ onto the generators of an algebra $\mathscr{A}_0 \in \mathfrak{A}_{(k,l)}$ can be extended to a homomorphism of $\mathscr{A}_{(m)}$ onto $\mathscr{A}_0$.*

To prove (*) we take the following notation

Definition 1. If ${}^{0}\Delta_0$ is a set of generators of $\mathscr{A}_0$, we denote by ${}^{0}\Delta_0 = {}^{0}A_0 \subset {}^{1}A_0 \subset {}^{2}A_0 \subset \ldots$ the sequence of sets defined inductively by

$${}^{n}A_0 = \bigcup_{\sigma,\varrho} \{\varphi_1(\sigma), \ldots, \varphi_l(\sigma), \omega_1(\varrho), \ldots, \omega_k(\varrho)\}$$

where σ and ϱ run over all k-sequences and l-sequences in ${}^{n-1}A_0$. We write ${}^{n}\Delta_0 = {}^{n}A_0 - {}^{n-1}A_0$ ($n \geqslant 1$).

Definition 2. If a mapping λ is defined on some elements $b_1, \ldots, b_r$, we define λ on the sequence $\beta = (b_1, \ldots, b_r)$ by $\lambda\beta = (\lambda b_1, \ldots, \lambda b_r)$.

We return to the proof of (*). We note first that ${}^{0}\Delta_0 = \{\lambda_1 a_1, \ldots, \lambda_1 a_m\}$ is a set of generators of $\mathscr{A}_0$. Using the decompositions $A = \bigcup_{i=0}^{\infty} {}^{i}\Delta$ and ${}^{n}A = \bigcup_{i=0}^{n} {}^{i}\Delta$, given above, we observe that it will be enough to prove that there is a mapping λ of A into $\bigcup_{i=0}^{\infty} {}^{i}A_0$ which is an extension of λ_1 such that, for every $n \geqslant 0$,

(**) *λ maps ${}^{n}A$ into ${}^{n}A_0$ and*

$$\lambda\varphi(\sigma) = \varphi(\lambda\sigma), \quad \lambda\omega(\varrho) = \omega(\lambda\varrho)$$

provided that $\sigma, \omega(\varrho) \in ({}^{n}A)^k$; $\varrho, \varphi(\sigma) \in ({}^{n}A)^l$.

We set $\lambda = \lambda_1$ on ${}^{0}A$. Thus (**) holds trivially for $n = 0$, since none of the sequences $\omega(\varrho)$, $\varphi(\sigma)$ is in ${}^{0}A$ (see (a); there are, by definition, no normal sequences in ${}^{0}\Delta = {}^{0}A$). Assuming that λ is defined on ${}^{n}A$ so that (**) holds, we extend this mapping to ${}^{n+1}\Delta$ as follows. By (i), every element of ${}^{n+1}\Delta = {}^{n+1}A - {}^{n}A$ belongs to a certain normal k-sequence σ or to a normal l-sequence ϱ which is entirely in ${}^{n+1}\Delta$. By condition (b) the sequence $\varphi(\sigma)$ or $\omega(\varrho)$ is in ${}^{n}A$ and thus $\lambda\varphi(\sigma)$ or $\lambda\omega(\varrho)$ is defined and is in ${}^{n}A_0$. We define

$$\lambda\sigma = \omega(\lambda\varphi(\sigma)), \quad \lambda\varrho = \varphi(\lambda\omega(\varrho)). \tag{10}$$

Then λ is uniquely extended to a mapping of ${}^{n+1}A$ into ${}^{n+1}A_0$. To prove that λ has on ${}^{n+1}A$ the property (**) we need to consider, by our inductive assumption, only these sequences σ, ϱ which have at least one term in ${}^{n}\Delta \cup {}^{n+1}\Delta$. Suppose first that σ or ϱ have terms in ${}^{n+1}\Delta$. Then, by (a), we have to assume that these sequences are normal (since otherwise $\omega(\varrho)$ and $\varphi(\sigma)$ are not in ${}^{n+1}\Delta$). Thus $\lambda\sigma$ and $\lambda\varrho$ are given by (10), and hence, by (2) and (3),

$$\varphi(\lambda\sigma) = \varphi\big(\omega(\lambda\varphi(\sigma))\big) = \lambda\varphi(\sigma),$$
$$\omega(\lambda\varrho) = \omega\big(\varphi(\lambda\omega(\varrho))\big) = \lambda\omega(\varrho).$$

