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Local properties of solutions of elliptic partial differential equations

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Introduction. The purpose of this paper* is to establish *pointwise* estimates for solutions of elliptic partial differential equations and systems. The results presented here differ from the familiar ones in that they give information about the behavior of solutions at individual points. More specifically, we obtain two kinds of results. On the one hand, we establish inequalities for solutions and their derivatives at isolated individual points. On the other, we also obtain results of the „almost everywhere" type. Theorems 1 and 2 below summarize the main results.

1. We begin with notations and definitions.

By $x, y, \ldots$ we denote respectively points $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n), \ldots$ of the n-dimensional Euclidean space E_n; dimension n is fixed throughout the paper. The class of measurable functions $f(x)$ such that $\|f(x)\|_p = \left(\int |f(x)|^p dx\right)^{1/p} < \infty$ will be denoted by $L^p(E_n)$ or, simply, L^p; here and elsewhere dx stands for the element of volume in E_n, and $\int$ means $\int\limits_{E_n}$. All functions we consider are complex-valued, unless otherwise stated. The symbols $x+y$ and λx, where λ is a scalar, have the usual meaning; we also use the notations $|x| = (x_1^2+x_2^2+\ldots+x_n^2)^{1/2}$, $x\cdot y = x_1y_1+x_2y_2+\ldots+x_ny_n$, and if $a = (a_1, a_2, \ldots, a_n)$, where the a_j are non-negative integers, we will write

$$|a| = a_1+a_2+\ldots+a_n, \qquad x^a = x_1^{a_1}x_2^{a_2}\ldots x_n^{a_n},$$

$$a! = a_1!a_2!\ldots a_n!, \qquad \left(\frac{\partial}{\partial x}\right)^a f = f_a = \left(\frac{\partial}{\partial x_1}\right)^{a_1}\left(\frac{\partial}{\partial x_2}\right)^{a_2}\ldots\left(\frac{\partial}{\partial x_n}\right)^{a_n} f.$$

Finally $f*g$ will stand for the convolution of f and g.

The symbol C with various subscripts will stand for a constant, not necessarily the same at each occurrence, which depends only (unless otherwise stated) on the variables displayed. Dependence on the dimension, though, will not be indicated. Thus C without subscripts will indicate either an absolute constant or a quantity depending on the dimension only.

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We denote by C_0^∞ the class of infinitely differentiable functions with compact support.

It seems that the notion of differentiability which is most suited to the treatment of the problems that concern us, is not the classical one. It appears that it is more convenient to estimate the remainder of the Taylor series in the mean with various exponents. This type of differentiability is much more stable than the classical one in the sense that it is preserved at individual points under various operations such as fractional integration and singular integral transformations.

Definition 1. Let u be any number $\geqslant -n/p$. By $T_u^p(x_0)$ we shall denote the class of functions $f(x)\in L^p(E_n)$ such that there exists a polynomial $P(x-x_0)$ of degree strictly less than u (in particular $P\equiv 0$ if $u\leqslant 0$) with the property that

$$(1.1)\qquad \Big(\varrho^{-n}\int_{|x-x_0|\leqslant\varrho}|f(x)-P(x-x_0)|^p\,dx\Big)^{1/p}\leqslant A\varrho^u\qquad (0<\varrho<\infty)$$

with A independent of ϱ. Here $1\leqslant p\leqslant\infty$; when $p=\infty$ the left side of (1.1) means, as usual, $\operatorname*{ess\,sup}_{|x-x_0|\leqslant\varrho}|f(x)-P(x-x_0)|$. Instead of $T_u^\infty(x_0)$ we shall simply write $T_u(x_0)$.

Definition 2. Let f be a function in $T_u^p(x_0)$. We shall say that f *belongs to* $t_u^p(x_0)$, if there exists a polynomial $P(x-x_0)$ of degree less than or equal to u such that

$$(1.2)\qquad \Big(\varrho^{-n}\int_{|x-x_0|\leqslant\varrho}|f(x)-P(x-x_0)|^p\,dx\Big)^{1/p}=o(\varrho^u)\qquad\text{as}\qquad \varrho\to 0.$$

As in definition 1, we shall denote $t_u^\infty(x_0)$ by $t_u(x_0)$.

In both definitions the polynomial P is unique, as easily seen by considering the difference P_1-P_2 of two such possible polynomials and observing that the inequalities (1.1) and (1.2) imply that it must vanish identically. Any function in L^p belongs to $T_u^p(x_0)$ with $u=-n/p$ for any x_0. The familiar result about the Lebesgue set of a function can be expressed by saying that if $f\in L^p$, $p<\infty$, then $f\in t_0^p(x_0)$ for almost all x_0.

The class $T_u^p(x_0)$ is a linear space in which we introduce a norm. It will be convenient to use the rather unorthodox notation $T_u^p(x_0,f)$ for the norm of an $f\in T_u^p(x_0)$. We define $T_u^p(x_0,f)$ as the sum of $\|f\|_p$, the moduli of the coefficients of the polynomial $P(x-x_0)$, and the least admissible value of A in (1.1).

In the definition of $T_u^p(x_0)$ or $t_u^p(x_0)$ the value of the function at x_0 is irrelevant. Nevertheless, if the function f in L^p belongs to $t_u^p(x_0)$ with $u\geqslant 0$ (and so also if f belongs to $T_u^p(x_0)$ with $u>0$) for each x_0 belonging to a set of positive measure, then at every x_0 in the Lebesgue set $f(x_0)$ coincides with the constant term of the corresponding $P(x-x_0)$; consequently be redefining f, if necessary, on a set of measure zero, we may assume that this condition is always satisfied.

Definition 3. Let Q be a closed set. We shall say that the *bounded function f belongs to the class $B_u(Q)$*, $u>0$, if there exist bounded functions f_α, $|\alpha|<u$, such that

$$f_\alpha(x+h)=\sum_{|\beta|<u-|\alpha|}\frac{h^\beta}{\beta!}f_{\alpha+\beta}(x)+R_\alpha(x,h)$$

for all x and $x+h$ in Q, with $|R_\alpha(x,h)|\leqslant C|h|^{u-|\alpha|}$. We say that f *belongs to* $b_u(Q)$, $u\geqslant 0$, if there exist functions f_α with $|\alpha|\leqslant u$ such that

$$f_\alpha(x+h)=\sum_{|\beta|\leqslant u-|\alpha|}\frac{h^\beta}{\beta!}f_{\alpha+\beta}(x)+R_\alpha(x,h)$$

for all x and $x+h$ in Q, with $|R_\alpha(x,h)|\leqslant C|h|^{u-|\alpha|}$ and, in addition, $R_\alpha(x,h)=o(|h|)^{u-|\alpha|}$ as $|h|\to 0$, uniformly in $x\in Q$.

Definition 4. Given a non-negative integer k, L_k^p will denote the class of functions in L^p with distribution derivatives of orders less than or equal to k in L^p. The norm $\|f\|_{p,k}$ of a function f in L_k^p is, by definition, the sum of the norms in L^p of f and its derivatives of orders less than or equal to k.

Definition 5. Let $f=(f_1,f_2,\dots,f_r)$ be a vector valued function and

$$\mathcal{L}f=\sum_{|\alpha|\leqslant m}a_\alpha(x)\Big(\frac{\partial}{\partial x}\Big)^\alpha f=g,$$

where $g=(g_1,g_2,\dots,g_s)$ is a vector valued function of s components, $s\geqslant r$, and $a_\alpha(x)$ are $s\times r$ matrices. We shall say that the operator $\mathcal{L}$, or the equation $\mathcal{L}f=g$, is *elliptic* at the point x_0 if the characteristic matrix

$$(1.3)\qquad \sum_{|\alpha|=m}a_\alpha(x_0)\,\xi^\alpha$$

has the property that for $|\xi|=1$

$$\det\Big[\Big(\sum_{|\alpha|=m}a_\alpha^*(x_0)\,\xi^\alpha\Big)\Big(\sum_{|\alpha|=m}a_\alpha(x_0)\,\xi^\alpha\Big)\Big]\geqslant\mu(x_0)>0,$$

where a^* denotes the conjugate transpose of a. We shall call the largest admissible value of $\mu(x_0)$ the *ellipticity constant of $\mathcal{L}$ at x_0*.

THEOREM 1. *Let $\mathcal{L}f=g$ be an equation of order m with coefficients in $T_u(x_0)$, $u>0$, which is elliptic at x_0 in the sense of definition 5. Let $1<p<\infty$, $u\geqslant v\geqslant -n/p$ and v be non-integral. If $f\in L_m^p$ and $g\in T_v^p(x_0)$ in the sense that their components belong to these spaces, then*

(i) *for* $j = 1, 2, \ldots, r$ *and* $|a| \leqslant m$,

$$T^q_{v+m-|a|}\left(x_0, \left(\frac{\partial}{\partial x}\right)^a f_j\right) \leqslant C\left[\sum_{i=1}^{s} T^p_v(x_0, g_i) + \sum_{i=1}^{r} \|f_i\|_{p,m}\right], \tag{1.4}$$

where $1/p \geqslant 1/q \geqslant 1/p-(m-|a|)/n$ *if* $1/p-(m-|a|)/n > 0$, $p \leqslant q \leqslant \infty$ *if* $1/p < (m-|a|)/n$, *or* $p \leqslant q < \infty$ *if* $1/p = (m-|a|)/n$, *and* C *depends only on* v, p, r, s, $\mu(x_0)$ *and the least upper bound of the norms in* $T_u(x_0)$ *of the coefficients of* $\mathfrak{L}$.

(ii) *If in addition the leading coefficients of the equation are uniformly continuous and the equation is uniformly elliptic in the sense that the constant of ellipticity* $\mu(x)$ *is bounded away from zero, then the quantity* $\sum_{i=1}^{r}\|f_i\|_{p,m}$ *on the right of* (1.4) *can be replaced by* $C\left[\sum_{i=1}^{r}\|f_i\|_p + \sum_{i=1}^{s}\|g_i\|_p\right]$, *where* C *depends on* $\mathfrak{L}$.

(iii) *If* $g \in t^p_v(x_0)$, *then* $(\partial/\partial x)^a f$ *belongs to* $t^q_{v+m-|a|}$ *with the same* q *as in part* (i).

THEOREM 2. *Let* $\mathfrak{L}f = g$ *be an equation of order* m *which is elliptic in the sense of definition* 5 *at all points* x_0 *belonging to a set* Q *of positive measure, and whose coefficients belong to* $T_u(x_0)$, $u \geqslant 1$, *for all* x_0 *in* Q. *Let* v *be a positive integer not larger than* u. *Then, if* $g \in T^p_v(x_0)$, $1 < p < \infty$, *for all* x_0 *in* Q, *and* $f \in L^p_m$, *the functions* $(\partial/\partial x)^a f$, $|a| \leqslant m$, *belong to* $t^q_{v+m-|a|}(x_0)$ *for almost all* x_0 *in* Q, *where* q *is the same as in part* (i) *of Theorem* 1.

THEOREM 3. *Let* $\mathfrak{L}f = g$ *be an equation of order* m *with coefficients in* $T_u(x_0)$, $u > 0$, *for all* x_0 *in a closed set* Q. *Let the norms of the coefficients in* $T_u(x_0)$ *be bounded in* Q *and* $\mathfrak{L}$ *be uniformly elliptic in* Q, *that is, let the constant of ellipticity* $\mu(x_0)$ *of* $\mathfrak{L}$ *be bounded away from zero in* Q. *Then if* $g \in T^p_v(x_0)$ *for all* x_0 *in* Q *and* $\sum_{i=1}^{s} T^p_v(x_0, g_i)$ *is bounded in* Q, $1 < p < \infty$, v *is positive and non integral,* $-m < v \leqslant u$, *and* $f \in L^p_m$, *the function* f *belongs to* $B_{v+m}(Q)$.

If in addition $g \in t^p_v(x_0)$ *for all* $x_0 \in Q$, *then* $f \in b_{v+m}(Q)$.

It has already been observed (see [7]), and the idea is basic for the present paper, that, roughly speaking, a differential operator is the composition of fractional differentiation and a singular integral transformation. Accordingly, our method will consist in deriving estimates for solutions of differential equations from estimates for fractional integrals and singular integral transforms.

Definition 6. Let f be a tempered distribution in E_n; the fractional integral of order u of f, denoted by $J^u f$, is defined by

$$\widehat{J^u f} = (1+4\pi^2|x|^2)^{-u/2}\hat{f},$$

where $\hat{f}$ stands for the Fourier transform of f, that is,

$$\hat{f}(x) = \int_{E_n} e^{-2\pi i(x\cdot y)} f(y)\,dy,$$

if f is an integrable function.

This definition of fractional integration is different from the familiar one due to M. Riesz. It has some advantages, namely it is defined for any u real or complex and thus is a one-parameter group of operations, and furthermore, for $u > 0$ it is a bounded operation on L^p, $1 \leqslant p < \infty$. This notion of fractional integration was introduced in [1] and [5; I, page 25].

THEOREM 4. *Let* $u \geqslant -n/p$, $v > 0$, $u+v \neq 0, 1, 2, \ldots$, $1 < p \leqslant \infty$. *Then* J^v *maps continuously*

(i) $T^p_u(x_0)$ *into* $T^q_{v+u}(x_0)$, *provided*

(a) $$\frac{1}{p} \geqslant \frac{1}{q} \geqslant \frac{1}{p} - \frac{v}{n}, \quad \text{if} \quad p < \frac{n}{v};$$

(b) $$p \leqslant q \leqslant \infty, \quad \text{if} \quad \frac{n}{v} < p \leqslant \infty;$$

(c) $$p \leqslant q < \infty, \quad \text{if} \quad \frac{n}{v} = p;$$

(ii) $t^p_u(x_0)$ *into* $t^q_{u+v}(x_0)$, *with* u *and* v *as in* (i).

THEOREM 5. *Let* u *and* v *be non-negative integers. Then if* $f \in T^p_u(x_0)$, $1 < p < \infty$, *for all* x_0 *in a set* Q *of positive measure, we have* $J^v f \in t^q_{u+v}(x_0)$ *for almost all* $x_0 \in Q$, p *and* q *being related as in part* (i) *of Theorem* 4. *If* $u+v \geqslant 1$, *the assertion is valid for* $1 < p \leqslant \infty$.

This theorem asserts in particular that if u is a positive integer and $1 < p \leqslant \infty$ the condition $f \in T^p_u(x_0)$ implies $f \in t^p_u(x_0)$ almost everywhere; the case $u = 1$ and $p = \infty$ is the familiar result of Rademacher-Stepanov; the case $u > 1$, $p = \infty$ was proved by Oliver [12]; related results are Theorem 11 and 12 below which extend some known results (see [8] and [4]).

We shall now consider singular integral operators of the following form:

$$\mathcal{K}f = a(x)f(x) + \int k(x, x-y) f(y)\,dy,$$

where $a(x)$ is a bounded measurable function and $k(x, z)$ is homogeneous of degree $-n$ with respect to z, that is, such that $k(x, \lambda z) = \lambda^{-n}k(x, z)$ for all $\lambda > 0$, and further $k(x, z)$ has for each x mean value zero on $|z| = 1$. In addition we shall assume that $k(x, z)$ is infinitely differentiable with respect to z and is uniformly bounded for $|z| = 1$. The preceding integral must of course be interpreted as a principal value integral. Associated with the operator $\mathcal{K}$ is its *symbol* $\sigma(\mathcal{K})$ which is defined as

$$\sigma(\mathcal{K}) = a(x) + k(x, \hat{z}),$$

where $k(x, \hat{z})$ is the Fourier transform of $k(x, z)$ with respect to z.

Definition 7. An operator $\mathcal{K}$ as above is said *to belong to the class* $T_u(x_0)$, $u \geqslant 0$, if $\sigma(\mathcal{K})$ and its derivatives with respect to coordinates of z of orders $\leqslant 2n+u+1$ belong to $T_u(x_0)$ for each $z \neq 0$, uniformly in $|z| = 1$. The norm $T_u(x_0, \mathcal{K})$ of an operator of class $T_u(x_0)$ is, by definition, the least upper bound of the norms in $T_u(x_0)$ of $\sigma(\mathcal{K})$ and its derivatives with respect to z of orders less than or equal to $2n+u+1$ evaluated in $|z| = 1$. Operators of class $t_u(x_0)$ are analogously defined.

