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Acyclicity of compact connected semigroups

by

A. D. Wallace (New Orleans, Louisiana)

0. For the purpose of this note a *semigroup* ([14]) is a non-void Hausdorff space together with a continuous associative multiplication, denoted by juxtaposition, $(x, y) \to xy$. In more detail a semigroup is such a function

$$m\colon\ S \times S \to S$$

that S is a non-void Hausdorff space, m is continuous and satisties

$$m\big(x, m(y, z)\big) = m\big(m(x, y), z\big)$$

for all $x, y, z \in S$. *In all that follows S will denote a semigroup.*

An element t of S will be termed a *left unit* (*left zero*) if it satisfies the equation $tx = x\,(tx = t)$ for each $x \in S$. It has been known for some time ([13]) that a compact connected semigroup with left zero and left unit has the chohomology groups of a point-space (is *acyclic*). Thus, for example, a simple closed curve cannot be a semigroup with left unit and left zero. Also, as may be shown from results in [15], a compact connected locally connected one-dimensional metrizable semigroup with (two-sided) unit is either a tree (dendrite) or contains exactly one simple closed curve.

Without some restriction on the algebraic structure of a semigroup it is not possible to conclude much about its topological structure. For example, for some point p in a simple closed curve we may define $xy = p$ for all points x and y.

Perhaps the first positive result which relates to the problem considered here is given in a paper by R. J. Koch and the author ([8]). More recently L. W. Anderson and L. E. Ward, Jr., have proved that if S is a compact connected locally connected one-dimensional semigroup which is commutative ($xy = yx$ for all $x, y \in S$) and idempotent ($x^2 = x$ for all $x \in S$) then S is a tree ([1]). Their methods are order-theoretic and do not seem to extend to the proofs of results given here.

We terminate the introduction with an indicative result of this paper. Adopting the notation

$$AB = m(A \times B)$$

for subsets $A, B \subset S$ and putting

$$E = \{x|\ x \in S \text{ and } x^2 = x\}$$

we have—

If S is a compact connected semigroup with zero which satisfies also the conditions (i) $S = ES = SE$ *and* (ii) E *is commutative, then S is acyclic.*

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1. We state here some definitions and preliminary results. The Alexander-Kolmogoroff-Spanier cohomology groups will be used as explicated in [9], [10] and [12], though we shall generally employ the terminology of [6] and [12]. The coefficient groups is fixed but anonymous throughout. The following is easily proved from the definition—

(1.1) *For the space X and any connected set $A \subset X$ the homomorphism i^* is an epimorphism and hence δ, the coboundary homomorphism, is 0—*

$$H^0(X) \xrightarrow{i^*} H^0(A) \xrightarrow{\delta} H^1(X, A) .$$

We recall ([11]) that a compact Hausdorff space is fully normal and hence that results of [12], in particular the Map Excision Theorem, may be used. From this it follows that the isomorphisms demanded in [6], p. 43, are available and we may then use the Mayer-Vietoris exact sequence give there—

(1.2) *If X is compact Hausdorff and if $X = A_1 \cup A_2$, with A_1 and A_2 closed then there is an exact sequence*

$$\dots H^p(A_1 \cap A_2) \xrightarrow{\Delta} H^{p+1}(X) \xrightarrow{J^*} H^{p+1}(A_1) \times H^{p+1}(A_2) \dots$$

such that $\Delta = 0$ if $p = 0$ and if $A_1 \cap A_2$ is connected.

The assertion that $\Delta = 0$ follows from (1.1) and the construction given in [6] for this homomorphism. (Our notation differs a little from that in [6].)

The following notation is convenient. If $P \subset Q$ and if $h \in H^p(Q)$ then $h|P$ denoted the image of h under the natural homomorphism induced by the inclusion map of P into Q. Using this notation the homomorphism J^* may be described thus—

If $h \in H^p(X)$ then $J^*(h) = (h|A_1, h|A_2)$.

We denote the *closure* of a set by affixing an asterisk, thus the closure of A is A^*.

It is convenient to state the Reduction Theorem ([12])—

(1.3) *If X is compact Hausdorff, if A is closed and if $h \in H^p(X)$ such that $h|A = 0$ then there is an open set U about A such that $h|U^* = 0$.*

A *tower* is such a family $\mathcal{A}$ of sets that if $A_1, A_2, \in \mathcal{A}$ then either $A_1 \subset A_2$ or $A_2 \subset A_1$. It is a consequence of the Hausdorff Maximality Principle that any tower is contained in a maximal such.

2. We shall need some results on semigroups.

(2.1) *If S is a connected semigroup with zero such that $S = SE$ then $AS \cap BS$ is connected for any subsets $A, B \subset S$.*

Proof. Upon writing $R = AS \cap BS$ we have $RS \subset R$. Moreover, if $x \in R$ then $x = ye$ for some $y \in S$ and some $e \in E$ so that

$$xe = ye^2 = ye = x$$

and thus

$$R \subset RE \subset RS \subset R .$$

We have

$$R = RS = \bigcup \{xS|\ x \in R\}$$

and S, being connected and multiplication continuous, we see that R is the union of a family of connected sets all of which contain the zero element of S.