Assuming that σ, ϱ have at least one term in ${}^{n}\Delta$, and no terms in ${}^{n+1}\Delta$, we can further assume, by (b), that they are not normal. Then, by (a), the sequence $\varrho_1 = \varphi(\sigma)$, $\sigma_1 = \omega(\varrho)$ are normal sequences in ${}^{n+1}\Delta$, and, as we have shown above, $\lambda\varphi(\sigma_1) = \varphi(\lambda\sigma_1)$, $\lambda\omega(\varrho_1) = \omega(\lambda\varrho_1)$ hold. Hence

$$\omega(\lambda\varphi(\sigma_1)) = \lambda\sigma_1, \quad \varphi(\lambda\omega(\varrho_1)) = \lambda\varrho_1,$$

i.e. $\omega(\lambda\varrho) = \lambda\omega(\varrho)$, $\varphi(\lambda\sigma) = \lambda\varphi(\sigma)$. This completes the proof of (**) and of Lemma 2.

DEFINITION 3. A set $S \subset A_m$ will be called *reduced* if there are no normal sequences in S.

Taking the notation of Lemma 2 we prove now

LEMMA 3. *If $\mathcal{A}_0$ is a subalgebra of the free algebra $\mathcal{A}_m$ and ${}^{0}\Delta_0$ is a reduced set of generators of $\mathcal{A}_0$, then*

$$ {}^{n}\Delta_0 \subset \bigcup_{i=n}^{\infty} {}^{i}\Delta \tag{11}$$

holds for $n = 0, 1, 2, \ldots$

Remark. Under the conditions of this lemma $\mathcal{A}_m$ cannot be generated by ${}^{0}\Delta_0$ unless ${}^{0}\Delta \subset {}^{0}\Delta_0$. Indeed, if $\mathcal{A}_0 = \mathcal{A}_m$, then, by (11) and by $A_m = \bigcup_{i=0}^{\infty} {}^{i}\Delta_0$,

$${}^{0}\Delta \subset \bigcup_{i=0}^{\infty} {}^{i}\Delta_0 \subset {}^{0}\Delta_0 \cup \bigcup_{i=1}^{\infty} {}^{i}\Delta$$

and since ${}^{0}\Delta \cap \bigcup_{i=1}^{\infty} {}^{i}\Delta = \emptyset$, we have ${}^{0}\Delta \subset {}^{0}\Delta_0$.

To prove the lemma we need the following notion of an $\mathcal{A}_0$-normal sequence.

DEFINITION 4. Using the notation of Definition 1, we call, for every $n \geq 1$, a sequence in ${}^{n}\Delta_0$ an $\mathcal{A}_0$-*normal* sequence, provided it is of the form $\varphi(\sigma)$ (or $\omega(\varrho)$) where σ (or ϱ) is in ${}^{n-1}A_0$.

We note that, by Definition 1, ${}^{n}\Delta_0$ is a union of A_0-normal sequences.

Proof of Lemma 3. We shall prove, by induction with respect to n, that both the inclusion (11) and the following assertion hold

(12) a sequence in ${}^{n}\Delta_0$ is $\mathcal{A}_0$-normal if and only if it is normal.

The case $n = 0$ is trivial because we have assumed that there are no normal sequences in ${}^{0}\Delta_0$, and clearly there are no $\mathcal{A}_0$-normal sequences in ${}^{0}\Delta_0$. Obviously (11) holds for $n = 0$.

Now suppose that (11) and (12) hold for a certain $n \geq 0$. Let $\varphi(\sigma)$ be $\mathcal{A}_0$-normal in ${}^{n+1}\Delta_0$. We prove that $\varphi(\sigma)$ is a normal sequence in $\bigcup_{i=n+1}^{\infty} {}^{i}\Delta$. We note first that the k-sequence σ in ${}^{n}A_0$ is not $\mathcal{A}_0$-normal, since otherwise we have an l-sequence ϱ in ${}^{n-1}A_0$ such that $\sigma = \omega(\varrho)$ and then $\varphi(\sigma) = \varphi(\omega(\varrho)) = \varrho$, by (2), contradicting our assumption that $\varphi(\sigma)$ is in ${}^{n+1}\Delta_0$. Since σ is not $\mathcal{A}_0$-normal, we have, by the inductive assumption that σ is not normal. Moreover we see, by Definition 1 that at least one term of σ belongs to ${}^{n}\Delta_0$. Thus, by the inductive assumption (11), at least one term of σ belongs to $\bigcup_{i=n}^{\infty} {}^{i}\Delta$. Applying Lemma 2 (a) we have that $\varphi(\sigma)$ is a normal sequence in $\bigcup_{i=n+1}^{\infty} {}^{i}\Delta$.