THEOREM 6. *Let $\mathcal{K}$ be a singular integral operator of class $T_u(x_0)$. If $1 < p < \infty$, and v is not equal to zero or a positive integer and is larger than or equal to $-n/p$, then $\mathcal{K}$ maps $T_v^p(x_0)$ continuously into $T_v^p(x_0)$, with norm less than or equal to $C_{p,u} T_u(x_0, \mathcal{K})$, provided $u \geqslant v$. The corresponding result for operators of class $t_u(x_0)$ and the spaces $t_v^p(x_0)$ is also valid.*

THEOREM 7. *Let u be a non negative integer and f a function belonging to $T_u^p(x_0)$, $1 < p < \infty$, for all x_0 in a set Q of positive measure. Then there exists a subset $\bar{Q}$ of Q such that $Q - \bar{Q}$ has measure zero and such that for every singular integral operator $\mathcal{K}$ belonging to $T_u(x_0)$, $x_0 \in \bar{Q}$, $\mathcal{K}f$ belongs to $T_u^p(x_0)$.*

2. In this section we establish certain properties of the spaces $T_u^p(x_0)$, $t_u^p(x_0)$ which we shall need later.

LEMMA 2.1. *If $-n/p \leqslant u \leqslant v$, $1 \leqslant p \leqslant \infty$, then $T_u^p(x_0) \supset T_v^p(x_0)$, and $T_u^p(x_0, f) \leqslant C T_v^p(x_0, f)$.*

Proof. Assume $u \geqslant 0$. Let P_u be the sum of the terms of degree $< u$ in the Taylor expansion of f, R_u the corresponding remainder, and let P_v and R_v be similarly defined. Then P_u is the sum of the terms of P_v of degree less than u. For $\varrho \leqslant 1$, we have $|P_v(h) - P_u(h)| \leqslant T_v^p(x_0, f)|h|^u$ and

$$\Big[\int_{|h|\leqslant\varrho} |R_u(h)|^p dh\Big]^{1/p} \leqslant \Big[\int_{|h|\leqslant\varrho} |P_v(h) - P_u(h)|^p dh\Big]^{1/p} + \Big[\int_{|h|\leqslant\varrho} |R_v(h)|^p dh\Big]^{1/q}$$
$$\leqslant C T_v^p(x_0, f)\varrho^{n/p+u} + T_v^p(x_0, f)\varrho^{n/p+v} \leqslant C T_v^p(x_0, f)\varrho^{n/p+u}.$$

And for $\varrho > 1$, since $|P_u(h)| \leqslant T_v^p(x_0, f)\varrho^u$ for $|h| \leqslant \varrho$, we have

$$\Big[\int_{|h|\leqslant\varrho} |R_u(h)|^p dh\Big]^{1/p} \leqslant \|f\|_p + \Big[\int_{|h|\leqslant\varrho} |P_u(h)|^p dh\Big]^{1/p}$$
$$\leqslant T_v^p(x_0, f) + C T_v^p(x_0, f)\varrho^{n/p+u} \leqslant C T_v^p(x_0, f)\varrho^{n/p+u},$$

and thus

$$\sup_\varrho \frac{1}{\varrho^{u+n/p}}\Big[\int_{|h|\leqslant\varrho} |R_u(h)|^p dh\Big]^{1/p} \leqslant C T_v^p(x_0, f).$$

This, as easily seen, implies the desired result.

For $u < 0$ we have, if $\varrho \leqslant 1$,

$$\frac{1}{\varrho^u}\Big[\frac{1}{\varrho^n}\int_{|h|\leqslant\varrho} |f(x_0+h)|^p dh\Big]^{1/p} \leqslant \frac{1}{\varrho^v}\Big[\frac{1}{\varrho^n}\int_{|h|\leqslant\varrho} |f(x_0+h)|^p dh\Big]^{1/p},$$

and, if $\varrho > 1$,

$$\frac{1}{\varrho^u}\Big[\frac{1}{\varrho^n}\int_{|h|\leqslant\varrho} |f(x_0+h)|^p dh\Big]^{1/p} \leqslant \|f\|_p,$$

which again implies the desired result.

LEMMA 2.2. *The spaces $T_u^p(x_0)$, $1 \leqslant p \leqslant \infty$, $u \geqslant -n/p$, are complete.*

Proof. Suppose that the sequence f_ν is such that $T_u^p(x_0, f_\nu - f_\mu) \to 0$ as ν and μ tend to infinity. Then, in the first place, f_ν converges in L^p to a limit f. Let $P = \lim_{\nu\to\infty} P_\nu$, where P_ν is the Taylor expansion of f_ν; P exists since the coefficients of P_ν converge. Then, for each ϱ,

$$\frac{1}{\varrho^u}\Big[\frac{1}{\varrho^n}\int_{|h|\leqslant\varrho} |[f(x_0+h) - f_\nu(x_0+h)] - [P(h) - P_\nu(h)]|^p dh\Big]^{1/p}$$
$$= \lim_{\mu\to\infty} \frac{1}{\varrho^u}\Big[\frac{1}{\varrho^n}\int_{|h|\leqslant\varrho} |(f_\mu - f_\nu) - (P_\mu - P_\nu)|^p dh\Big]^{1/p} \leqslant \varliminf_{\mu\to\infty} T_u^p(x_0, f_\mu - f_\nu) < \infty.$$

This shows that $f \in T_u^p(x_0)$, and as ν tends to infinity we find

$$\sup_\varrho \frac{1}{\varrho^u}\Big[\frac{1}{\varrho^n}\int_{|h|\leqslant\varrho} |(f - f_\nu) - (P - P_\nu)|^p dh\Big]^{1/p} \to 0.$$

From this it follows that $T_u^p(x_0, f - f_\nu) \to 0$ as $\nu \to \infty$.

LEMMA 2.3. *Let $-n/p \leqslant u$, $1 \leqslant p \leqslant \infty$.*

(i) *The space $t_u^p(x_0)$ is a closed subspace of $T_u^p(x_0)$.*

(ii) *If f is a function in $t_u^p(x_0)$ and $\varphi(x)$ is a function in C_0^∞ such that $\int\varphi(x)dx = 1$, then $f^\lambda = \lambda^n\varphi(\lambda x) * f(x)$ converges to f in $T_u^p(x_0)$ as λ tends to infinity.*

(iii) *The space C_0^∞ is dense in $t_u^p(x_0)$.*

Proof. Suppose $f_\nu \epsilon t_u^p(x_0)$ and $T_u^p(x_0, f_\nu - f) \to 0$ as $\nu \to \infty$. Then, if R_ν and R are the respective remainders,

$$\sup \frac{1}{\varrho^u}\left[\frac{1}{\varrho^n}\int_{|x-x_0|\leqslant\varrho} |R_\nu - R|^p dx\right]^{1/p} \leqslant T_u^p(x_0, f_\nu - f).$$

Consequently, the left-hand side is less than any preassigned ε if ν is sufficiently large. Since

$$\lim_{\varrho\to 0} \frac{1}{\varrho^u}\left[\frac{1}{\varrho^n}\int_{|x-x_0|\leqslant\varrho} |R_\nu|^p dx\right]^{1/p} = 0,$$

it follows that

$$\overline{\lim_{\varrho\to 0}} \frac{1}{\varrho^u}\left[\frac{1}{\varrho^n}\int_{|x-x_0|\leqslant\varrho} |R(x)|^p dx\right]^{1/p} \leqslant \varepsilon.$$

Since ε is arbitrary, this implies that $f \epsilon t_u^p(x_0)$.

To prove (ii) we will assume, without loss of generality, that $\varphi(x)$ vanishes for $|x| \geqslant 1$, and that $x_0 = 0$.

Denote by P^λ the sum of terms of the Taylor expansion of $f^\lambda(x)$ at $x_0 = 0$ of degree $\leqslant u$. Set $R^\lambda = f^\lambda - P^\lambda$. Then, by Hölder's inequality, we have

$$\frac{1}{\varrho^n}\int_{|x|\leqslant\varrho} |R(x)|\,dx \leqslant C\left[\frac{1}{\varrho^n}\int_{|x|\leqslant\varrho} |R(x)|^p dx\right]^{1/p} \leqslant C\varepsilon(\varrho)\,\varrho^u,$$

where R is the remainder in the expansion of f at $x_0 = 0$ and $\varepsilon(\varrho)$ is bounded and tends to zero as ϱ tends to zero. Without loss of generality we may assume that $\varepsilon(\varrho)$ decreases to 0.

We first show that the coefficients of P^λ converge to the coefficients of P. By differentiation we obtain

$$\left(\frac{\partial}{\partial x}\right)^\alpha f^\lambda(0) = \int \lambda^n \varphi(-\lambda y) P_\alpha(y)\,dy + \int \lambda^{n+|\alpha|}\varphi_\alpha(-\lambda y) R(y)\,dy,$$

where $f = P + R$, $P_\alpha = (\partial/\partial x)^\alpha P$ and $\varphi_\alpha = (\partial/\partial x)^\alpha \varphi$. The first integral converges to $P_\alpha(0)$ as $\lambda \to \infty$, and for the second we have

$$\left|\lambda^{n+|\alpha|}\int \varphi_\alpha(-\lambda y) R(y)\,dy\right| \leqslant N\lambda^{n+|\alpha|}\int_{|y|\leqslant 1/\lambda} |R(y)|\,dy \leqslant CN\varepsilon\left(\frac{1}{\lambda}\right)\left(\frac{1}{\lambda}\right)^{u-|\alpha|},$$

where N is a bound for $|\varphi_\alpha|$, $|\alpha| \leqslant u$, and so it tends to zero as $\lambda \to \infty$.

Since $\|f^\lambda - f\|_p$ tends to zero as $\lambda \to \infty$, it remains to show that

$$\sup_\varrho \frac{1}{\varrho^u}\left[\frac{1}{\varrho^n}\int_{|x|\leqslant\varrho} |R^\lambda(x) - R(x)|^p\,dx\right]^{1/p}$$

tends to zero as $\lambda \to \infty$. Let $\eta > 0$ and let m_λ be the sum of the absolute values of the coefficients of $P_\lambda - P$. Then for $\varrho > \eta$ we have

$$\frac{1}{\varrho^u}\left[\frac{1}{\varrho^n}\int_{|x|\leqslant\varrho} |R^\lambda - R|^p dx\right]^{1/p} \leqslant \frac{1}{\eta^{n/p+u}}\|f^\lambda - f\|_p + \frac{1}{\varrho^u}\left[\frac{1}{\varrho^n}\int_{|x|\leqslant\varrho} |P^\lambda - P|^p\,dx\right]^{1/p}$$

$$\leqslant \frac{1}{\eta^{n/p+u}}\|f^\lambda - f\|_p + \frac{m_\lambda}{\varrho^{n/p+u}}\left[\int_{|x|\leqslant\varrho} (1 + |x|^{up})\,dx\right]^{1/p},$$

and the right hand side tends to zero as $\lambda \to \infty$, uniformly in $\varrho > \eta$.

On the other hand, we have

$$R^\lambda(x) = \int\left[\lambda^n\varphi[\lambda(x-y)] - \sum_{|\alpha|\leqslant u}\lambda^{n+|\alpha|}\frac{x^\alpha}{\alpha!}\varphi_\alpha(-\lambda y)\right][P(y) + R(y)]\,dy.$$

Since $\lambda^n\varphi(\lambda x) * P$ is a polynomial Q, the contribution of P to this integral is Q minus its Taylor expansion at $x = 0$, and so is zero. Thus P may be dropped in the preceding expression. If $\varrho \geqslant 1/\lambda$ we have, using Young's inequality,

$$\left[\int_{|x|\leqslant\varrho} |R^\lambda(x)|^p dx\right]^{1/p} \leqslant \left[\int_{|x|\leqslant 2\varrho} |R(x)|^p dx\right]^{1/p} +$$

$$+ \sum_\alpha \lambda^{n+|\alpha|}\left|\int \varphi_\alpha(-\lambda y) R(y)\,dy\right|\left[\int\!\!\int_{|x|\leqslant\varrho} |x|^{|\alpha|p}\,dx\right]^{1/p}$$

$$\leqslant C_u\varepsilon(2\varrho)\,\varrho^{n/p+u} + C\sum N\varepsilon\left(\frac{1}{\lambda}\right)\left(\frac{1}{\lambda}\right)^{u-|\alpha|}\varrho^{|\alpha|+n/p}$$

$$\leqslant C_u\varepsilon(2\varrho)\,\varrho^{n/p+u}.$$

If $\varrho \leqslant 1/\lambda$ and if N now denotes a bound for $|\varphi_\alpha(x)|$, $|\alpha| = [u+1]$, we have

$$\left|\lambda^n\varphi[\lambda(x-y)] - \sum_{|\alpha|\leqslant u}\lambda^{n+|\alpha|}\frac{x^\alpha}{\alpha!}\varphi_\alpha(-\lambda y)\right| \leqslant C_u N(\lambda|x|)^{[u+1]}\lambda^n,$$

whence

$$|R^\lambda(x)| \leqslant C_u N(\lambda|x|)^{[u+1]}\lambda^n\int_{|y|\leqslant 2/\lambda} |R(y)|\,dy \leqslant C_u\varepsilon\left(\frac{2}{\lambda}\right)\lambda^{[u+1]-u}|x|^{[u+1]},$$

and

$$\left[\int_{|x|\leqslant\varrho} |R^\lambda(x)|^p dx\right]^{1/p} \leqslant C_u\varepsilon\left(\frac{2}{\lambda}\right)\lambda^{[u+1]-u}\varrho^{n/p+[u+1]} \leqslant C_u\varepsilon\left(\frac{2}{\lambda}\right)\varrho^{n/p+u}.$$

This combined with the inequality obtained above gives

$$\frac{1}{\varrho^u}\left[\frac{1}{\varrho^n}\int_{|x|\leqslant\varrho} |R^\lambda(x)|^p\,dx\right]^{1/p} \leqslant C_u\left[\varepsilon(2\varrho) + \varepsilon\left(\frac{2}{\lambda}\right)\right]$$

for any ϱ and λ. Consequently,

$$\frac{1}{\varrho^u}\left[\frac{1}{\varrho^n}\int_{|x|\leqslant\varrho}|R^\lambda-R|^p dx\right]^{1/p}\leqslant C_u\left[\varepsilon(2\varrho)+\varepsilon(\varrho)+\varepsilon\left(\frac{2}{\lambda}\right)\right]\leqslant C_u\left[\varepsilon(2\varrho)+\varepsilon\left(\frac{2}{\lambda}\right)\right],$$

and thus the left hand side converges to zero uniformly in ϱ as $\lambda\to\infty$. This proves (ii). Part (iii) is a consequence of the fact that functions f in $t_u^p(x_0)$ with compact supports are dense in $t_u^p(x_0)$, and for each such f, f^λ belongs to C_0^∞.

LEMMA 2.4. *Let $f\in T_u^p(x_0)$, $1\leqslant p\leqslant\infty$, $u\geqslant -n/p$, and $g\in T_v(x_0)$, $v\geqslant u$, $v\geqslant 0$. Then $fg\in T_u^p(x_0)$ and $T_u^p(x_0,fg)\leqslant CT_u^p(x_0,f)\,T_v(x_0,g)$. If $f\in t_u^p(x_0)$ and $g\in t_v(x_0)$, then $fg\in t_u^p(x_0)$.*

Proof. If $u\leqslant 0$, our assertion is obvious. Suppose that $u>0$, and let r be the largest integer less than u. In view of Lemma 2.1, we may assume that $v=u$. Then $g(x_0+h)=P_1(h)+R_1(h)$, where $P_1(h)$ is polynomial of degree r and $|R_1(h)|\leqslant T_u(x_0,g)\,|h|^u$. On the other hand, $f(x_0+h)=P_2(h)+R_2(h)$, where

$$\left[\frac{1}{\varrho^n}\int_{|h|\leqslant\varrho}|R_2(h)|^p dh\right]^{1/p}\leqslant T_u^p(x_0,f)\,\varrho^u.$$

Let now $\bar P$ denote the sum of terms of degree $\leqslant r$ in P_1P_2. Then

$$f(x_0+h)g(x_0+h)=\bar P(h)+(R_1P_2+R_2g+P_1P_2-\bar P)=\bar P+\bar R,$$

say. Since $|g|\leqslant T_u(x_0,g)$ and $P_2(h)\leqslant T_u^p(x_0,f)$ for $|h|\leqslant 1$, and since the sum of the absolute values of the coefficients of P_1P_2 does not exceed $T_u(x_0,g)T_u^p(x_0,f)$, we have $|P_1P_2-\bar P|\leqslant|h|^{r+1}T_u(x_0,g)\,T_u^p(x_0,f)$ for $|h|\leqslant 1$. From this and the estimates for R_1 and R_2 we obtain, for $\varrho\leqslant 1$,

$$\varrho^{-u}\left[\frac{1}{\varrho^n}\int_{|h|\leqslant\varrho}|\bar R(h)|^p dh\right]^{1/p}\leqslant CT_u(x_0,g)T_u^p(x_0,f),\qquad \varrho\leqslant 1.$$

For $\varrho\geqslant 1$, $|h|\leqslant\varrho$, we use the inequalities $|\bar R|\leqslant|f|\,|g|+|\bar P|$ and $|\bar P(h)|\leqslant\varrho^r T_u(x_0,g)\,T_u^p(x_0,f)$, and obtain

$$\varrho^{-u}\left[\frac{1}{\varrho^n}\int_{|h|\leqslant\varrho}|\bar R(h)|^p dh\right]^{1/p}\leqslant\varrho^{-u}\|f\|_p\|g\|_\infty+C\varrho^{-u+r}T_u(x_0,g)\,T_u^p(x_0,f)$$
$$\leqslant CT_u(x_0,g)\,T_u^p(x_0,f).$$

The other terms that enter in the norm of fg are also majorized by $CT_u(x_0,g)\,T_u^p(x_0,f)$.

When dealing with the spaces $t_u^p(x_0)$ and $t_u(x_0)$ we take r to be the largest integer less than or equal to u. We then have $|R_1(h)|=o(|h|^u)$, $|P_1P_2-\bar P|=o(|h|^u)$ and

$$\left[\frac{1}{\varrho^n}\int_{|h|\leqslant\varrho}|R_2(h)|^p dh\right]^{1/p}=o(\varrho^u),$$

for $|h|$ and $\varrho\to 0$. From this the desired result follows at once.

LEMMA 2.5. *Let $f\in T_u^p(x_0)$, $1\leqslant p\leqslant\infty$, $u\geqslant -n/p$, and let $g\in T_v(x_0)$, $v>0$, $v\geqslant u$, and $g(x_0)=0$. Then $fg\in T_w^p(x_0)$ and $T_w^p(x_0,fg)\leqslant CT_u^p(x_0,f)\,T_v(x_0,g)$, where*

(i) $w=\min(u+v,v)$ *if* $v\leqslant 1$,

(ii) $w=\min(u+1,v)$ *if* $v\geqslant 1$,

and

(iii) $w=u+1$ *if* $u>0$, $v\geqslant 1$ *and* $f(x_0)=0$.