(2.2) *If E is commutative, if $E^2 = E$, and if $S = ES$, then for any sets $A, B \subset E$ there is a set $C \subset E$ such that $AS \cap BS = CS$.*

Proof. If $a \in A$ and $b \in B$, then $ab = ba$ and $ab \in Ea$, by assumption. It is clear from this that $ab \in aS \cap bS$ and thus

$$abS \subset (aS \cap bS)S \subset aS^2 \cap bS^2 \subset aS \cap bS .$$

Also $abS \subset aS^2 \subset aS$ and similarly, $baS \subset bS$. It follows that $aS \cap bS = abS$. Now

$$\begin{aligned} AS \cap BS &= \bigcup\{aS \cap bs|\ a \in A \text{ and } b \in B\} \\ &= \bigcup\{abS|\ a \in A \text{ and } b \in B\} \\ &= ABS . \end{aligned}$$

A subset $A \subset S$ is a *subgroup* if it is not empty and if

$$xA = A = Ax$$

for each $x \in A$. From [3] or [7] it may be seen that each subgroup of S is contained in a maximal such and that no two maximal subgroups intersect.

(2.3) *If S is the union of its subgroups then $S = ES = SE$ and if A and B are subsets of E there is a subset C of E such that $AS \cap BS = CS$.*

Proof. If $x \in S$ then there is one and only one maximal subgroup, $G(x)$, which contains x; let $u(x)$ be the unit of $G(x)$. Then $x = xu(x) \in Su(x) \subset SE$ and thus $S = SE$. Similarly we see that $S = ES$. Now if A and B are subsets of E we shall prove that $AS \cap BS = u(AS \cap BS)S$.

For this purpose it is enough to show that $aS \cap bS = u(aS \cap bS)S$ for each $a \in A$ and each $b \in B$. Let $x \in aS \cap bS$ and let x' be the inverse of x in $G(x)$ so that $xx' = u(x) = x'x$. Then, from $x \in aS \cap bS$, we obtain

$$u(x) = xx' \subset (aS \cap bS)x' \subset aSx' \cap bSx' \subset aS \cap bS$$

so that $u(aS \cap bS) \subset aS \cap bS$ and right multiplication by S gives one inclusion necessary for the desired equality. Moreover,

$$x = u(x)x \in u(x)S \subset u(aS \cap bS)S$$

so that $aS \cap bS \subset u(aS \cap bS)S$ and the proposition is proved.

This result is implicit in [3].

From the continuity of the multiplication in S we readily obtain the following—

(2.4) *If A is compact and if U is an open set containing AS, then there is an open set V containing A such that $V^*S \subset U$.*

3. The principal result of this note follows—

THEOREM. *If S is a compact connected semigroup with zero, if $S = SE$ and if E_0 is a closed subset of E and*

(*) *If A and B are closed subsets of E_0 then $AS \cap BS = CS$ for some closed subset C of E_0*

then PS is acyclic for any closed subset P of E_0.

Proof. From (2.1) we know that PS is connected and hence has the 0-dimensional groups of a point. We proceed inductively, letting n be a positive integer.

If $h \in H^n(PS)$ and if $h \neq 0$ let $\mathcal{Q}$ be a maximal tower of closed subsets of P such that $h|QS \neq 0$ for any $Q \in \mathcal{Q}$. Denoting by R the intersection of all the members of $\mathcal{Q}$ we show that $h|RS \neq 0$. In the contrary case, that is if $h|RS = 0$, there is an open set $U \supset RS$ such that $h|U^* = 0$, by the Reduction Theorem (1.3). From $RS \subset U$ we obtain, by (2.4), an open set V about R with $V^*S \subset U$. Since $\mathcal{Q}$ is a tower whose intersection is contained in V, there is an element Q of $\mathcal{Q}$ contained in V and thus $QS \subset U$ so that $h|QS = 0$. Since this contradicts the definition of $\mathcal{Q}$, we conclude that $h|RS \neq 0$.

We consider first the case in which $\operatorname{card} R = 1$. We know that RS is a compact connected semigroup with unit (the element of R) and zero (the zero of S) and thus that $H^n(RS) = 0$, by [13]. But then $h|RS = 0$, a contradiction.

Thus $R = M \cup N$ where M and N are closed proper subsets of R and hence $RS = MS \cup NS$ so that upon putting $h' = h|RS$,

$$h'|MS = 0 = h'|NS.$$

We apply (1.2) taking $X = RS$, $A_1 = MS$, $A_2 = NS$ and $p = n-1$. From the last equation of the preceding paragraph we have $J^*(h') = 0$. If $n = 1$ then $A_1 \cap A_2$ is connected by (2.1) and thus $\Delta = 0$ and J^* is a monomorphism. If $n > 1$, $A_1 \cap A_2 = CS$ for some closed set $C \subset E_0$ by (*) and, by the inductive hypothesis, we have $H^{n+1}(CS) = 0$ so that $h|RS = 0$ and again J^* is a monomorphism. We conclude that $h' = 0$, i.e. $h|RS = 0$ and this contradiction completes the proof.