In exactly the same way we show that an $\mathcal{A}_0$-normal sequence $\omega(\varrho)$ in ${}^{n+1}\Delta_0$ is a normal sequence in $\bigcup_{i=n+1}^{\infty} {}^{i}\Delta$. It is clear that each element of ${}^{n+1}\Delta_0$ belongs to an $\mathcal{A}_0$-normal sequence. Thus (11) holds for $n+1$ instead of n.

Since all normal sequences are disjoint, we have that all $\mathcal{A}_0$-normal sequences in ${}^{n+1}\Delta_0$ are normal and disjoint. Moreover ${}^{n+1}\Delta_0$ is the union of these sequences. This shows that every normal sequence in ${}^{n+1}\Delta_0$ is $\mathcal{A}_0$-normal. Hence (12) holds when n is replaced by $n+1$. This completes the proof of Lemma 3.

4. Bases of $\mathcal{A}_m$. We use the notion of independence introduced by E. Marczewski in [5]. For our purpose the following definition of this notion is most suitable.

DEFINITION 5. We say that the elements $c_1, \ldots, c_r$ of a certain algebra $\mathcal{A}$ are *independent* if the subalgebra generated by them is the free $\{\mathcal{A}\}$-algebra with the free generators $c_1, \ldots, c_r$.

A set of independent generators of $\mathcal{A}$ will be called a *basis*. A sequence composed exactly of all elements of a basis (i.e. an ordered basis) will be called a *basic sequence*. We note that in a basic sequence no element is repeated. If the basis is a reduced set (Def. 3), the corresponding basic sequence will be called *reduced*. Given any sequences $\gamma = (c_1, \ldots, c_s)$, $\delta = (d_1, \ldots, d_t)$, we define the composition $\gamma \ast \delta$ as follows

$$\gamma \ast \delta = (c_1, \ldots, c_s, d_1, \ldots, d_t) .$$

Definition 6. We denote by $\equiv$ the equivalence relation between sequences in A_m which is determined by the conditions

(I) $\delta \equiv \delta'$ when δ' is a permutation of δ,

(II) if $\delta = \sigma \ast \alpha$, $\sigma \in (A_m)^k$, then $\delta \equiv \varphi(\sigma) \ast \alpha$,

(III) if $\delta = \varrho \ast \beta$, $\varrho \in (A_m)^l$, then $\delta \equiv \omega(\varrho) \ast \beta$.

Remark. It follows that if an r-sequence and a t-sequence are equivalent, then $t-r$ is a multiple of $d = l-k$.

Lemma 4. *All basic sequences of the algebra $\mathcal{A}_m$ form a single equivalence class with respect to $\equiv$.*

Proof. We show first that if δ is a basic sequence and $\delta \equiv \delta'$, then δ' is basic. Clearly we can assume that this equivalence is of the type (II) or (III). Suppose that $\delta \equiv \delta'$ holds by (II), i.e. $\delta = \sigma \ast \alpha$, $\delta' = \varphi(\sigma) \ast \alpha$. Then the elements of δ' generate $\mathcal{A}_m$ because $\omega(\varphi(\sigma)) = \sigma$. To prove that the elements of δ' are independent we have to show that every mapping λ_1 of the sequence δ' (cf. Definitions 2 and 5) into $\mathcal{A}_m$ can be extended to an endomorphism. Define the mapping μ_1 of δ into $\mathcal{A}_m$ by

$$\mu_1 \sigma = \omega \lambda_1 \varphi(\sigma) , \qquad \mu_1 \alpha = \lambda_1 \alpha . \tag{13}$$

By assumption μ_1 can be extended to an endomorphism μ. It is easily seen that μ satisfies $\mu\varphi(\sigma) = \lambda_1\varphi(\sigma)$, by (2) and (13). Hence, by $\mu\alpha = \lambda_1\alpha$, the mapping $\lambda = \mu$ is the desired extension of λ_1. If $\delta \equiv \delta'$ holds by (III), then the proof is analogous.