Proof. The argument is parallel to that of the preceding proof. The case $u\leqslant 0$ is immediate. Assume therefore that $u>0$, and let R_1, P_1, R_2, P_2 be as in the preceding proof, and r the largest integer less than w. Then

$$f(x_0+h)g(x_0+h)=\bar P(h)+(R_1P_2+R_2g+P_1P_2-\bar P)=\bar P+\bar R,$$

where $\bar P$ is the sum of terms of P_1P_2 of degrees less than or equal to r.

We then have the inequalities

$$|g(x_0+h)|\leqslant T_v(x_0,g)\,|h|^v,\qquad\text{if}\quad v\leqslant 1;$$
$$|g(x_0+h)|\leqslant T_v(x_0,g)\max(1,|h|),\qquad\text{if}\quad v\geqslant 1;$$
$$|\bar P(h)|\leqslant T_v(x_0,g)\,T_u^p(x_0,f)\max(1,|h|^w).$$

Consequently, if $\varrho\geqslant 1$,

$$\left[\frac{1}{\varrho^n}\int_{|h|\leqslant\varrho}|\bar R(h)|^p dh\right]^{1/p}\leqslant\left[\frac{1}{\varrho^n}\int_{|h|\leqslant\varrho}|g(x_0+h)f(x_0+h)|^p dh\right]^{1/p}+$$
$$+\left[\frac{1}{\varrho^n}\int_{|h|\leqslant\varrho}|\bar P(h)|^p dh\right]^{1/p}\leqslant CT_v(x_0,g)\,T_u^p(x_0,f)\,\varrho^w.$$

If $\varrho\leqslant 1$, we have

$$\left[\frac{1}{\varrho^n}\int_{|h|\leqslant\varrho}|R_2(h)g(x_0+h)|^p dh\right]^{1/p}\leqslant CT_v(x_0,g)\,T_u^p(x_0,f)\,\varrho^w.$$

On the other hand, if $|h|\leqslant 1$ we have $|P_2(h)|\leqslant T_u^p(x_0,f)$ in the cases (i) and (ii), and $|P_2(h)|\leqslant T_u^p(x_0,f)\,|h|$ in the case (iii). Thus, since $|R_1(h)|\leqslant|h|^v T_v(x_0,g)$, we have

$$\left[\frac{1}{\varrho^n}\int_{|h|\leqslant\varrho}|R_1(h)P_2(h)|^p dh\right]^{1/p}\leqslant CT_v(x_0,g)\,T_u^p(x_0,f)\,\varrho^w,$$

if $\varrho \leqslant 1$. Finally, since $|P_1P_2-\bar{P}| \leqslant T_v(x_0, g)T_u^p(x_0, f)|h|^{r+1}$ for $|h| \leqslant 1$, we have

$$\left[\frac{1}{\varrho^n}\int_{|h|\leqslant\varrho}|P_1P_2-\bar{P}|^p\,dh\right]^{1/p} \leqslant CT_v(x_0, g)T_u^p(x_0, f)\,\varrho^w$$

for $\varrho \leqslant 1$. Collecting results we find that

$$\left[\frac{1}{\varrho^n}\int_{|h|\leqslant\varrho}|\bar{R}(h)|^p\,dh\right]^{1/p} \leqslant CT_v(x_0, f)T_u^p(x_0, f)\,\varrho^w$$

for all ϱ. Since the sum of the moduli of the coefficients of $\bar{P}$ does not exceed $T_u^p(x_0, f)\,T_v(x_0, g)$, and since $\|gf\|_p \leqslant T_u^p(x_0, f)\,T_v(x_0, g)$, the lemma follows.

LEMMA 2.6. *Given an integer m, $m \geqslant 0$, there exists a function $\varphi(x)$ infinitely differentiable with support in $|x| \leqslant 1$, such that for every $\lambda > 0$ and every polynomial P of degree $\leqslant m$,*

$$\int \lambda^n\varphi[\lambda(x-y)]P(y)\,dy = P(x)$$

holds.

Proof. Consider the class of all infinitely differentiable functions $\varphi(x)$ supported by $|x| \leqslant 1$, and the mapping of this linear space into the vector space V of points $\{\xi_a\}$, $0 \leqslant |a| \leqslant m$, given by

$$\xi_a = \int \varphi(x)x^a\,dx.$$

If the range of this mapping is not the entire space V, then there exist numbers η_a, $0 \leqslant |a| \leqslant m$ not all zero, such that

$$\sum \eta_a\xi_a = \int\varphi(x)\sum\eta_a x^a\,dx = 0$$

for all φ. In particular, if $\psi(x)$ is an infinitely differentiable function supported by $|x| \leqslant 1$ which is positive in $|x| < 1$, and $\varphi(x)$ is $\psi(x)\sum\eta_a x^a$, we obtain

$$\int\psi(x)\left|\sum\eta_a x^a\right|^2 dx = 0.$$

This implies that $\sum\eta_a x^a = 0$ in $|x| < 1$ and consequently $\eta_a = 0$, $0 \leqslant |a| \leqslant m$, which contradicts our assumption. That is, the range of the mapping is all of V, and therefore there exists a function $\varphi(x)$ such that

$$\int\varphi(x)\,dx = 1, \qquad \int\varphi(x)x^a\,dx = 0, \qquad 0 < |a| \leqslant m.$$

If Q is any polynomial of degree $\leqslant m$, then evidently $\int\varphi(z)Q(z)\,dz = Q(0)$. Given x and λ we change variables in the preceding integral by setting $z = \lambda x - \lambda y$ and replace $Q(z)$ by $Q(z) = P(x - z/\lambda)$, and the desired result follows.

The next proposition seems to be of independent interest.

THEOREM 8. *Let Q be a closed set in E_n and f a function in E_n, such that $f \in T_u^p(x_0)$, $1 \leqslant p \leqslant \infty$, $u > 0$, for all $x_0 \in Q$, with $T_u^p(x_0, f) \leqslant M < \infty$, for $x_0 \in Q$. Then $f \in B_u(Q)$. If in addition $f \in t_u^p(x_0)$ for all $x_0 \in Q$, then $f \in b_u(Q)$.*

Proof. Denote by $f_a(x_0)$ the coefficient of $(x-x_0)^a/a!$ in the expansion of f at x_0, and let $x = x_0 + h$ be another point of Q. Let φ be the function of the preceding lemma with $m > u$, and $\varphi_a(x) = (\partial/\partial x)^a\varphi$. Then if $P(x)$ is any polynomial of degree $\leqslant m$ we have

$$\int\lambda^n\varphi[\lambda(x-y)]P(y)\,dy = P(x),$$

and by differentiating under the integral sign we obtain

$$\int\lambda^{n+|a|}\varphi_a[\lambda(x-y)]P(y)\,dy = \left(\frac{\partial}{\partial x}\right)^a P(x).$$

Consider the case when $f \in T_u^p(x_0)$ for all $x_0 \in Q$. Then we have

$$f(y) = \sum_{|a|<u}\frac{1}{a!}f_a(x_0)(y-x_0)^a + R(x_0, y),$$

$$f(y) = \sum_{|a|<u}\frac{1}{a!}f_a(x)(y-x)^a + R(x, y),$$

where

$$\left[\int_{|y-x_0|\leqslant\varrho}|R(x_0, y)|^p dy\right]^{1/p} \leqslant M\varrho^{n/p+u},$$

and similarly for $R(x, y)$. Set now $\lambda^{-1} = |h| = |x-x_0|$ and given β, $0 \leqslant |\beta| < u$, consider the expression

$$I = \int\lambda^{n+|\beta|}\varphi_\beta[\lambda(x-y)]f(y)\,dy.$$

If we replace here $f(y)$ by the first of its expressions above we get

$$I = \sum_{|a|<u}\frac{1}{a!}f_a(x_0)\left(\frac{\partial}{\partial x}\right)^\beta(x-x_0)^a + \int\lambda^{n+|\beta|}\varphi_\beta[\lambda(x-y)]R(x_0, y)\,dy$$

$$= \sum_{|\gamma|<u-|\beta|}\frac{1}{\gamma!}f_{\beta+\gamma}(x_0)(x-x_0)^\gamma + \int\lambda^{n+|\beta|}\varphi_\beta[\lambda(x-y)]R(x_0, y)\,dy.$$

On the other hand, if we use the second expression for $f(y)$, we find

$$I = f_\beta(x) + \int\lambda^{n+|\beta|}\varphi_\beta[\lambda(x-y)]R(x, y)\,dy.$$

Consequently,

$$f_\beta(x) = \sum_{|\gamma|<u-|\beta|} \frac{1}{\gamma!} f_{\beta+\gamma}(x_0)(x-x_0)^\gamma + \int \lambda^{n+|\beta|}\varphi_\beta[\lambda(x-y)][R(x,y)- -R(x_0,y)]dy.$$

Since $\varphi(x)$ vanishes for $|x|\geqslant 1$, and $\lambda = |x-x_0|^{-1} = |h|^{-1}$, if N is a bound for $|\varphi_\beta|$ we find that the last integral is dominated by

$$\Big[\int_{|y-x_0|\leqslant 2|h|} |R(x_0,y)|\,dy + \int_{|y-x|\leqslant |h|} |R(x,y)|dy\Big]\,|h|^{-n-|\beta|}N,$$

and an application of Hölder's inequality shows that the last expression is less than or equal to $C_{\beta u}M\,|h|^{u-|\beta|}$.

A parallel argument gives the desired result in the case $f\in t_u^p(x_0)$. It is not difficult to see that in this case the theorem holds also if $u=0$.

3. The results of this section will only be needed in the proof of theorems 2, 3, 5 and 7. Essentially they are reformulations and slight extensions of theorems of Whitney [14] and Marcinkiewicz [10]. The proof of theorem 9 is taken from [5; II, p. 57].

Lemma 3.1. *Let Q be a closed set in E_n and U the neighborhood of Q consisting of all points of E_n whose distance from Q is less than 1. Then there is a covering of $U-Q$ by means of non-overlapping closed cubes K with the property that $\frac{1}{2}\leqslant d_j/e_j\leqslant 1+\sqrt{n}$, where e_j is the length of the edge of K_j and d_j is the distance between K_j and Q.*

Proof. Consider the subdivisions π_l of E_n into the cubes $m_i/2^l\leqslant x_i \leqslant (m_i+1)/2^l$, $l=0,1,\ldots$ Select all cubes in π_0 which intersect $U-Q$ and are at distance not less than $\frac{1}{2}$ from Q. Having already selected cubes from π_{j-1}, select all cubes in π_j which are not contained in the previously selected cubes which intersect $U-Q$, and which are at distance no less than 2^{-j-1} from Q. The collection K_j of cubes thus obtained has the required properties. First of all, every $x_0\in U-Q$ is contained in one of the K_j; for if d is the distance between x_0 and Q, and $\bar{K}$ is the cube in π_ν, with $d>2^{-\nu+1}\sqrt{n}$, which contains x_0, then the distance between $\bar{K}$ and Q is not less than $2^{-\nu+1}\sqrt{n}-2^{-\nu}\sqrt{n} = 2^{-\nu}\sqrt{n}$, whence it follows that if $\bar{K}$ was not selected at the ν^{th} step then $\bar{K}$ was contained in one of the previously selected cubes. It is clear that the distance between each K_j and Q is not less than one half the length of the edge of K_j. Finally consider K_j and let $e_j=2^{-\nu}$ be the length of its edge. Let $\bar{K}$ be the cube in $\pi_{\nu-1}$ which contains K_j. Since $\bar{K}$ was not selected the distance between $\bar{K}$ and Q must be less than $2^{-\nu}$, consequently the distance d_j between K_j and Q is less than $2^{-\nu}(1+\sqrt{n})$, whence $d_j/e_j<1+\sqrt{n}$.

Lemma 3.2. *Let Q and U be as in the preceding lemma, and let $d(x)$ denote the distance between x and Q. Then there exists an infinitely differentiable function $\delta(x)$ defined in $U-Q$ and a positive number ε, $\varepsilon<1$, such that*

$$d(x)\,\varepsilon\leqslant \delta(x)\leqslant \frac{1}{\varepsilon}\,d(x),\qquad x\in U-Q,$$

$$\left|\left(\frac{\partial}{\partial x}\right)^\alpha \delta(x)\right| \leqslant C_\alpha d(x)^{1-|\alpha|},\qquad x\in U-Q.$$

Proof. Let K_j be the covering of $U-Q$ of the preceding lemma, and $x^{(j)}$ the center of K_j. Let $\eta(x)\geqslant 0$ be an infinitely differentiable function which vanishes outside the cube

$$-\frac{1}{2}-\frac{1}{4\sqrt{n}}\leqslant x_j\leqslant \frac{1}{2}+\frac{1}{4\sqrt{n}},\qquad j=1,2,\ldots,n,$$

and which exceeds 1 in $-\frac{1}{2}\leqslant x_j\leqslant\frac{1}{2}$. Set

$$\delta(x) = \sum_{j=1}^{\infty} e_j\,\eta\left(\frac{x-x^{(j)}}{e_j}\right).$$

We shall show that $\delta(x)$ has the required properties. It is readily verified that $\eta\left(\frac{x-x^{(j)}}{e_j}\right)$ vanishes in Q. On the other hand, as we will presently prove, there is an integer m such that no more than m terms of the series are distinct from zero at each point.

For let $x_0\in U-Q$ and let $\eta\left(\frac{x_0-x^{(j)}}{e_j}\right)\neq 0$. Then the distance between x_0 and K_j does not exceed $e_j/4$ and since $\frac{1}{2}e_j\leqslant d_j\leqslant(1+\sqrt{n})e_j$ we find that $d(x_0)$, that is the distance between x_0 and Q, satisfies the inequalities

$$(1+\sqrt{n})e_j+\frac{e_j}{4}+e_j\sqrt{n}\geqslant d(x_0)\geqslant\frac{e_j}{4},$$

where $e_j\sqrt{n}$ is the diameter of K_j. This means that a sphere with center at x_0 and radius $d(x_0)(1+4\sqrt{n})$ contains all the cubes K_j corresponding to terms of the series not vanishing at x_0. Since these cubes are disjoint and their edges are not less than $d(x_0)[5/4+2\sqrt{n}]^{-1}$, it follows that their number does not exceed a fixed integer m.

Let N_α be a bound for $(\partial/\partial x)^\alpha\eta(x)$. Then differentiating the series term by term we find that

$$\left|\left(\frac{\partial}{\partial x}\right)^\alpha\delta(x)\right|\leqslant\sum e_j^{1-|\alpha|}N_\alpha,$$

the sum being extended over all j for which $\eta\left(\dfrac{x-x^{(j)}}{e_j}\right)$ does not vanish. But for these terms we have $e_j \leqslant 4d(x)$, from which it follows that

$$\left|\left(\frac{\partial}{\partial x}\right)^{\alpha}\delta(x)\right| \leqslant 4mN_{\alpha}d(x)^{1-|\alpha|},$$

if $|\alpha|\leqslant 1$; and similarly for $|\alpha|>1$, since $e_j \geqslant Cd(x)$ (see below). Finally, if $x\in K_j$, then clearly

$$\delta(x) \geqslant e_j\eta\left(\frac{x-x^{(j)}}{e_j}\right) = e_j,$$

and since $x\in K_j$ implies that $d(x)\leqslant d_j+e_j\sqrt{n}\leqslant(1+2\sqrt{n})e_j$, it follows that $\delta(x)\geqslant d(x)(1+2\sqrt{n})^{-1}$. The lemma is thus established.

THEOREM 9. *Let f be a function in L^p, $1\leqslant p\leqslant\infty$, such that $f\in T_u^p(x_0)$ and $T_u^p(x_0,f)\leqslant M<\infty$ for all x_0 in a closed set Q. Then there exists a function $\bar{f}$ in $B_u(E_n)$ such that, for $|\beta|<u$, $(\partial/\partial x)^{\beta}\bar{f}(x_0)=f_{\beta}(x_0)$ for $x_0\in Q$. If in addition $f\in t_u^p(x_0)$ for all $x_0\in Q$, then $\bar{f}$ can be chosen to be in $b_u(E_n)$ in such a way that $(\partial/\partial x)^{\beta}\bar{f}(x_0)=f_{\beta}(x_0)$ for $|\beta|\leqslant u$, and all $x_0\in Q$.*

Proof. Let U be a neighborhood of Q as in Lemma 3.2 and $\delta(x)$ the corresponding function. For $x\in Q$ define $\bar{f}(x)=f(x)$, and for $x\in U-Q$, set

$$\bar{f}(x) = \delta(x)^{-n}\int\varphi[(x-y)\delta^{-1}(x)]f(y)\,dy,$$

where $\varphi(x)$ is the function in Lemma 2.6.