COROLLARY 1. *If S is a compact connected semigroup with zero and if either S is the union of its subgroups or if $S = ES = SE$ and E is commutative then S is acyclic.*

Proof. The first case follows from the theorem and (2.3). The second follows from the theorem and (2.2) if it is observed that when E is commutative the product of any two idempotents is again an idempotent.

Recall that a *semilattice* is a commutative semigroup S for which $S = E$. We improve slightly the result of Anderson and Ward ([1])—

COROLLARY 2. *If S is a compact connected semilocally connected semilattice of codimension* 1 ([4]) *then S is a tree.*

Proof. First of all, we know that S has a zero, which is indeed the unique element of the set

$$\bigcap \{xS \mid x \in S\}.$$

Since S is compact and $\operatorname{card} S = 1$, we have at once that $H^1(A) = 0$ (see Cohen [4]) for any closed subset $A \subset S$. By a result of Borsuk [2] and Čech [5] we know that any subcontinuum of S is unicoherent. If S is *not* a tree then, by definition, there exist two points a and b such that no point separates a and b in S (in the strong sence). By a suitable modification of an argument of Whyburn's ([17], p. 50), it follows that there is, for any p different from both a and b, a continuum P irreducible from a to b and which does not contain p. If q is a element of P distinct from a and from b there is also a continuum Q irreducible from a to b and which does not contain q. But then $P \cap Q$ is a subcontinuum of S which is not unicoherent because $P \cap Q$ is clearly not connected. This completes the proof. Actually, it is not necessary to assume commutativity to obtain the stronger result but a different argument must be used.

We recall that I is an *ideal* of S if I is not void and if $SI \subset I \supset IS$. The next result can be improved somewhat but, since it is unlikely that even the improved version is in any sense a final result, we remain content to state it in its present form, which is somewhat easier to prove than the more general version.

COROLLARY 3. *Assuming the hypotheses of Corollary* 1 *but with the deletion of the assumption that S has a zero, we have $H^p(S)$ naturally isomorphic with $H^p(I)$ for any closed ideal I of S.*

Proof. Upon endowing $S\times S$ with coordinatewise multiplication we see that it is a semigroup. Let D be its diagonal and let R be a closed reflexive symmetric transitive subset of $S\times S$ which satisfies also the condition

$$DR\subset R\supset RD\,,$$

which is equivalent, under the assumptions on R, to $R^2\subset R$. Let S/R denote the set of equivalence classes for the relation R and let f denote the function which assigns to each $x\in S$ the equivalence class $f(x)\in S/R$ which contains x. For simplicity of writing we put $T=S/R$ and observe that there is one and only one multiplication on T, denoted by juxtaposition, such that

$$f(xy)=f(x)f(y)$$

for all $x, y\in S$. Since S is compact and since R is closed, it follows that T is a semigroup, termed the *quotient of S by R*. Clearly the natural map f is a homomorphism of S onto T.

Adhering to the notation and assumptions of the preceding paragraph it may be shown without difficulty that if S is the union of its subgroups then so is T and also that when S satisfies the second set of conditions in Corollary 1 then these are also satisfied by T.

Now let I be a closed ideal of S and let

$$R=I\times I\cup D\,.$$

It is readily verified that R satisfies all of the hypotheses of the first paragraph. Moreover, f collapses I to a point q, $f(I)=q$, and f is a homeomorphism on $S\setminus I$ (the complement of I in S) onto $T\setminus q$. The Map Excision Theorem ([12]) may be applied and, denoting by f^* the appropriate homomorphism induced by f, we have

$$f^*\colon\ H^p(T, q)\approx H^p(S, I)$$

for any non-negative integer p. Now it is readily verified from the fact that f is a homomorphism that q is the zero of T and thus, since T satisfies the conditions of Corollary 1, we know that it is acyclic. Using the acyclicity of T and of q and the exact sequence for the pair (T, q) we have $H^p(T, q)=0$ for all non-negative integers p and thus $H^p(S, I)=0$ for the range of p. From the exact sequence for the pair (S, I) it is readily seen that $H^p(S)$ is naturally isomorphic with $H^p(I)$, as was to be shown.

Employing the above corollary and the results of [16] we may prove the following—

(i) *If S is a compact connected manifold without boundary, if S is the union of its subgroups and if S is prime in the sense of Borsuk then either S is a Lie group or else has one of the multiplications*

$$xy=x \quad\text{or}\quad xy=y \qquad\text{for all}\quad x, y\in S\,.$$

(ii) *If S is a compact connected manifold without boundary, if $S=ES=SE$ with E commutative and if S is prime then S is a Lie group.*

Since the above results are unlikely to be final, we omit the proofs.

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