We prove now that every basic sequence is equivalent to a (basic) reduced sequence. Suppose we have given a basic sequence δ which is not reduced. Let n be the greatest number such that all terms of some normal k-sequence σ or l-sequence ϱ in ${}^n\Delta$ appear in δ. Then, replacing δ, if necessary, by a permutation of this sequence, we may assume that δ is of the form $\sigma \ast \alpha$ or $\varrho \ast \beta$. By Lemma 2 (b), the sequences $\varphi(\sigma)$, $\omega(\varrho)$ are in $\bigcup_{i=0}^{n-1} {}^i\Delta$. Thus we have a sequence $\delta' = \varphi(\sigma) \ast \alpha$ or $\delta' = \omega(\varrho) \ast \beta$ which is equivalent to δ and which has the property that the number of normal sequences in ${}^n\Delta$ which have all their terms appearing in δ' is smaller than the corresponding number for δ. Applying the above process a finite number of times, we arrive to a reduced basic sequence.

To complete the proof of Lemma 4, it is sufficient to show that whenever δ and δ' are reduced basic sequences, one is a permutation of the other. This is indeed so because, by Lemma 3 (Remark), every reduced basic sequence δ is composed exactly of all the free generators $a_1, \ldots, a_m$ of $\mathcal{A}_m$ (there are no other elements in δ since a proper subset of a basis is never a basis). This completes the proof of Lemma 4.

Calling the number of elements of a basis of an algebra a *rank* of this algebra, we prove the following

Lemma 5. *A free $\mathfrak{A}_{(k,l)}$-algebra $\mathcal{A}_m$ is isomorphic to a free $\mathfrak{A}_{(k,l)}$-algebra $\mathcal{A}_n$ if and only if n is a rank of $\mathcal{A}_m$.*

Proof. Suppose that $\mathcal{A}_m \simeq \mathcal{A}_n$ and let τ be the isomorphism of $\mathcal{A}_n$ onto $\mathcal{A}_m$. If $b_1, \ldots, b_n$ are the free generators of $\mathcal{A}_n$, then $\tau b_1, \ldots, \tau b_n$ generate $\mathcal{A}_m$ freely, i.e. every mapping of these elements into any algebra $\mathcal{A} \in \mathfrak{A}_{(k,l)}$ can be extended to a homomorphism of $\mathcal{A}_m$ into $\mathcal{A}$. In particular any such mapping into $\mathcal{A}_m$ can be extended to an endomorphism and this proves that $\tau b_1, \ldots, \tau b_n$ are independent generators of $\mathcal{A}_m$.

To prove the 'if' part of our lemma consider the following assertion.

(A) *If δ is a sequence in A_m such that every mapping μ_1 of δ into an algebra $\mathcal{A} \in \mathfrak{A}_{(k,l)}$ can be extended to a homomorphism μ of $\mathcal{A}_m$ into $\mathcal{A}$, then, for $\delta' \equiv \delta$, every mapping λ_1 of δ' into $\mathcal{A}$ can be extended to a homomorphism λ of $\mathcal{A}_m$ into $\mathcal{A}$.*

The proof of (A) is analogous to the first part of the proof of Lemma 4 (where one has to replace mappings into $\mathcal{A}_m$ by mappings into $\mathcal{A}$).

Suppose now that δ' is a basic n-sequence in A_m. We have to show that every mapping λ_1 of δ' into an algebra $\mathcal{A} \in \mathfrak{A}_{(k,l)}$ can be extended to a homomorphism of $\mathcal{A}_m$ into $\mathcal{A}$. This is certainly the case when δ' is replaced by the sequence $\delta = (a_1, \ldots, a_m)$ of the free generators of $\mathcal{A}_m$. Since we have $\delta' \equiv \delta$, by Lemma 4, our assertion follows by (A). This completes the proof of Lemma 5.