Let us consider the case $f\in T_u^p(x_0)$ for all $x_0\in Q$. We shall study first the behaviour of $\bar{f}$ near points x in $U-Q$. By differentiation under the integral sign it is readily seen that $\bar{f}(x)$ is infinitely differentiable in $U-Q$. Let x be given and $\bar{x}$ be a point in Q such that $|x-\bar{x}|=d(x)$. Then we can write

$$f(x) = \sum_{|\alpha|<u}\frac{f_{\alpha}(\bar{x})}{\alpha!}(x-\bar{x})^{\alpha}+R(\bar{x},x).$$

Let $\Phi_{\beta}(x,y)=(\partial/\partial x)^{\beta}\{\delta(x)^{-n}\varphi[(x-y)\delta(x)^{-1}]\}$. By differentiation under the integral sign we obtain

$$\left(\frac{\partial}{\partial x}\right)^{\beta}\bar{f}(x) = \left(\frac{\partial}{\partial x}\right)^{\beta}\left[\delta(x)^{-n}\int\varphi[(x-y)\delta^{-1}(x)]\left[\sum_{|\alpha|<u}\frac{f_{\alpha}(\bar{x})}{\alpha!}(y-\bar{x})^{\alpha}\right]dy\right]+$$
$$+\int\Phi_{\beta}(x,y)R(\bar{x},y)\,dy.$$

According to Lemma 2.6, the first term on the right is equal to

$$\left(\frac{\partial}{\partial x}\right)^{\beta}\left[\sum_{|\alpha|<u}\frac{f_{\alpha}(\bar{x})}{\alpha!}(x-\bar{x})^{\alpha}\right].$$

Consequently we have

$$\left(\frac{\partial}{\partial x}\right)^{\beta}\bar{f}(x) = \sum_{|\gamma|<u-|\beta|}\frac{f_{\beta+\gamma}(\bar{x})}{\gamma!}(x-\bar{x})^{\gamma}+\int\Phi_{\beta}(x,y)R(\bar{x},y)\,dy. \tag{3.1}$$

To estimate the remainder here, we need estimates for $\Phi_{\beta}(x,y)$. It is readily verified by induction that $\Phi_{\beta}(x,y)$ is a sum of terms of the form

$$\mathrm{const}\cdot\delta^{-n-r}(x)\varphi_{\gamma}[(x-y)\delta^{-1}]\cdot p\cdot(x-y)^{\alpha},$$

where p is a product of derivatives of δ, such that if t is the number of factors in p and w the sum of its orders, then $r+w-t-|\alpha|=|\beta|$, and φ_{γ} is a derivative of φ with $|\gamma|\leqslant|\beta|$. Since $\varphi[(x-y)\delta^{-1}(x)]$ vanishes for $|x-y|>\delta(x)$, we have $|(x-y)^{\alpha}|\leqslant\delta(x)^{|\alpha|}$ on the support of $\Phi_{\beta}(x,y)$. From this and the estimates for $\delta(x)$ and its derivatives it follows that

$$|\Phi_{\beta}(x,y)|\leqslant C_{\beta}d(x)^{-n-|\beta|}.$$

Consequently we have

$$\left|\int\Phi_{\beta}(x,y)R(\bar{x},y)\,dy\right|\leqslant C_{\beta}d(x)^{-n-|\beta|}\int|R(\bar{x},y)|\,dy,$$

where the integral on the right is extended over $|y-x|\leqslant\delta(x)$, which is contained in the sphere $|y-\bar{x}|\leqslant cd(x)$, since $|x-\bar{x}|=d(x)$ and $\delta(x)\leqslant\leqslant\dfrac{1}{\varepsilon}d(x)$. Now

$$\int\limits_{|y-\bar{x}|\leqslant cd(x)}|R(\bar{x},y)|\,dy\leqslant CT_u^p(\bar{x},f)[cd(x)]^{n+u}\leqslant CM[cd(x)]^{n+u},$$

and from this it follows that

$$\left|\left(\frac{\partial}{\partial x}\right)^{\beta}\bar{f}(x)-\sum_{|\gamma|<u-|\beta|}\frac{f_{\beta+\gamma}(\bar{x})}{\gamma!}(x-\bar{x})^{\gamma}\right|\leqslant C_{\beta u}M|x-\bar{x}|^{u-|\beta|}. \tag{3.2}$$

Let us consider now any point $\bar{x}_1$, in Q. Given the assumptions on f, it follows from Theorem 8 that

$$\left|f_{\alpha}(\bar{x})-\sum_{|\gamma|<u-|\alpha|}\frac{f_{\alpha+\gamma}(\bar{x}_1)}{\gamma!}(\bar{x}-\bar{x}_1)^{\gamma}\right|\leqslant C_{\alpha u}M|\bar{x}-\bar{x}_1|^{u-|\alpha|}.$$

If we replace these values of $f_{\alpha}(\bar{x})$ in (3.2) we obtain

$$\left|\left(\frac{\partial}{\partial x}\right)^{\beta}\bar{f}(x)-\sum_{|\gamma|<u-|\beta|}\frac{(x-\bar{x})^{\gamma}}{\gamma!}\sum_{|\eta|<u-|\beta+\gamma|}\frac{f_{\beta+\gamma+\eta}(\bar{x}_1)}{\eta!}(\bar{x}-\bar{x}_1)^{\eta}\right|$$
$$\leqslant C_{\beta u}M|x-\bar{x}|^{u-|\beta|}+\sum_{|\gamma|<u-|\beta|}C_{\beta+\gamma,u}M|x-\bar{x}|^{|\gamma|}|\bar{x}-\bar{x}_1|^{u-|\beta|-|\gamma|}.$$

Since $|x-\bar{x}| \leqslant |x-\bar{x}_1|$ and $|\bar{x}-\bar{x}_1| \leqslant |\bar{x}-x|+|x-\bar{x}_1| \leqslant 2|x-\bar{x}_1|$, using Taylor's expansion for polynomials this inequality gives

$$\left|\left(\frac{\partial}{\partial x}\right)^{\beta}\bar{f}(x)-\sum_{|\gamma|<u-|\beta|}\frac{f_{\beta+\gamma}(\bar{x}_1)}{\gamma!}(x-\bar{x}_1)^{\gamma}\right| \leqslant C_{\beta u}M|x-\bar{x}_1|^{u-|\beta|}, \tag{3.3}$$

which holds for any $x \in U-Q$ and any $\bar{x}_1 \in Q$. By Theorem 8 the inequality also holds if x and $\bar{x}_1$ belong both to Q. This shows that the functions $(\partial/\partial x)^{\beta}\bar{f}(x)$, $x \in U-Q$, are continuous bounded extensions of the functions $f_{\beta}(x)$, $x \in Q$, and that these extensions have a Taylor expansion at each point of Q. Since $\bar{f}$ is infinitely differentiable in $U-Q$, it follows that $\bar{f}$ has continuous bounded derivatives of orders less than u in U. It remains to show that the highest order derivatives of $\bar{f}$ satisfy a Hölder condition of appropriate order.

Let r be the largest integer less than u. Then, if $|\beta| = r$ and $x_1 \in Q$, the inequality (3.3) gives

$$\left|\left(\frac{\partial}{\partial x}\right)^{\beta}\bar{f}(x_1)-\left(\frac{\partial}{\partial x}\right)^{\beta}\bar{f}(x_2)\right| \leqslant C_u M|x_1-x_2|^{u-r}.$$

If both x_1 and x_2 belong to $U-Q$, we distinguish two cases. First, if $|x_1-x_2| \geqslant d(x_1)/2$, then if $\bar{x} \in Q$ is such that $|x_1-\bar{x}| \leqslant 2|x_1-x_2|$, we have $|x_2-\bar{x}| \leqslant |x_2-x_1|+|x_1-\bar{x}| \leqslant 3|x_1-x_2|$; and from

$$\left|\left(\frac{\partial}{\partial x}\right)^{\beta}\bar{f}(x_1)-\left(\frac{\partial}{\partial x}\right)^{\beta}\bar{f}(\bar{x})\right| \leqslant C_u M|x_1-\bar{x}|^{u-r},$$

$$\left|\left(\frac{\partial}{\partial x}\right)^{\beta}\bar{f}(x_2)-\left(\frac{\partial}{\partial x}\right)^{\beta}\bar{f}(\bar{x})\right| \leqslant C_u M|x_2-\bar{x}|^{u-r}$$

we obtain

$$\left|\left(\frac{\partial}{\partial x}\right)^{\beta}\bar{f}(x_1)-\left(\frac{\partial}{\partial x}\right)^{\beta}\bar{f}(x_2)\right| \leqslant C_u M|x_2-x_1|^{u-r}.$$

If, on the other hand, $|x_1-x_2| < d(x_1)/2$, we let $\bar{x}$ be a point in Q such that $|x_1-\bar{x}| = d(x_1)$, and represent $(\partial/\partial x)^{\beta}\bar{f}$ by means of (3.1). Using the mean value theorem we obtain

$$\left|\left(\frac{\partial}{\partial x}\right)^{\beta}\bar{f}(x_1)-\left(\frac{\partial}{\partial x}\right)^{\beta}\bar{f}(x_2)\right| \leqslant |x_1-x_2|\sum_{j=1}^{n}\int\left|\frac{\partial}{\partial x_j}\Phi(x_0,y)\right||R(\bar{x},y)|\,dy,$$

where x_0 is a point of the segment x_1, x_2.

Now $\partial\Phi(x_0,y)/\partial x_j$ vanishes outside the sphere $|y-x_0| \leqslant cd(x_0)$ and is dominated in absolute value by $C_{\beta}d(x_0)^{-n-|\beta|-1} = C_{\beta}d(x_0)^{-n-r-1}$, whence

$$\begin{aligned}\left|\left(\frac{\partial}{\partial x}\right)^{\beta}f(x_1)-\left(\frac{\partial}{\partial x}\right)^{\beta}\bar{f}(x_2)\right| \\ \leqslant C_{\beta}|x_1-x_2|\,d(x_0)^{-n-r-1}\int_{|y-x_0|\leqslant cd(x_0)}|R(\bar{x},y)|\,dy.\end{aligned} \tag{3.4}$$

Since $d(x_0) \geqslant d(x_1)-|x_1-x_2| > d(x_1)/2 > |x_1-x_2|$, the sphere $|y-x_0| \leqslant cd(x_0)$, is contained in the sphere $|y-\bar{x}| \leqslant cd(x_0)+|x_0-\bar{x}| \leqslant cd(x_0)+|x_1-x_2|+d(x_1) \leqslant (c+3)d(x_0)$, and therefore

$$\int_{|y-x_0|\leqslant cd(x_0)} R(\bar{x},y)\,dy \leqslant \int_{|y-\bar{x}|\leqslant(c+3)d(x_0)}|R(\bar{x},y)|\,dy \leqslant M[(c+3)d(x_0)]^{n+u}.$$

This, combined with $d(x_0) > |x_1-x_2|$ shows that the left-hand side of (3.4) is dominated by $C_u M|x_1-x_2|^{u-r}$.

The function $\bar{f}$ as constructed above is defined only in U; finding an $\bar{f}$ which is defined everywhere now offers little difficulty: we merely multiply the $\bar{f}$ already obtained by an infinitely differentiable function $\eta(x)$ with bounded derivatives of all orders and which is equal to 1 for $d(x) \leqslant \frac{1}{2}$ and vanishes for $d(x) \geqslant \frac{3}{4}$.

The case when $f \in t_u^p(x_0)$ is treated in exactly the same way, and further explanations do not seem to be necessary.

Corollary. *Let $f \in T_u^p(x_0)$, $T_u^p(x_0,f) \leqslant M < \infty$, $1 \leqslant p < \infty$, for all x_0 in a closed set Q. Then $f = f_1+f_2$, where $f_1 \in B_u(E_n)$ and $f_2 \in T_u^p(x_0)$ for all $x_0 \in Q$, and*

$$\left[\frac{1}{\varrho^n}\int_{|x-x_0|\leqslant\varrho}|f_2(x)|^p\,dx\right]^{1/p} \leqslant C_u M\varrho^u$$

for all $x_0 \in Q$ and all $\varrho > 0$. If, in addition, $f \in t_u^p(x_0)$ for all $x_0 \in Q$, then the left-hand side of the inequality above is $o(\varrho^u)$ as $\varrho \to 0$.

This is merely a reformulation of the preceding theorem with $f_1 = \bar{f}$.

Theorem 10. *Let $f \in L^p$, $1 \leqslant p \leqslant \infty$, be such that*

$$\left[\frac{1}{\varrho^n}\int_{|x-x_0|\leqslant\varrho}|f(x)|^p\,dx\right]^{/p} = O(\varrho^u), \qquad u > 0,$$

for all x_0 in a measurable set S. Then

$$\text{(i)}\quad \int\frac{|f(x)|}{|x-x_0|^{n+u}}\,dx < \infty, \qquad \text{(ii)}\quad \left[\frac{1}{\varrho^n}\int_{|x-x_0|\leqslant\varrho}|f(x)|^p\,dx\right]^{1/p} = o(\varrho^u), \quad \varrho \to 0,$$

for almost all x_0 in S.

Proof. We may assume without loss of generality that the set S is bounded. Given $\varepsilon > 0$ we can find a closed subset Q of S such that $S-Q$ has measure less than ε and that

$$\left[\frac{1}{\varrho^n}\int_{|x-x_0|\leqslant\varrho}|f(x)|^p\,dx\right]^{1/p} \leqslant M\varrho^u, \qquad M < \infty,$$

for all x_0 in Q and all $\varrho > 0$. Let U be a neighborhood of Q and K_j a covering of $U-Q$ as in Lemma 3.1. Let $d(x)$ denote the distance between

x and Q. Since the distance between the complement $\overline{U}$ of U and Q is 1, it follows that

$$\int_{\overline{U}} \frac{|f(x)|}{|x-x_0|^{n+u}}\,dx < \infty$$

for all $x_0 \in Q$. On the other hand,

$$(3.5)\qquad \int_Q dx_0 \int_{\overline{U}} \frac{|f(x)|}{|x-x_0|^{n+u}}\,dx = \int_Q dx_0 \sum_j \int_{K_j} \frac{|f(x)|}{|x-x_0|^{n+u}}\,dx$$

$$= \sum_j \int_{K_j} |f(x)| \Big[\int_Q \frac{dx_0}{|x-x_0|^{n+u}}\Big]\,dx.$$

If e_j denotes the edge of K_j then, according to Lemma 3.1, $|x-x_0| \geqslant e_j/2$. Further, since the distance between K_j and Q does not exceed $(1+\sqrt{n})e_j$, if $\bar{x}$ in Q is within that distance from K_j we have, setting $\varrho = (1+2\sqrt{n})\,e_j$,

$$\int_{K_j} |f(x)|\,dx \leqslant e_j^{n/q}\Big[\int_{K_j} |f(x)|^p\,dx\Big]^{1/p} \leqslant e_j^{n/q}\Big[\int_{|x-\bar{x}|\leqslant\varrho} |f(x)|^p\,dx\Big]^{1/p}$$

$$\leqslant e_j^{n/q}[(1+2\sqrt{n})\,e_j]^{n/p+u}M, \quad \text{where} \quad q = \frac{p}{p-1}.$$

Consequently,

$$\int_{K_j} |f(x)|\,dx \leqslant CMe_j^{n+u},$$

and since $|x-x_0| \geqslant e_j/2$ for $x \in K_j$,

$$\int_Q \frac{dx_0}{|x-x_0|^{n+u}} \leqslant C_u e_j^{-u},$$

which combined with the previous inequality gives

$$\int_{K_j} |f(x)| \Big[\int_Q \frac{dx_0}{|x-x_0|^{n+u}}\Big]\,dx \leqslant C_u M e_j^n.$$

Summing over j we find that the left-hand side of (3.5) is less than $C_u M \sum_j e_j^n = C_u M\,|U-Q|$, which is finite. Hence the inner integral there is finite for almost all $x_0 \in Q$, and (i) is established.

To prove (ii) in the case $p<\infty$, we merely apply (i) to the function $g = |f|^p$ and conclude that for almost all x_0 in Q we have

$$\int \frac{g(x)}{|x-x_0|^{n+pu}}\,dx = \int \frac{|f(x)|^p}{|x-x_0|^{n+pu}}\,dx < \infty.$$

At every point where this holds (ii) is evidently valid. If $p=\infty$, then $|f(x)| \leqslant M\,d(x)^u$, and (ii) is satisfied at every point of density of Q.

4. We pass to the study of the properties of the fractional integration introduced in definition 5.

LEMMA 4.1. *Let $0<u<n+1$, and let f be a tempered distribution. Then $J^u f = G_u * f$, where*

$$G_u(x) = \gamma(u)\,e^{-|x|}\int_0^{\infty} e^{-|x|t}\Big(t+\frac{t^2}{2}\Big)^{(n-u-1)/2}dt,$$

$$\gamma(u)^{-1} = (2\pi)^{(n-1)/2}\,2^{u/2}\,\Gamma\Big(\frac{u}{2}\Big)\,\Gamma\Big(\frac{n-u+1}{2}\Big).$$

Proof. The inverse Fourier transform of $(1+4\pi^2|x|^2)^{-u/2}$, $u>n\geqslant 2$, can be calculated in polar coordinates by

$$\int_{E_n} (1+4\pi^2|y|^2)^{-u/2}e^{2\pi i(x\cdot y)}\,dy = \int_\Sigma\int_0^\infty (1+4\pi^2\varrho^2)^{-u/2}e^{2\pi i r\varrho\cos\theta}\varrho^{n-1}\,d\varrho\,d\sigma,$$

where $|y|=\varrho$, $|x|=r$, $x\cdot y = r\varrho\cos\theta$, Σ is the sphere $|y|=1$ and $d\sigma$ is the element of area of Σ. If we set

$$\varphi(s) = \int_0^\pi e^{is\cos\theta}(\sin\theta)^{n-2}\,d\theta,$$

and denote by ω_n the area of the unit sphere in E_n, the last integral becomes

$$\omega_{n-1}\int_0^\infty (1+4\pi^2\varrho^2)^{-u/2}\varrho^{n-1}\varphi(2\pi r\varrho)\,d\varrho.$$

Using successively the formulas (6) page 48, (2) page 434 and (4) page 172 of [13], and setting $t=s+1$ in the integral in the last formula we find that the Fourier transform of $(1+4\pi^2|x|^2)^{-u/2}$ is the function $G_u(x)$ of the lemma, provided $u<n+1$.

Consider now the function $G_u(x)$ for $0<u<n+1$. Then, since

$$e^{-|x|}\int_0^\infty e^{-|x|t}\Big(t+\frac{t^2}{2}\Big)^{(n-u-1)/2}dt$$

$$\leqslant e^{-|x|}\int_0^1 \Big(t+\frac{t^2}{2}\Big)^{(n-u-1)/2}dt + C_u e^{-|x|}\int_1^\infty e^{-|x|t}t^{n-u-1}\,dt$$

$$\leqslant C_u e^{-|x|}\big(1+|x|^{-n+u}+\big|\log|x|\big|\big),$$

it follows that $G_u(x)$ is integrable for $0<u<n+1$. In addition $G_u(x)$ is, for each fixed x, an analytic function of u in $0<R(u)<n+1$, which in a neighborhood of each u of the strip is majorized by an integrable

function of x independent of u. From this it follows that the Fourier transform of $G_u(x)$ is also an analytic function of u in $0 < R(u) < n+1$, and consequently it coincides with $(1+4\pi^2|x|^2)^{-u/2}$ there. The assertion of the lemma follows now from convolution theorem for distributions.

The formula for G_u is still valid in the case $n = 1$, for $0 < u < 2$. This follows as before if we use the formulas (28), page 14, of [2_{I}] and formula (19), page 82, of [3_{II}].

Lemma 4.2. *The function $G_u(x)$ of Lemma 1 is non-negative, has integral over E_n equal to 1 and satisfies the following inequalities:*

$$G_u(x) \leqslant C_u e^{-|x|}(1+|x|^{-n+u}), \quad \text{for} \quad 0 < u < n;$$

$$G_u(x) \leqslant Ce^{-|x|}\left(1+\log\frac{1}{|x|}\right), \quad \text{for} \quad u = n;$$

$$\left|\left(\frac{\partial}{\partial x}\right)^a G_u(x)\right| \leqslant C_{u,a}e^{-|x|}(1+|x|^{-n+u-|a|}), \quad |a| > 0, \quad 0 < u < n+1.$$

Proof. We have

$$G_u(x) = \gamma(u)e^{-|x|}\int_0^\infty e^{-|x|t}\left(t+\frac{t^2}{2}\right)^{(n-u-1)/2}dt$$

$$\leqslant C_u e^{-|x|}\left[\int_0^1\left(t+\frac{t^2}{2}\right)^{(n-u-1)/2}dt+\int_1^\infty e^{-|x|t}t^{n-u-1}dt\right],$$

from which the first two inequalities follow. Differentiating the expression for $G_u(x)$ we find by induction that $(\partial/\partial x)^a G_u(x)$ is a sum of terms of the form

$$e^{-|x|}g_r(x)\int_0^\infty e^{-|x|t}t^s\left(t+\frac{t^2}{2}\right)^{(n-u-1)/2}dt$$

where $g_r(x)$ is a homogeneous function of degree $-r$ and $r+s \leqslant |a|$. The desired estimate follows now by decomposing the integral as before.

Definition 8. Let $\mathcal{R}_j$ be the operator on L^p defined by

$$\mathcal{R}_j f = -i\pi^{-(n+1)/2}\Gamma\left(\frac{n+1}{2}\right)\lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon}\frac{x_j-y_j}{|x-y|^{n+1}}f(y)\,dy.$$

Then Λ is the operator on L_1^p given by

$$\Lambda f = i\sum_{j=1}^n \mathcal{R}_j\frac{\partial}{\partial x_j}f.$$

Lemma 4.3. *If $f\in L^p$, $1 \leqslant p < \infty$, $\mathcal{R}_j f$ is defined almost everywhere as an ordinary limit. The operation $\mathcal{R}_j$ transforms L_k^p continuously into*

L_k^p, $k \geqslant 0$, $1 < p < \infty$. The operation Λ transforms L_{k+1}^p continuously into L_k^p, $k \geqslant 0$, $1 < p < \infty$. Furthermore we have

$$\frac{\partial}{\partial x_j}\mathcal{R}_k = \mathcal{R}_k\frac{\partial}{\partial x_j}; \qquad \frac{\partial}{\partial x_j} = -i\mathcal{R}_j\Lambda.$$

Proof. According to [7], Theorem 1, $\mathcal{R}_j$ is defined almost everywhere and is continuous in L^p, and transforms L_k^p into L_k^p. The continuity of $\mathcal{R}_j$ in L_k^p is an immediate consequence of the fact that

$$\frac{\partial}{\partial x_i}\mathcal{R}_j f = \mathcal{R}_j\frac{\partial}{\partial x_i}f \quad \text{for} \quad f\in L_1^p$$

which was established loc. cit. The identity $\partial/\partial x_j = -i\mathcal{R}_j\Lambda$ was proved in [7], p. 309. Since evidently $\partial/\partial x_j$ maps L_{k+1}^p continuously in L_k^p, the same holds for Λ in view of its definition.

Lemma 4.4. *Let $f\in L^p$, $1 \leqslant p \leqslant \infty$, and let $g_j = \partial G_1/\partial x_j$, then the ordinary limit*

$$\lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon} g_j(x-y)f(y)\,dy$$

exists for almost all x.

Proof. Differentiating with respect to x_j the expression for G_1 in Lemma 4.1 and setting $s = t+1$ in the integral one obtains

$$\frac{\partial}{\partial x_j}G_1(x) = g_j(x) = -\gamma(1)2^{(2-n)/2}\frac{x_j}{|x|}\int_1^\infty e^{-|x|s}s(s^2-1)^{(n-2)/2}ds.$$

Now, for $s \geqslant 1$ we have $s(s^2-1)^{(n-2)/2} = s^{n-1}+O(s^{n-2})$ and consequently,

$$g_j(x) = C\frac{x_j}{|x|^{n+1}}+r(x),$$

where $r(x) = O(|x|^{-n+1})$ as $|x|\to 0$ if $n > 1$, or $r(x) = O(\log|x|)$ if $n = 1$. In either case r is locally integrable.

We may assume that f has compact support. Then

$$\int_{|x-y|>\varepsilon} g_j(x-y)f(y)\,dy = C\int_{|x-y|>\varepsilon}\frac{x_j-y_j}{|x-y|^{n+1}}f(y)\,dy+\int_{|x-y|>\varepsilon}r(x-y)f(y)\,dy.$$

The second integral on the right is absolutely convergent for $\varepsilon = 0$ and almost all x. On the other hand, according to Lemma 4.3, the first integral has a limit as ε tends to zero for almost all x.

Lemma 4.5. *If $f\in L_1^2$, then $\widehat{\Lambda f} = 2\pi|x|\hat{f}$.*

Proof. It is well known, see e. g., [9] or [7], p. 914, that if $g \in L^2$, then

$$\widehat{\mathcal{R}_j g} = -\frac{x_j}{|x|}\hat{g},$$

and this combined with the definition of Λ gives the desired identity.

In what follows we will write systematically J for J^1.

Lemma 4.6. *The operator J transforms L^p continuously into L_1^p, $1 < p < \infty$. Furthermore for $m \geqslant 0$ we have*

$$\Lambda J = I + a_1 J^2 + a_2 J^4 + \ldots + a_m J^{2m} - Q_m,$$

where $a_j < 0$ for all j, $\sum_{j=1}^{\infty} a_j = -1$ and the operation Q_m is convolution with a positive integrable function with derivatives which are integrable up to order $2m+1$, and bounded and continuous up to order $2m+1-n$, if $2m+1 \geqslant n$. In particular, $I - \Lambda J$ is convolution with a positive integrable function of integral 1, and with integrable first order derivatives.

Proof. Let f be a function in C_0^∞, that is, f is infinitely differentiable and has compact support. According to Lemmas 4.1 and 4.2, $Jf = G_1 * f$, where G_1 is a positive integrable function of integral equal to 1. By differentiating $G_1 * f$ under the integral sign it follows that Jf is in L_1^2. Consequently, by Lemma 4.5, we have

$$\widehat{\Lambda J f} = 2\pi|x|(1+4\pi^2|x|^2)^{-1/2}\hat{f}.$$

Set $u = (1+4\pi^2|x|^2)^{-1/2}$. Then

$$\begin{aligned}2\pi|x|(1+4\pi^2|x|^2)^{-1/2} &= \sqrt{1-u^2} = 1 + a_1 u^2 + a_2 u^4 + \ldots + a_m u^{2m} + \ldots \\ &= 1 + a_1 u^2 + \ldots + a_m u^{2m} - R_m(u).\end{aligned}$$

The coefficients a_j are all negative and $\sum_{j=1}^{\infty} a_j = -1$; and $0 \leqslant R_m(u) \leqslant u^{2m+2}$ for $0 \leqslant u \leqslant 1$. Consequently

$$1 - 2\pi|x|(1+4\pi^2|x|^2)^{-1/2} = -\sum_{j=1}^{\infty} a_j[(1+4\pi^2|x|^2)^{-1/2}]^{2j},$$

and this shows that $I - \Lambda J$ is convolution with a positive integrable function of integral equal to 1. Furthermore,

$$R_m[(1+4\pi^2|x|^2)^{-1/2}] = -\sum_{j=m+1}^{\infty} a_j[(1+4\pi^2|x|^2)^{-1/2}]^{2j},$$

and this in turn shows that Q_m is also a convolution with a positive integrable function h_m. We shall now show that h_m has the properties stated in the lemma.

We have in fact $\hat{h}_m(x) = R_m(u) \leqslant u^{2m+2}$, where $u = (1+4\pi^2|x|^2)^{-1/2}$. Consequently $x^a \hat{h}_m(x)$ is integrable for $|a| \leqslant 2m+1-n$ and therefore $h_m(x)$ has continuous bounded derivatives up to order $2m+1-n$. Further, from the last inequality of Lemma 4.2, it follows that $\partial G_u(x)/\partial x_j$ is integrable for $1 < u < n+1$, that is $x_j(1+4\pi^2|x|^2)^{-u/2}$ has an integrable inverse Fourier transform. Since $(1+4\pi^2|x|^2)^{-1/2}$ also has an integrable Fourier transform we find that the inverse Fourier transform of $x^a(1+4\pi^2|x|^2)^{-m-1}$ is integrable for $|a| \leqslant 2m+1$. From this it follows that $(\partial/\partial x)^a h_m$, whose Fourier transform is

$$(2\pi i x)^a (1+4\pi^2|x|^2)^{-m-1} \sum_{m+1}^{\infty} a_j (1+4\pi^2|x|^2)^{-j+m+1},$$

is integrable for $|a| \leqslant 2m+1$.

It remains to show that J transforms L^p into L_1^p. If $f \in C_0^\infty$, then $Jf = G_1 * f \in L_1^p$ as we already saw, and according to Lemma 4.3, $\partial Jf/\partial x_j = -i\mathcal{R}_j \Lambda J f$. Since $\Lambda J - I$ is convolution with an integrable function and $\mathcal{R}_j$ is continuous in L^p, $1 < p < \infty$, it follows that $\|\partial Jf/\partial x_j\|_p \leqslant C_p\|f\|_p$. If f is now a function in L^p and f_n is a sequence of functions in C_0^∞ converging to f in L^p, Jf_n converges in L_1^p. This shows that $Jf \in L_1^p$.

Lemma 4.7. *If m is an integer, $m \geqslant -k$, and $1 < p < \infty$ then J^m transforms L_k^p continuously onto L_{k+m}^p.*

Proof. Since J transforms L^p continuously into L_1^p (see Lemma 4.6) and since $(\partial/\partial x)^a J = J(\partial/\partial x)^a$, as seen by taking Fourier transforms, it follows that J transforms L_k^p continuously into L_{k+1}^p. On the other hand, since $J^{-1} = (I-\Delta)J$ and since $(1-\Delta)$ maps L_{k+2}^p continuously into L_k^p we find that J^{-1} maps L_{k+1}^p continuously into L_k^p. From this the lemma follows.

Lemma 4.8. *If $f \in L_k^p$ then $f \in L^q$ with $1/q = 1/p - k/n$ if $1 \leqslant p < n/k$, or q is any number $p \leqslant q < \infty$ if $p = n/k$, or $q = \infty$ if $p > n/k$.*

Proof. The case $1 \leqslant p < n/k$ is an immediate consequence of Soboleff's theorem which also holds for $p = 1$ (see [11]). The case $p = n/k$ follows from the fact that $f = J^k g = G_k * g$, where $g \in L^p$, from the inequalities for G_k given in Lemma 4.2 and Young's theorem on convolutions. The case $p > n/k$ is obtained by applying Hölder's inequality to $G_k * g$.

Proof of Theorem 4. Let $f \in T_u^p(x_0)$ and assume for simplicity that $x_0 = 0$. Then $f = P + R$ where P is a polynomial of degree $< u$ if $u > 0$, or zero if $u \leqslant 0$ and $R(x)$ is such that

$$\left[\frac{1}{\varrho^n}\int_{|x|\leqslant\varrho} |R(x)|^p\,dx\right]^{1/p} \leqslant T_u^p(x_0, f)\,\varrho^u. \tag{4.1}$$

We shall first consider the case where $0 < v < n$. We have

$$J^v f = G_v * f = G_v * P + G_v * R.$$

Since, according to Lemma 4.2, $G_v(x)$ decreases exponentially at infinity, both convolutions on the right of the preceding equation are meaningful. Furthermore, by differentiating under the integral sign one sees readily that $G_v * P$ is a polynomial of degree $< u$ whose coefficients are dominated in absolute value by $C_{u,v} T_u^p(x_0, f)$.

Consider now the integrals

$$\int\limits_{|x|\leqslant\varrho} |R(x)|\,|x|^{-r}dx, \qquad \int\limits_{|x|\geqslant\varrho} |R(x)|\,|x|^{-r}dx.$$

If we set

$$\varphi(\varrho) = \int\limits_{|x|\leqslant\varrho} |R(x)|\,dx$$

and use Hölder's inequality and (4.1) we obtain

$$\varphi(\varrho) \leqslant C\Big[\int\limits_{|x|\leqslant\varrho} |R(x)|^p dx\Big]^{1/p} \varrho^{\frac{n(p-1)}{p}} \leqslant C T_u^p(x_0, f)\,\varrho^{n+u}.$$

Hence, if $n+u-r>0$

$$\int\limits_{\varepsilon<|x|<\varrho} |R(x)|\,|x|^{-r}dx = \int_\varepsilon^\varrho s^{-r}d\varphi(s) \leqslant r\int_0^\varrho \varphi(s)s^{-r-1}ds + \varphi(\varrho)\varrho^{-r}$$

$$\leqslant C T_u^p(x_0, f)\Big[r\int_0^\varrho s^{n+u-r-1}ds + \varrho^{n+u-r}\Big] \leqslant C_{r,u} T_u^p(x_0, f)\,\varrho^{n+u-r},$$

that is, if $n+u-r>0$, then

$$\int\limits_{|x|\leqslant\varrho} |R(x)|\,|x|^{-r}dx \leqslant C_{r,u} T_u^p(x_0, f)\,\varrho^{n+u-r}. \tag{4.2}$$

Similarly, if $n+u-r<0$, then

$$\int\limits_{|x|\geqslant\varrho} |R(x)|\,|x|^{-r}dx = \int_\varrho^\infty s^{-r}d\varphi(s) \leqslant r\int_\varrho^\infty s^{-r-1}\varphi(s)\,ds,$$

so that

$$\int\limits_{|x|\geqslant\varrho} |R(x)|\,|x|^{-r}dx \leqslant C_{r,u} T_u^p(x_0, f)\,\varrho^{n+u-r}. \tag{4.3}$$

Let us write g for G_v and g_a for $(\partial/\partial x)^a G_v$, and assume that $2|x|\leqslant\varrho$. Then, by the mean-value theorem,

$$(G_v * R)(x) = g * R = \int g(x-y)R(y)\,dy = \int\limits_{|y|\leqslant\varrho} g(x-y)R(y)\,dy +$$

$$+\sum_{|a|\leqslant u+v}\frac{x^a}{a!}\int g_a(-y)R(y)\,dy - \sum_{|a|\leqslant u+v}\frac{x^a}{a!}\int\limits_{|y|\leqslant\varrho} g_a(-y)R(y)\,dy + \tag{4.4}$$

$$+\sum_{|a|=[u+v]+1}\frac{x^a}{a!}\int\limits_{|y|>\varrho} g_a(\Theta x-y)R(y)\,dy\ ^{(1)},$$

(1) If $u+v<0$ we merely decompose the integral of $g(x-y)R(y)$ into two, extended over $|y|\leqslant\varrho$ and $|y|\geqslant\varrho$ respectively, and the argument simplifies.

where $0<\Theta<1$. Now, according to Lemma 4.1, $|g_a(-y)|\leqslant C_{a,v}[1+|y|^{-n+v-|a|}]e^{-|y|}\leqslant C_{av}|y|^{-n+v-|a|}$ whence setting $r=n-v+|a|$ in (4.2) it follows that for $|a|<u+v$ the integral

$$\int g_a(-y)R(y)\,dy \tag{4.5}$$

is absolutely convergent and is dominated in absolute value by $C_{uv}T_u^p(x_0, f)$. Furthermore

$$\Big|\int\limits_{|y|\leqslant\varrho} g_a(-y)R(y)\,dy\Big| \leqslant C_{uv}T_u^p(x_0, f)\,\varrho^{u+v-|a|} \tag{4.6}$$

for $|a|\leqslant u+v$.