5. Proof of Theorem 2. We observe first that if $1 \leqslant m < k$, then $\mathcal{A}_m$ has no other basis except the set of the free generators $\{a_1, \ldots, a_m\}$. Indeed, no basic sequence δ' can be equivalent to $\delta = (a_1, \ldots, a_m)$, by (II) or by (III), since this would imply $m \geqslant k$. Thus, by Lemma 4, every basic sequence is a permutation of $(a_1, \ldots, a_m)$. It follows now, by Lemma 5, that none of the algebras $\mathcal{A}_m$, $1 \leqslant m < k$, is isomorphic to an algebra $\mathcal{A}_n$, where $n \neq m$.

To complete the proof we consider the algebra $\mathcal{A}_k$. We show first that each rank of $\mathcal{A}_k$ is of the form $k+qd$, where $d = l-k$; $q = 0, 1, 2, \ldots$ It is clear, by the remark following Definition 6, and by Lemma 4 that a rank of $\mathcal{A}_k$ must be of the form $k \pm qd$. Supposing that a number $k-qd$ $(q > 0)$ is a rank of $\mathcal{A}_k$, we have that $\mathcal{A}_{k-qd} \simeq \mathcal{A}_k$ holds, by Lemma 5. As we noticed above, this is impossible. Using again Lemma 5, we see that if $\mathcal{A}_k \simeq \mathcal{A}_m$, $m \neq k$, holds, then $m = k+qd$ for some $q > 0$. Since $l = k+d$, the algebra $\mathcal{A}_k$ cannot have any rank s, such that $k < s < l$. It remains to prove that l is a rank of $\mathcal{A}_k$. Indeed, if $\delta = (a_1, \ldots, a_k)$ is the basic sequence of the free generators of $\mathcal{A}_k$, then the sequence $\varphi(\delta)$ defined by (7) is a basic sequence composed of l elements (cf. Lemma 4). This completes the proof of Theorem 2.

References

[1] G. Birkhoff, *Lattice theory*, Amer. Math. Soc. Colloquium Public., New York, 25 (1948).

[2] A. Goetz and Cz. Ryll-Nardzewski, *On bases of abstract algebras*, Bull. Acad. Pol. Sci., Série des sci. math., astr. et phys. 8 (1960), pp. 157-162.

[3] J. A. Green, *A duality in abstract algebra*, Journ. Lond. Math. Soc. 27 (1952), pp. 64-76.

[4] B. Jónsson and A. Tarski, *Two general theorems concerning free algebras*, Bull. Amer. Math. Soc. 62 (1956), p. 554.

[5] E. Marczewski, *A general scheme of the notions of independence in mathematics*, Bull. Acad. Pol. Sci., Série des sci. math., astr. et phys. 6 (1958), pp. 731-736.

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Independence and homomorphisms in abstract algebras

by

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I have shown in 1958 that many notions called *independence* in different branches of mathematics are particular cases of a certain general notion defined in terms of abstract algebra ([4]).

This general concept of independence has subsequently been treated by several authors. They have discussed some of its properties in finitely generated algebras (Świerczkowski [10]), algebras in which all elements are independent and, more generally, algebras in which every n elements form a basis (i.e. a set of independent generators; Świerczkowski [10], [12], Marczewski and Urbanik [7]), bases of an algebra and the set of their cardinal numbers (Goetz and Ryll-Nardzewski [2], Świerczkowski [13]), and self-dependent elements of an algebra (Goetz and Ryll-Nardzewski [2], Nitka [9]). The study of algebras in which independence has the properties of linear independence (Marczewski [5], Urbanik [14]) constitutes a special domain in this research. A discussion of independence in the algebras of sets and Boolean algebras (Marczewski [6]) is the first step in the study of this general concept in particular classes of algebras usually considered.

The purpose of this paper is to formulate and to prove explicitly several simple but fundamental properties of the notion of independence, no special hypotheses about the algebra in question being postulated. Some concrete algebras quoted below serve merely as counterexamples.

Chapter 1 contains preliminaries without any new results. Chapter 2 contains some lemmas on the extension of mappings to homomorphisms (2.1), the definition of independence, and some equivalence theorems (2.2); one of these (iii) enables us to define the notion of independence by that of homomorphism (¹). The following section (2.3) treats of some properties connected with the idea of independence but defined by the notion of the algebraic closure only, i.e. without the use of algebraic operations. It seems interesting that these properties (as well as other

(¹) The notion of independence is related to that of free algebra, and the equivalence theorem 2.2 (iii) is a particular case of the known equivalence of two definitions of a free algebra.