For the functions $g_a(\Theta x-y)$, $|a|=[u+v]+1$, we have $|g_a(\Theta x-y)|\leqslant C_{av}|\Theta x-y|^{-n+v-|a|}\leqslant C_{av}|y|^{-n+v-|a|}$ if $|y|\geqslant\varrho\geqslant 2|x|$. Consequently it follows from (4.3) on setting $r=n-v+|a|$,

$$\Big|\int\limits_{|y|>\varrho} g_a(\Theta x-y)R(y)\,dy\Big| \leqslant C_{u,v}\,T_u^p(x_0, f)\,\varrho^{u+v-|a|}. \tag{4.7}$$

It remains to estimate the first term of the right-hand side of (4.4). If $1/p-v/n<0$ the inequality $|g(y)|\leqslant C_v|y|^{-n+v}$, (4.1) and Hölder's inequality give, with $q=p/(p-1)$,

$$\begin{aligned}\Big|\int\limits_{|y|\leqslant\varrho} g(x-y)R(y)\,dy\Big| &\leqslant \Big[\int\limits_{|y|\leqslant\varrho}|R(y)|^p dy\Big]^{1/p}\Big[\int\limits_{|y|\leqslant 2\varrho}|g(y)|^q dy\Big]^{1/q}\\ &\leqslant C_{pv}T_u^p(x_0, f)\,\varrho^{u+v}.\end{aligned} \tag{4.8}$$

If, on the other hand, $1/p-v/n>0$, replacing $g(x-y)$ by $C_v|y|^{-n+v}$ and applying Soboleff's theorem with $1/q=1/p-v/n$ we obtain

$$\begin{aligned}\Big[\int dx\Big|\int\limits_{|y|\leqslant\varrho} g(x-y)R(y)\,dy\Big|^q\Big]^{1/q} &\leqslant C_{pv}\Big[\int\limits_{|y|\leqslant\varrho}|R(y)|^p dy\Big]^{1/p}\\ &\leqslant C_{pv}T_u^p(x_0, f)\,\varrho^{n/p+u} = C_{pv}T_u^p(x_0, f)\,\varrho^{n/q+u+v}.\end{aligned} \tag{4.9}$$

It now follows from the estimates (4.6) to (4.9) that the assertion of the theorem is valid if $f\in T_u^p(x_0)$ with $1/q=1/p-v/n$ if $1/p-v/n>0$, or $q=\infty$ if $1/p-v/n<0$, provided $0<v<n$. This result can now be extended to general v by repeated application of the case $0<v<n$ using the group properties of J^v.

To cover the case $1/p=v/n$ and the other values of q in the other cases we argue as follows.

Suppose that $f\in T_u^p(x_0)$ and $J^v f\in T_{u+v}^r(x_0)$ with $r\geqslant p$. Then $J^v f\in T_{u+v}^s(x_0)$ for all s, $p\leqslant s\leqslant r$, and

$$T_{u+v}^s(x_0, J^v f)\leqslant C[T_{u+v}^r(x_0, J^v f)+\|f\|_p].$$

This is an immediate consequence of the inequalities

$$\left[\frac{1}{\varrho^n}\int\limits_{|x-x_0|\leqslant\varrho}|R(x)|^s dx\right]^{1/s}\leqslant C\left[\frac{1}{\varrho^n}\int\limits_{|x-x_0|\leqslant\varrho}|R(x)|^r dx\right]^{1/r},$$

$$\|J^v f\|_s\leqslant\|J^v f\|_r^{\Theta}\|J^v f\|_p^{1-\Theta}\leqslant\|J^v f\|_r^{\Theta}\|f\|_p^{1-\Theta}\leqslant\|J^v f\|_r+\|f\|_p,$$

which are obtained by applying Hölder's inequality, and where R is the remainder of the expansion if $J^v f$ at x_0, and $0\leqslant\theta\leqslant 1$.

This combined with the results already obtained gives: if $1<p\leqslant\infty$ and $u+v\neq 0,1,2,\ldots$, then J^v maps $T_u^p(x_0)$ continuously into $T_{u+v}^q(x_0)$ provided

(a) $\dfrac{1}{p}\geqslant\dfrac{1}{q}\geqslant\dfrac{1}{p}-\dfrac{v}{n}$, if $\dfrac{1}{p}>\dfrac{v}{n}$,

(b) $p\leqslant q\leqslant\infty$, if $\dfrac{1}{p}<\dfrac{v}{n}$,

(c) $p\leqslant q<\infty$, if $\dfrac{1}{p}=\dfrac{v}{n}$.

Only (c) requires additional explanation. If $1/p=v/n$ and $0<\varepsilon<v$, then, according to (a), J^ε maps $T_u^p(x_0)$ continuously into $T_{u+\varepsilon}^p(x_0)$ and $J^{v-\varepsilon}$ maps $T_{u+\varepsilon}^p(x_0)$ continuously into $T_{u+\varepsilon}^q(x_0)$ with $1/p\geqslant 1/q\geqslant 1/p-(v-\varepsilon)/n=\varepsilon/n$. Thus $J^v=J^{v-\varepsilon}J^\varepsilon$ maps $T_u^p(x_0)$ continuously into $T_{u+v}^q(x_0)$ for all q such that $1/p\geqslant 1/q\geqslant\varepsilon/n$.

To prove that $J^v f\epsilon t_{u+v}^q(x_0)$ if $f\epsilon t_u^p(x_0)$ it is enough to observe that if $f\epsilon C_0^\infty$ then $J^v f$ is infinitely differentiable and thus belongs to $t_{u+v}^q(x_0)$. Since, according to Lemma 2.3, C_0^∞ is dense in $t_u^p(x_0)$ and $t_{u+v}^q(x_0)$ is a closed subspace of $T_{u+v}^q(x_0)$ the desired conclusion is obtained by a passage to the limit.

Remark. If $u+v$ is a non-negative integer then the preceding argument shows that $J^v f\epsilon T_{u+v}^q(x_0)$ (or $t_{u+v}^q(x_0)$) if we assume in addition that

$$\int\frac{|R(y)|}{|y-x_0|^{n+u}}dy<\infty.$$

This validates (4.4) for $|\alpha|=u+v$, which is what fails otherwise if $u+v$ is an integer. An alternative assumption could be $f(x_0+h)=f(x_0-h)$ if $u+v$ is odd, or $f(x_0+h)=-f(x_0-h)$ if $u+v$ is even. This makes the left-hand side of (4.6) vanish if $|\alpha|=u+v$.

The proof of Theorem 5 will be based on Theorem 11 below. The latter is analogous to the special case of Theorem 4 when $v=1$ and makes stronger assumptions, but in return its conclusions are also stronger: the case $p=1$ is included, and the only exceptional u is $u=-1$.

THEOREM 11. *Let f have first order derivatives f_j in L^p, and let $f_j\epsilon T_u^p(x_0)$, $j=1,2,\ldots,n$, $-n/p\leqslant u\neq -1$. If*

1° $1\leqslant p<n$, and $f\epsilon L^q$ ([2]) with $1/q=1/p-1/n$, then $f\epsilon T_{u+1}^q(x_0)$ and

$$T_{u+1}^q(x_0,f)\leqslant C_{pu}\sum_{j=1}^n T_u^p(x_0,f_j);$$

2° if $n<p\leqslant\infty$ and $f\epsilon B(E_n)$, then $f\epsilon T_{u+1}(x_0)$ and

$$T_{u+1}(x_0,f)\leqslant C_{pu}\Big[B(f)+\sum_{j=1}^n T_u^p(x_0,f_j)\Big],$$

where $B(f)$ is the essential least upper bound of $|f|$;

3° if $f\epsilon L^r$ with $1/p\geqslant 1/r>1/p-1/n$, then $f\epsilon T_{u+1}^r(x_0)$ and

$$T_{u+1}^r(x_0,f)\leqslant C_{ru}\Big[\|f\|_r+\sum_{j=1}^n T_u^p(x_0,f_j)\Big];$$

4° the preceding statements hold with the spaces T replaced by the spaces t.

Proof. We will first prove the inequality in 3° with $p=r$ assuming that $f\epsilon C_0^\infty$. Let

$$k_j(x)=\frac{1}{\omega_n}\frac{x_j}{|x|^n},$$

where ω_n is the surface area of the unit sphere $|x|=1$. Then

$$\sum_1^n\left(\frac{\partial}{\partial x_j}\right)k_j(x)=0,$$

and consequently if $f\epsilon C_0^\infty$ using Green's formula we obtain

$$\sum_{j=1}^n\int\limits_{|x-y|\geqslant\varrho}k_j(x-y)f_j(y)\,dy=\frac{1}{\omega_n}\int\limits_{|y-x|=\varrho}\frac{f(y)}{|x-y|^{n-1}}d\sigma,$$

where $d\sigma$ is the area element of the sphere $|x-y|=\varrho$. As ϱ tends to zero the right-hand side tends to $f(x)$. Thus we have the following representation of $f(x)$:

$$f(x)=\sum_{j=1}^n\int k_j(x-y)f_j(y)\,dy.$$

([2]) The assumption $f\epsilon L^q$ is almost superfluous. For if $f_j\epsilon T_u^p(x_0)$, then $f_j\epsilon L^p$ and using Soboleff's Theorem one can show that f differs by an additive constant from a function in L^q. Our assumption makes it certain that this constant is zero.

Let us consider first the case $-n/p \leqslant u < -1$. Let $|x| \leqslant \varrho$ and set

$$f(x) = \sum \int_{|x_0-y|\leqslant 2\varrho} k_j(x-y) f_j(y)\,dy + \\ + \sum_j \int_{|x_0-y|\geqslant 2\varrho} k_j(x-y) f_j(y)\,dy = f_1(x)+f_2(x),$$

say. Then an application of Young's theorem on convolution gives

$$\text{(4.10)}\quad \begin{aligned} \Big[\frac{1}{\varrho^n}\int_{|x_0-y|\leqslant\varrho} |f_1(y)|^p dy\Big]^{1/p} &\leqslant \sum \Big[\int_{|y|\leqslant 4\varrho} |k_j(y)|\,dy\Big]\times \\ &\times\Big[\frac{1}{\varrho^n}\int_{|x_0-y|\leqslant 2\varrho} |f_j(y)|^p dy\Big]^{1/p} \leqslant C\varrho \sum_{j=1}^{n} T_n^p(x_0, f_j)\,\varrho^u. \end{aligned}$$

To estimate $f_2(x)$ we proceed as follows. First we observe that

$$\frac{1}{\varrho^n}F_j(\varrho) = \frac{1}{\varrho^n}\int_{|x_0-y|\leqslant\varrho} |f_j(y)|\,dy \leqslant C\Big[\frac{1}{\varrho^n}\int_{|x_0-y|\leqslant\varrho} |f_j(y)|^p dy\Big]^{1/p} \leqslant C T_u^p(x_0, f_j)\,\varrho^u,$$

whence

$$\Big|\int_{|x_0-y|\geqslant 2\varrho} k_j(x-y) f_j(y)\,dy\Big| \leqslant C\int_{2\varrho}^{\infty} \frac{dF_j(s)}{s^{n-1}} \leqslant C_u T_u^p(x_0, f_j)\,\varrho^{u+1}.$$

Consequently,

$$\text{(4.11)}\qquad \Big[\frac{1}{\varrho^n}\int_{|x_0-y|\leqslant\varrho} |f_2(y)|^p dy\Big]^{1/p} \leqslant C_u \sum T_u^p(x_0, f_j)\,\varrho^{u+1}.$$

From this and (4.10) we obtain

$$\text{(4.12)}\qquad \Big[\frac{1}{\varrho^n}\int_{|x_0-y|\leqslant\varrho} |f(y)|^p dy\Big]^{1/p} \leqslant C_u \sum T_u^p(x_0, f_j)\,\varrho^{u+1}.$$

We next prove a similar inequality for $-1 < u < 0$. We have again

$$\begin{aligned} f(x)-f(x_0) &= \sum \int_{|x_0-y|\leqslant 2\varrho} k_j(x-y) f_j(y)\,dy + \\ &\quad + \sum \int_{|x_0-y|\geqslant 2\varrho} [k_j(x-y)-k_j(x_0-y)] f_j(y)\,dy - \\ &\quad - \sum \int_{|x_0-y|\leqslant 2\varrho} k_j(x_0-y) f_j(y)\,dy \\ &= f_1(x)+f_2(x)+\varepsilon(\varrho), \end{aligned}$$

say. Then, as before, $f_1(x)$ satisfies (4.10). If $F_j(\varrho)$ is defined as in the previous case, then, since $|k_j(x-y)-k(x_0-y)| \leqslant C\varrho\,|x_0-y|^{-n}$ for $|x_0-y| \geqslant 2\varrho$, we see readily that $f_2(x)$ satisfies (4.11). Finally,

$$\Big|\int_{|x_0-y|\leqslant 2\varrho} k_j(x_0-y) f_j(y)\Big| = C\int_0^r \frac{\varrho\,dF_j(s)}{s^{n-1}} \leqslant C_u T_u^p(x_0, f_j)\,\varrho^{u+1}.$$

Combining these results we obtain

$$\text{(4.13)}\qquad \Big[\frac{1}{\varrho^n}\int_{|x_0-y|\leqslant\varrho} |f(y)-f(x_0)|^p dy\Big]^{1/p} \leqslant C_u \sum T_u^p(x_0, f_j)\,\varrho^{u+1}.$$

We now consider the case $u \geqslant 0$. Let P denote the sum of the terms of degree less than $u+1$ of the Taylor expansion of f at x_0; set $\bar f = f-P$ and $\bar f_j = f_j - P_j$. Then, using polar coordinates,

$$\begin{aligned} \Big[\int_{|y-x_0|\leqslant\varrho} |\bar f(y)|^p dy\Big]^{1/p} &\leqslant \Big[\int_{r\leqslant\varrho} r^{n-1}\Big|\int_0^r \sum \bar f_j(s,\omega)\,a_j\,ds\Big|^p dr\,d\omega\Big]^{1/p} \\ &\leqslant C\sum_j \Big[\int_{r\leqslant\varrho} r^{n+p-2}\int_0^r |\bar f_j(s,\omega)|^p ds\,dr\,d\omega\Big]^{1/p} \\ &\leqslant C\varrho^{(n+p-1)/p}\sum_j \Big[\int_{s\leqslant\varrho} |\bar f_j(s,\omega)|^p ds\,d\omega\Big]^{1/p} \\ &= C\varrho^{(n+p-1)/p}\sum_j \Big[\int_{|y-x_0|\leqslant\varrho} \frac{|\bar f_j(y)|^p}{|y-x_0|^{n-1}}\,dy\Big]^{1/p}. \end{aligned}$$

On the other hand, setting

$$\int_{|y-x_0|\leqslant\varrho} |\bar f_j(y)|^p dy = F_j(\varrho),$$

we find that $F_j(\varrho) \leqslant T_u^p(x_0, f_j)^p\,\varrho^{n+pu}$ and

$$\int_{|y-x_0|\leqslant\varrho} \frac{|\bar f_j(y)|^p}{|y-x_0|^{n-1}}\,dy = \int_0^\varrho \frac{dF_j(s)}{s^{n-1}} \leqslant C T_u^p(x_0, f_j)\,\varrho^{pu+1}.$$

This combined with the inequality obtained previously gives

$$\text{(4.14)}\qquad \Big[\frac{1}{\varrho^n}\int_{|y-x_0|\leqslant\varrho} |f(y)-P(y)|^p dy\Big]^{1/p} \leqslant C\sum_j T_u^p(x_0, f_j)\,\varrho^{u+1}.$$

Since all the coefficients of P except the constant term enter in the $T_u^p(x_0, f_j)$ it remains to estimate $P(x_0) = f(x_0)$. Let $\varphi(x)$, $|\varphi| \leqslant 1$, be a function in C_0^∞ which is equal to 1 in $|x| \leqslant 1$ and vanishes in $|x| \geqslant 2$. Set $\psi(x) = \varphi(x-x_0)$. Then

$$f(x_0) = f(x_0)\psi(x_0) = \sum_j \int k_j(x_0-y)\psi(y)f_j(y)\,dy + \sum_j \int k_j(x_0-y)f(y)\psi_j(y)\,dy,$$

whence

$$|f(x_0)| \leqslant \sum_j \int_{|y-x_0|\leqslant 2} \frac{|f_j(y)|\,dy}{|y-x_0|^{n-1}} + C\int_{|y-x_0|\leqslant 2} |f(y)|\,dy.$$

Let $u > -1$ and $v = \min(u, 0)$; then, by Lemma 2.1,

$$F_j(\varrho) = \int_{|y-x_0|\leqslant\varrho} |f_j(y)|\,dy \leqslant C\varrho^n\Big[\frac{1}{\varrho^n}\int_{|y-x_0|\leqslant\varrho} |f_j(y)|^p\,dy\Big]^{1/p} \leqslant CT_v^p(x_0, f_j)\,\varrho^{n+v}$$
$$\leqslant CT_u^p(x_0, f_j)\,\varrho^{n+v}.$$

Consequently,

$$|f(x_0)| \leqslant \sum_j \int_0^2 \frac{dF_j(s)}{s^{n-1}} + C\int_{|y-x_0|\leqslant 2} |f(y|\,dy,$$

$$(4.15)\qquad |f(x_0)| \leqslant C_u \sum T_u^p(x_0, f_j) + C\,\|f\|_p.$$

Now the inequalities (4.12) to (4.15) clearly imply that

$$(4.16)\qquad T_{u+1}^p(x_0, f) \leqslant C_u \sum T_u^p(x_0, f_j) + C\,\|f\|_p.$$

The factor C_u on the right tends to infinity as u tends to -1, but is bounded away from $u = -1$. The argument given above covers also the case $p = \infty$.

To prove 1° let us denote $f(x)-P(x)$ by $\bar{f}(x)$, and let $\varphi(x)$ be again a function which is equal to 1 for $|x| < 1$ and vanishes for $|x| \geqslant 2$. Then

$$\frac{\partial}{\partial y_j}\Big[\bar{f}(y)\varphi\Big(\frac{y-x_0}{\varrho}\Big)\Big] = \bar{f}_j(y)\varphi\Big(\frac{y-x_0}{\varrho}\Big) + \frac{1}{\varrho}\bar{f}(y)\varphi_j\Big(\frac{y-x_0}{\varrho}\Big),$$

and if $1 \leqslant p < n$, Soboleff's theorem, which is also valid when $p = 1$ (see [11]), gives, with $1/q = 1/p - 1/n$,

$$\Big[\frac{1}{\varrho^n}\int_{|y-x_0|\leqslant\varrho} |\bar{f}(y)|^q\,dy\Big]^{1/q} \leqslant \Big[\frac{1}{\varrho^n}\int\Big|\bar{f}(y)\varphi\Big(\frac{y-x_0}{\varrho}\Big)\Big|^q dy\Big]^{1/q}$$
$$\leqslant C_p\varrho\sum_{j=1}^n\Big[\frac{1}{\varrho^n}\int\Big|\bar{f}_j(y)\varphi\Big(\frac{y-x_0}{\varrho}\Big)\Big|^p dy\Big]^{1/p} +$$
$$+ C_p\sum_{j=1}^n\Big[\frac{1}{\varrho^n}\int\Big|\bar{f}(y)\varphi_j\Big(\frac{y-x_0}{\varrho}\Big)\Big|^p dy\Big]^{1/p},$$

and applying (4.12), (4.13) or (4.14) to the second sum, as the case may be, we obtain

$$(4.17)\qquad \Big[\frac{1}{\varrho^n}\int_{|y-x_0|\leqslant\varrho} |f(y)-P(y)|^q\,dy\Big]^{1/q} \leqslant C_{pu}\sum_{j=1}^n T_u^p(x_0, f_j)\,\varrho^{u+1}.$$

On the other hand, since $f_j \in L^p$ it follows from Soboleff's theorem that $f \in L^q$ and $\|f\|_q \leqslant C_p\sum\|f_j\|_p$. Furthermore it is easy to verify that (4.15) holds with $\|f\|_p$ replaced by $\|f\|_q$. Combining this with (4.17) we obtain

$$(4.18)\qquad T_{u+1}^q(x_0, f) \leqslant C_{pu}\sum_{j=1}^n T_u^p(x_0, f_j).$$

If $p > n$, then instead of applying Soboleff's theorem we use the representation

$$(4.19)\ \bar{f}(x)\varphi\Big(\frac{x-x_0}{\varrho}\Big) = \sum_j\int k_j(x-y)\Big[\bar{f}_j(y)\varphi\Big(\frac{y-x_0}{\varrho}\Big) + \frac{1}{\varrho}\bar{f}(y)\varphi_j\Big(\frac{y-x_0}{\varrho}\Big)\Big]dy$$

and Hölder's inequality, and obtain instead of (4.17)

$$\operatorname*{ess\,sup}_{|y-x_0|\leqslant\varrho} |f(y)-P(y)| \leqslant C_{pu}\sum_{j=1}^n T_u^p(x_0, f_j)\,\varrho^{u+1},$$

and thus, if $p > n$,

$$T_{u+1}(x_0, f) \leqslant C_{pu}\sum_{j=1}^n T_u^p(x_0, f_j) + CB(f).$$

To prove the inequality in 3° for general r we use the representation (4.19) for $\bar{f}(x)\varphi\Big(\dfrac{x-x_0}{\varrho}\Big)$ and apply to it Young's theorem on convolution with exponent r for the left side and exponent s, $1/r = 1/p+1/s-1$, for k_j on the right. Then using (4.12), (4.13) or (4.14), as the case may be, we obtain

$$\Big[\frac{1}{\varrho^n}\int_{|y|\leqslant\varrho} |f(y)-P(y)|^r\,dy\Big]^{1/r} \leqslant C_{ru}\sum_j T_u^p(x_0, f_j)\,\varrho^{u+1}.$$

This combined with (4.15) where we can replace $\|f\|_p$ on the right by $\|f\|_r$, gives the inequality in 3°.

So far we have been considering functions in C_0^∞. We now extend these results to functions in $t_u^p(x_0)$. Let f have compact support and have first order derivatives in $t_u^p(x_0)$, and let $f^\lambda = \lambda^n\varphi(\lambda x) * f$ where φ is in C_0^∞ and has integral equal to 1. Then $f^\lambda \in C_0^\infty$ and $f_j^\lambda = \lambda^n\varphi(\lambda x) * f_j$, and, by Lemma 2.3, f_j^λ converges in $T_u^p(x_0)$ to f_j. Applying to f^λ the inequalities we have obtained for functions in C_0^∞ we see that under the various hypothesis of the theorem, f^λ converges in the corresponding space $T_{u+1}^q(x_0)$ with appropriate q (according to Lemma 2.2, $T_{u+1}^q(x_0)$ is complete). On the other hand, f^{λ} converges to f in L^q, consequently $f\in t^q_{u+1}(x_0)$, since it is the limit in $T^q_{u+1}(x_0)$ of functions in C_0^∞ which belong to $t^q_{u+1}(x_0)$ (see Lemma 2.3). In passing to the limit we see that the inequalities also hold for f.

We now consider the case of general f. Let $\varphi(x)$ be a function of C_0^∞ which is equal to 1 in $|x|<1$ and vanishes in $|x|\geqslant 2$. We consider the function $f^\varepsilon(x)=f(x)\varphi[\varepsilon(x-x_0)]$. If $f_j\in t^p_u(x_0)$ then also $f^\varepsilon_j\in t^p_u(x_0)$, provided f belongs to L^p on bounded sets. Now this is part of our hypothesis in all cases. Furthermore, f^ε_j converges to f_j in $T^p_u(x_0)$. To see this we must verify that on the one hand $\|f^\varepsilon_j-f_j\|_p\to 0$ as $\varepsilon\to 0$, and that on the other

$$\sup_\varrho \frac{1}{\varrho^u}\left[\frac{1}{\varrho^n}\int\limits_{|x-x_0|\leqslant\varrho}|R^\varepsilon_j(x)-R_j(x)|^p\,dx\right]^{1/p} \tag{4.18}$$

tends to zero with ε, where R_j and R^ε_j are the remainders of f_j and f^ε_j respectively. Now

$$R^\varepsilon_j(x)-R_j(x)=f^\varepsilon_j(x)-f_j(x)=-f_j(x)[1-\varphi[\varepsilon(x-x_0)]]+\varepsilon f(x)\varphi_j[\varepsilon(x-x_0)],$$

and the first term on the right clearly converges to zero in L^p. For the second term we have

$$\left[\int|\varepsilon f\varphi_j[\varepsilon(x-x_0)]|^p\,dx\right]^{1/p}\leqslant C\varepsilon\left[\int\limits_{1/\varepsilon<|x-x_0|<2/\varepsilon}|f(x)|^p\,dx\right]^{1/p}$$

$$\leqslant C\varepsilon\left[\int\limits_{1/\varepsilon\leqslant|x-x_0|\leqslant 2/\varepsilon}|f(x)|^r\,dx\right]^{1/r}\left(\frac{1}{\varepsilon^n}\right)^{1/p-1/r}=o(\varepsilon^{1-n/p+n/r})=o(1),$$

where r is the exponent of the class to which f belongs (we have, in all cases, $1-n/p+n/\lambda>0$). This shows that $\|f^\varepsilon_j-f_j\|_p\to 0$ as $\varepsilon\to 0$. Since $R^\varepsilon_j(x)-R_j(x)$ vanishes for $|x|<1/\varepsilon$ this also shows that (4.18) tends to zero with ε. Consequently f^ε converges to f in $T^q_{u+1}(x_0)$ and 4° is established.

There only remains the case of $f_j\in T^p_u(x_0)$. By Lemma 2.1 evidently $f_j\in t^p_{u-\varepsilon}(x_0)$ and $T^p_{u-\varepsilon}(x_0,f_j)\leqslant CT^p_u(x_0,f_j)$ for every sufficiently small positive ε ([3]). Consequently $f\in t^q_{u+1-\varepsilon}(x_0)$ with appropriate q and we have that the inequalities of theorem 11 hold for f with u replaced by $u-\varepsilon$ on the left and in the constants. These constants are bounded functions of u for u away from -1. Now it is easy to see that if $T^q_{u+1-\varepsilon}(x_0,f)\leqslant M$ for sufficiently small positive ε, then $f\in T^q_{u+1}(x_0)$ and $T^q_{u+1}(x_0,f)\leqslant M$. Thus we can pass to the limit in the inequalities by letting ε tend to zero. This completes the proof of the theorem ([4]).

THEOREM 12. *If $f\in L^p_k$, $1\leqslant p<\infty$, $k=0,1,2,\ldots$ then $f\in t^q_k(x_0)$ for almost all x_0 with $1/p\geqslant 1/q\geqslant 1/p-k/n$ if $p<n/k$; $p\leqslant q\leqslant\infty$ if $p>n/k$, and $p\leqslant q<\infty$ if $p=n/k$.*

([3]) This presupposes that $u>-n/p$. Observe, however, that $T^p_{-n/p}(x_0)=t^p_{-n/p}(x_0)$.

([4]) Examples showing that the theorem is false for $u=-1$ can be easily constructed by means of the function $\log|x|$.

Proof. The case $k=0$ is the familiar theorem about the Lebesgue set of a function in L^p. The general case is obtained by induction on k using Theorem 11 and noting that in the first place $f_j\in L^p_{k-1}$, secondly that, according to Lemma 4.8, f is bounded if $1/p<k/n$ and $f\in L^r$ for all r, $p\leqslant r<\infty$, if $1/p=n/k$.

LEMMA 4.9. *Let $f\in t^p_k(x_0)$, $1<p\leqslant\infty$, and let*

$$\int\frac{|f(x)|}{|x-x_0|^{n+k}}\,dx<\infty.$$

Then, for $v\geqslant 0$, we have $J^vf\in t^q_{u+v}(x_0)$, where $1/p\geqslant 1/q\geqslant 1/p-v/n$ if $p<n/v$, $p\leqslant q\leqslant\infty$ if $p>n/v$, or $p\leqslant q<\infty$ if $p=n/v$.

Proof. The case $v=0$ is obvious. The finiteness of the integral above implies that the Taylor expansion of f reduces to the remainder $R(x)$, and therefore

$$\int\frac{|R(x)|}{|x-x_0|^{n+k}}\,dx<\infty.$$

Now the assertion of the lemma follows from the remark to the proof of Theorem 4.

Proof of Theorem 5. We shall distinguish two cases namely, $u\geqslant 1$ and $u=0$.

Suppose that $f\in T^p_u(x_0)$, $u\leqslant 1$, $1<p\leqslant\infty$, for all x_0 in a set S of positive measure. We may assume without loss of generality that $T^p_u(x_0,f)$ is bounded on S, and that S itself is closed and bounded. This presupposes the measurability of $T^p_u(x_0,f)$ as a function of x_0; we assume this for the moment. Then according to the corollary of Theorem 9, f can be written as f_1+f_2, where $f_1\in B_u(E_n)$ and f_2 satisfies the hypothesis of Theorem 10 on S. Further f_1 can be chosen to have compact support. But then, if $v\geqslant 0$, J^vf_1 belongs to L^p_{u+v} for all p, and Theorem 12 asserts that $J^vf_1\in t_{u+v}(x_0)$ for almost all x_0. Since f_1 has compact support, $J^vf_1\in L^q$ and thus $J^vf_1\in t^q_{u+v}(x_0)$ for almost all x_0. On the other hand, since f_2 satisfies the hypothesis of Theorem 10 on S, $f\in t^p_u(x_0)$ for almost all x_0 in S and, by Lemma 4.9, $J^vf_2\in t^q_{u+v}(x_0)$ for almost all $x_0\in S$; therefore $J^vf\in t^q_{u+v}(x_0)$ at every point x_0 where $J^vf_1\in t^q_{u+v}(x_0)$ and $J^vf_2\in t^q_{u+v}(x_0)$.

There remains the case $u=0$. If $v=0$, then since $f\in L^p$, $1<p<\infty$, it follows that

$$\left[\frac{1}{\varrho^n}\int\limits_{|h|\leqslant\varrho}|f(x_0+h)-f(x_0)|^p\,dh\right]^{1/p}=o(1),\qquad \varrho\to 0,$$

for almost all x_0, that is $f\in t^p_0(x_0)$ for almost all x_0. If $v\geqslant 1$, then $J^vf\in L^p_v$ and Theorem 12 gives the desired result.

We now sketch briefly the proof of the measurability of $T_u^p(x_0, f)$ as a function of x_0. Let φ be the function of Lemma 2.6 and let

$$f_\lambda(x) = \int \lambda^n \varphi[\lambda(x-y)] f(y)\,dy.$$

Then if $f \in T_u^p(x_0)$ and $f_\alpha(x_0)$ is one of the coefficients of the Taylor expansion of f at x_0 we have

$$\left(\frac{\partial}{\partial x}\right)^\alpha f_\lambda(x_0) = f_\alpha(x_0) + \int \lambda^{n+|\alpha|} \varphi_\alpha[\lambda(x_0-y)] R(y)\,dy, \quad \text{where} \quad \varphi_\alpha = \left(\frac{\partial}{\partial x}\right)^\alpha \varphi,$$

and R is the remainder in the Taylor expansion of f. The integral above is majorized by

$$\lambda^{n+|\alpha|} C_\alpha \int\limits_{|y-x_0| \leqslant 1/\lambda} |R(y)|\,dy = O(\lambda^{|\alpha|-u}),$$

which tends to zero as $\lambda \to \infty$. Consequently the function $f_\alpha(x_0)$ are limits of infinitely differentiable functions on the set where $T_u^p(x_0, f)$ is finite, which was assumed to be measurable, and therefore are measurable. From this the measurability of $T_u^p(x_0, f)$ follows without difficulty.

We conclude this section with a theorem which may be interpreted as an extension of the well known theorem of Lusin on the structure of measurable functions.

THEOREM 13. *Let $f \in L_k^p$, $1 \leqslant p < \infty$, then given $\varepsilon > 0$, there is a function $g(x)$ with continuous derivatives of orders $\leqslant k$, such that $f(x) = g(x)$ outside a set of measure $\leqslant \varepsilon$.*

Proof. According to Theorem 12, $f \in t_k^q(x_0)$ for almost all x_0, where q is some exponent larger than or equal to 1. Since $T_k^q(x_0, f)$ is a measurable function, given ε we can find an open set O such that $T_k^q(x_0, f)$ is bounded outside O, and whose measure is less than ε, and applying Theorem 9 to f and the complement of O the desired result follows.

5. In this section we study the effect of singular integral operators in the classes $T_u^p(x_0)$ and $t_u^p(x_0)$. We will use properties of singular integral operators which were established in [6].

LEMMA 5.1. *Let $\mathcal{K}$ be a convolution singular integral operator defined by*

$$\mathcal{K}f = \lim_{\varepsilon \to 0} \int\limits_{|x-y|>\varepsilon} k(x-y) f(y)\,dy,$$

where $k(x)$ is homogeneous of degree $-n$, is infinitely differentiable in $|x| \neq 0$, and has mean value zero on $|x| = 1$.

Then if $1 < p < \infty$, $-n/p \leqslant u \neq 0, 1, 2, \ldots$, and $f \in T_u^p(x_0)$, we also have $\mathcal{K}f \in T_u^p(x_0)$ and

$$T_u^p(x_0, \mathcal{K}f) \leqslant C_{up} M T_u^p(x_0, f),$$

where M is a bound for the absolute values $|(\partial/\partial x)^\alpha k(x)|$ on $|x| = 1$, $0 \leqslant |\alpha| \leqslant u+1$, if $u > 0$ and $|\alpha| = 0$ if $u \leqslant 0$.

If, in addition, $f \in t_u^p(x_0)$, then $\mathcal{K}f \in t_u^p(x_0)$.

Proof. We assume for simplicity that $x_0 = 0$. We choose once for all a function φ in C_0^∞ which is equal to 1 for $|x| \leqslant 1$, and we set $f = f_1 + f_2$ with $f_1 = \varphi P$ where P is the Taylor expansion of f at 0. Then $f_1 \in T_u^p(x_0)$ and it is not difficult to verify that $T_u^p(x_0, f_1) \leqslant C T_u^p(x_0, f)$; consequently f_2 also belongs to $T_u^p(x_0)$ and $T_u^p(x_0, f_2) \leqslant C T_u^p(x_0, f)$. We will apply $\mathcal{K}$ to f_1 and f_2 separately.

First let us observe that if ψ is a function in C_0^∞ which vanishes outside $|x| \leqslant 2$, then we have

$$\mathcal{K}\psi = \lim_{\varepsilon \to 0} \int\limits_{|y|>\varepsilon} k(y)\psi(x-y)\,dy = \lim_{\varepsilon \to 0} \int\limits_{|y|>\varepsilon} k(y)[\psi(x-y) - \psi(x)]\,dy,$$

which shows that the first integral converges uniformly as $\varepsilon \to 0$ and that

$$|\mathcal{K}\psi| \leqslant C_\psi M,$$

where C_ψ depends on ψ. From the uniform convergence of the integral it follows that

$$\frac{\partial}{\partial x_j} \mathcal{K}\psi = \mathcal{K}\left(\frac{\partial}{\partial x_j}\right)\psi,$$

and thus $\mathcal{K}\psi$ is an infinitely differentiable function and

$$\left|\left(\frac{\partial}{\partial x}\right)^\alpha \mathcal{K}\psi\right| \leqslant C_{\alpha\psi} M.$$

Furthermore $(\mathcal{K}\psi)(x) \leqslant M C_\psi |x|^{-n}$ for $|x| \geqslant 3$. This and the preceding inequalities show that $\|\mathcal{K}\psi\|_p \leqslant M C_{p,\psi}$. It is easy to see now that $\mathcal{K}\psi \in T_u^p(x_0)$ and that

$$T_u^p(x_0, \mathcal{K}\psi) \leqslant C_{up\psi} M.$$

Applying this result to $x^\alpha \varphi(x)$ we find that

$$T_u^p(x_0, \mathcal{K}x^\alpha \varphi) \leqslant C_{p,\alpha} M$$

since the function φ is fixed. Consequently, if $P = \sum a_\alpha x^\alpha$, we have

$$T_u^p(x_0, \mathcal{K}f_1) \leqslant \sum_{|\alpha|<u} |a_\alpha| T_u^p(x_0, \mathcal{K}x^\alpha \varphi) \leqslant C_{up} M T_u^p(x_0, f).$$

This is of course trivial if $u \leqslant 0$, since then $f_1 = 0$.

Consider next the function f_2. Its Taylor expansion vanishes and therefore the inequality $T_u^p(x_0, f_2) \leqslant C T_u^p(x_0, f)$ implies that

$$(5.1) \qquad \left[\frac{1}{\varrho^n} \int\limits_{|y| \leqslant \varrho} |f_2(y)|^p\,dy\right]^{1/p} \leqslant C T_u^p(x_0, f)\,\varrho^u.$$

From this we obtain the following inequalities, which are analogues of (4.2) and (4.3) in the proof of Theorem 4:

$$\int_{|y|\leqslant\varrho} |f_2(y)|\,|y|^{-r}\,dy \leqslant C_{ru} T_u^p(x_0,f)\,\varrho^{n+u-r} \tag{5.2}$$

if $n+u-r>0$, and

$$\int_{|y|\geqslant\varrho} |f_2(y)|\,|y|^{-r}\,dy \leqslant C_{ru} T_u^p(x_0,f)\,\varrho^{n+u-r} \tag{5.3}$$

if $n+u-r<0$. Expanding k by Taylor's formula we can write, if $u\geqslant 0$,

$$\begin{aligned}\int k(x-y)f_2(y)\,dy = &\int_{|y|\leqslant\varrho} f_2(y)\,k(x-y)\,dy + \\ &+\sum_{|a|\leqslant u}\frac{x^a}{a!}\int f_2(y)\,k_a(-y)\,dy+ \\ &+\sum_{u<|a|\leqslant u+1}\frac{x^a}{a!}\int_{|y|\geqslant\varrho} f_2(y)\,k_a(\Theta x-y)\,dy- \\ &-\sum_{|a|\leqslant u}\frac{x^a}{a!}\int_{|y|\leqslant\varrho} f_2(y)\,k_a(-y)\,dy,\end{aligned} \tag{5.4}$$

where the first two integrals are taken in the principal value sense. Since $|k_a(-y)|\leqslant M|y|^{-n-|a|}$ and on account of (5.2), the integrals in the first sum are absolutely convergent near zero. Further $\|f_2\|_p\leqslant CT_u^p(x_0,f)$ and using Hölder's inequality we see that those integrals are also absolutely convergent at infinity. Combining this with (5.2) we see that the first sum on the right of (5.4) is a polynomial $P(x)$ whose coefficients are dominated by $C_{pu}MT_u^p(x_0,f)$.

If we assume that $2|x|\leqslant\varrho$, then $|k_a|(\Theta x-y)|\leqslant C_uM|y|^{-n-|a|}$ for $|y|\geqslant\varrho$. Consequently, it follows from (5.3) that the second sum in (5.4) is dominated by $C_uMT_u^p(x_0,f)\,\varrho^{u-|a|}|x|^{|a|}$.

From (5.2) and the inequality for k_a it follows that the last sum is dominated by $C_uMT_u^p(x_0,f)\,\varrho^{u-|a|}|x|^{|a|}$.

Finally, let us consider the first term on the right of (5.4). From the remark on page 306 of [6] it follows that the norm in L^p of this term is dominated by

$$C_pM\Big[\int_{|y|\leqslant\varrho}|f_2(y)|^p\,dy\Big]^{1/p}\leqslant C_pMT_u^p(x_0,f)\,\varrho^{u+n/p}.$$

Combining these results we finally obtain

$$\Big[\int_{|x|<\varrho/2}|\mathcal{K}f_2-P(x)|^p\,dx\Big]^{1/p}\leqslant C_{up}MT_u^p(x_0,f)\,\varrho^{u+n/p}.$$

Since $\|\mathcal{K}f_2\|_p\leqslant C_pM\|f_2\|_p$ (see [6], loc. cit.), combining this with estimates for the coefficients of $P(x)$ we find that $T_u^p(x_0,\mathcal{K}f_2)\leqslant C_{up}MT_u^p(x_0,f)$. Since the same inequality has been established for $\mathcal{K}f_1$, the proof of the part of the lemma concerning $T_u^p(x_0)$ is complete if $u\geqslant 0$.

If $u<0$, instead of (5.4) we write

$$\int k(x-y)f_2(y)\,dy = \int_{|y|\leqslant\varrho}k(x-y)f_2(y)\,dy+\int_{|y|\geqslant\varrho}k(x-y)f_2(y)\,dy.$$

The first term on the right can be estimated as before, and for $|x|\leqslant\varrho/2$ and $|y|\geqslant\varrho$ we have $|k(x-y)|\leqslant CM|y|^{-n}$, and thus on account of (5.3) with $r=n$ we find that the second term is dominated by $C_uMT_u^p(x_0,f)\,\varrho^u$.

To prove that $\mathcal{K}f\in t_u^p(x_0)$ it is enough to observe that if $f\in C_0^\infty$ then $\mathcal{K}f$ is infinitely differentiable and thus belongs to $t_u^p(x_0)$. Since $\mathcal{K}$ is continuous in $T_u^p(x_0)$ and, according to Lemma 2.3, C_0^∞ is dense in $t_u^p(x_0)$ we conclude that $\mathcal{K}$ maps $t_u^p(x_0)$ into itself.

LEMMA 5.2. *If u is a non-negative integer and the other assumptions of Lemma 5.1 are satisfied, then $\mathcal{K}f\in T_u^p(x_0)$ provided*

$$\int\frac{|f(x)|}{|x-x_0|^{n+u}}\,dx = N<\infty. \tag{5.5}$$

Furthermore,

$$T_u^p(x_0,\mathcal{K}f)\leqslant C_{up}MT_u^p(x_0,f)+MN.$$

If u is odd and $f(x_0+x)$ and $k(x)$ are of the same parity, that is, they are both even or both odd functions of x, then the conclusions of Lemma 5.1 hold without the additional assumption (5.5). If u is even we get the same conclusion provided $f(x_0+x)$ and $k(x)$ are of opposite parity.

Proof. We merely observe that the proof of Lemma 5.1 applies also if u is a non-negative integer, except at one point: the integrals of the functions

$$f_2(y)\,k_a(-y),\qquad |a|=u,$$

are no longer convergent. However, under the additional assumption (5.5) of the present lemma the Taylor expansion P of f must vanish and thus $f_2=f$ and the integrals above converge. Under the other hypotheses f_2 has the same parity as f, if we assume, as we may, that φ is even, and the integrals above can be dropped.

LEMMA 5.3. *Let $u=0$ and let the other assumptions of Lemma 5.1 be satisfied. Let*

$$(\mathcal{K}f)^*(x_0)=\sup_\varepsilon\Big|\int_{|y-x_0|\geqslant\varepsilon}k(x_0-y)f(y)\,dy\Big|<\infty.$$

Then $\mathcal{K}f\in T_0^p(x_0)$ and

$$T_0^p(x_0,\mathcal{K}f)\leqslant C_pMT_0^p(x_0,f)+C(\mathcal{K}f)^*(x_0).$$

Proof. Let $\varrho > 0$ be given and let $f = f_1 + f_2$ where $f_1(x) = f(x)$ if $|x-x_0| \leqslant 2\varrho$, $f_1(x) = 0$ otherwise. Let $g_i = \mathcal{K} f_i$, $i = 1, 2$; $g = \mathcal{K} f$. Then

$$\Big[\int_{|x-x_0|\leqslant\varrho} |g(x)|^p dx\Big]^{1/p} \leqslant \Big[\int_{|x-x_0|\leqslant\varrho} |g_1(x)|^p dx\Big]^{1/p} + \Big[\int_{|x-x_0|\leqslant\varrho} |g_2(x)|^p dx\Big]^{1/p},$$

and since $\mathcal{K}$ is continuous in L^p (see [6], loc. cit.) the right-hand side is less than or equal to

$$C_p M \|f_1\|_p + \Big[\int_{|x_0-x|\leqslant\varrho} dx \Big| \int_{|y-x_0|\geqslant 2\varrho} [k(x-y)-k(x_0-y)] f(y)\, dy\Big|^p\Big]^{1/p} + \tag{5.6}$$
$$+ C\varrho^{n/p} (\mathcal{K} f)^*(x_0).$$

We have here $|k(x-y)-k(x_0-y)| \leqslant CM|x_0-y|^{-n-1}|x-x_0|$. Let

$$F(\varrho) = \int_{|x-x_0|\leqslant\varrho} |f(x)|\, dx.$$

Then Hölder's inequality shows that $F(\varrho) \leqslant C T_0^p(x_0, f)\varrho^n$. Consequently,

$$\Big|\int_{|y-x_0|\geqslant 2\varrho} [k(x-y)-k(x_0-y)] f(y)\, dy\Big| \leqslant CM|x-x_0| \int_{2\varrho}^{\infty} \frac{dF(s)}{s^{n+1}}$$
$$\leqslant CMT_0^p(x_0, f)|x-x_0|\,\varrho^{-1}.$$

Substituting the left-hand side in (5.6) and observing that $\|f_1\|_p \leqslant \varrho^{n/p} T_0^p(x_0, f)$, we obtain the inequality of the lemma.

LEMMA 5.4. *Let $f \in L_k^p$, $1 \leqslant p \leqslant \infty$, and denote $\left(\frac{\partial}{\partial x}\right)^a f$ by f_a. Then*

$$T_k^p(x_0, f) \leqslant C_{kp}\Big[\sum_{|a|\leqslant k-1} \|f_a\|_p + \sum_{|a|=k} T_0^p(x_0, f_a)\Big]$$

for almost all x_0.

Proof. The statement is obtained at once by induction on k from 3° of Theorem 11 with $r = p$ if $p < \infty$, or from 2° if $p = \infty$.

Definition 9. We denote by $\mathcal{R}_{lm}$ a convolution singular integral operator whose kernel is of the form $Y_{lm}(x|x|^{-1})|x|^{-n}$, where Y_{lm} is a complete orthonormal system of spherical harmonics and m denotes the degree of the harmonic.

LEMMA 5.5. *With the notation of Lemma 5.3 we have, for $1 < p < \infty$ and $-n/p \leqslant u \neq 0, 1, 2, \ldots$,*

$$\|\mathcal{R}_{lm} f\|_p \leqslant C_p \|f\|_p; \quad \|(\mathcal{R}_{lm} f)^*\|_p \leqslant C_p \|f\|_p.$$
$$T_u^p(x_0, \mathcal{R}_{lm} f) \leqslant C_{up} m^v T_u^p(x_0, f),$$

where $v = (n-2)/2 + [u+1]$ if $u \geqslant 0$ and $v = (n-2)/2$ if $u < 0$.

Proof. The first two inequalities follow immediately from [6], p. 306 and 307, Remark, taking into account that the Y_{lm} are normalized. The third inequality is a consequence of Lemma 5.1 and of the inequalities

$$\Big|\left(\frac{\partial}{\partial x}\right)^a Y_{lm}(x|x|^{-1})|x|^{-n}\Big| \leqslant C_a m^{(n-2)/2+|a|}, \qquad |x| \geqslant 1,$$

which follow immediately from the inequality (4) of [7].

The theorem which follows has Theorem 6 as an immediate corollary.

THEOREM 14. *Let $\mathcal{K}$ be a singular integral operator of class $T_u(x_0)$, $u \geqslant 0$, with kernel $k(x, x-y)$ and symbol $h(x, z)$. Let*

$$a_{lm}(x) = (-1)^v \gamma_m^{-1} [m(m+n-2)]^{-v} \int_{|z|=1} Y_{lm}(z) L^v h(x, z)\, d\sigma, \tag{5.7}$$

where $Lg(z) = |z|^2 \Delta g(z)$, $\gamma_m = i^m \pi^{m/2} \Gamma(\frac{1}{2}m)/\Gamma(\frac{1}{2}m+\frac{1}{2}n)$, $v = [n+(u+1)/2]$. Then

(i) *$a_{lm}(x) \in T_u(x_0)$ and $T_u(x_0, a_{lm}) \leqslant Cm^{\frac{1}{2}n-2v} T_u(x_0, \mathcal{K})$;*

(ii) *$\mathcal{K} = a(x) + \sum a_{lm}(x) \mathcal{R}_{lm}$ in the following senses:*

(a) *for $f \in L^p$, $1 < p < \infty$, and for almost every x, the principal value integrals in $\mathcal{K} f$ and $\mathcal{R}_{lm} f$ exist and the series $a(x)f(x) + \sum a_{lm}(x) \mathcal{R}_{lm} f$ converges absolutely to $\mathcal{K} f$.*

(b) *$\mathcal{K}$ is a bounded operator in L^p and $T_u^p(x_0)$, provided $u \neq 0, 1, 2, \ldots$, and the series converges to $\mathcal{K} f$ absolutely in the operator norm. The norms of $\mathcal{K}$ in L^p and $T_u^p(x_0)$ do not exceed $C_p T_u(x_0, \mathcal{K})$ and $C_{up} T_u(x_0, \mathcal{K})$ respectively.*

(iii) *If $\mathcal{K} \in t_u(x_0)$, then $a_{lm}(x) \in t_u^p(x_0)$ and if $f \in t_u^p(x_0)$, $u \neq 0, 1, 2, \ldots$, the function $\mathcal{K} f$ also belongs to $t_u^p(x_0)$.*

Proof. The $a_{lm}(x)$ are nothing but the coefficients of the expansion of the kernel $k(x, z)$ in series of spherical harmonics $Y_{lm}(z)$ on $|z| = 1$. The expression (5.7) for the $a_{lm}(x)$ given above was obtained in [7], p. 913. Part (i) of the theorem follows from (5.7) taking into account that for each z, $|z| = 1$, the derivatives $\left(\frac{\partial}{\partial z}\right)^a h(x, z)$, $|a| \leqslant 2n+u$, belong to $T_u(x_0)$ and their norms in $T_u(x_0)$ are dominated by $T_u(x_0, \mathcal{K})$ (see definition 6).

To prove part (a) of (ii) let us consider $(\mathcal{R}_{lm} f)^*$. Then since, by Lemma 5.5, $\|(\mathcal{R}_{lm} f)^*\|_p \leqslant C_p \|f\|_p$ and since for each m the number of distinct spherical harmonics of degree m is of the order m^{n-2}, it follows from (i) that $\sum |a_{lm}(x)| \leqslant CT_u(x_0, \mathcal{K})$ and consequently the series

$$\sum |a_{lm}(x)| (\mathcal{R}_{lm} f)^* \tag{5.8}$$

is finite almost everywhere. On the other hand, since $|Y_{lm}(z)| \leqslant Cm^{(n-2)/2}$

(see [7], p. 903], and since the number of distinct spherical harmonics of degree m is of the order m^{n-2}, it again follows from (i) that

$$\sum |a_{lm}(x)|\,|Y_{lm}(z)| \leqslant CT_u(x_0, \mathcal{K}).$$

But $k(x, z)|z|^n = \sum a_{lm}(x) Y_{lm}(z)$ and $f(y)|x-y|^{-n}$ is, by Hölder's inequality, integrable in $|x-y| > \varepsilon$, whence

$$(5.9)\qquad \int\limits_{|x-y|>\varepsilon} k(x, x-y) f(y)\,dy = \sum a_{lm}(x) \int\limits_{|x-y|>\varepsilon} Y_{lm}\left(\frac{x-y}{|x-y|}\right) |x-y|^{-n} f(y)\,dy.$$

If x is point where $\mathcal{R}_{lm}f$ exists as principal value integral for all l and m, and where (5.8) is finite, then, since the terms on the right of (5.9) are dominated by the corresponding terms of (5.8), we can pass to the limit termwise on the right of (5.9), as ε tends to zero. Thus (ii) (a) is established.

To prove (ii) (b) we merely have to observe that, by Lemma 5.5, the norm in L^p of $a_{lm}(x)\mathcal{R}_{lm}$ is $\leqslant C_p \sup\limits_x |a_{lm}(x)| \leqslant C_p T_u(x_0, a_{lm})$ and thus, by (i), the series of the norms of these operators is finite.

Similarly, by Lemmas 5.5 and 2.4, the norm of $a_{lm}(x)\mathcal{R}_{lm}$ in $T_u^p(x_0)$ is less than or equal to $C_{up} m^v T_u(x_0, a_{lm})$, where v is as in Lemma 5.5. It now follows again from (i) that the series of the norms in $a_{lm}(x)\mathcal{R}_{lm}$ in $T_u(x_0)$ is finite.

The proof of (iii) is merely a repetition of the preceding one and rests on the completeness of $t_u^p(x_0)$.

Proof of Theorem 7. Since, as it was shown in the proof of Theorem 5, the norm $T_u^p(x_0, f)$ is a measurable function of x_0 on Q, we may assume without loss of generality that Q is compact and that $T_u^p(x_0, f)$ is a bounded function of x_0 on Q. We first assume that u is a positive integer. Then according to the corollary of Theorem 9 and Theorem 10 we can decompose f as $f = f_1 + f_2$, where $f_1 \in B_u(E_n)$ and has compact support and f_2 is such that

$$(5.10)\qquad \int \frac{|f_2(x)|}{|x-x_0|^{n+u}}\,dx = N_2(x_0) < \infty$$

for almost all x_0 in Q. Since the function f_1 belongs to $B_u(E_n)$ and has compact support it belongs to L_u^p. Let $f_{1a} = (\partial/\partial x)^a f_1$. Then since, according to Lemma 5.5, $\|(\mathcal{R}_{lm} f_{1a})^*\|_p \leqslant C_p \|f_{1a}\|_p$ and the number of distinct $\mathcal{R}_{lm}$ for each m is of the order m^{n-2}, the sum

$$\sum_{|a|=u} \sum_{l,m} m^{-n} (\mathcal{R}_{lm} f_{1a})^*(x_0) = N_1(x_0)$$

is finite for almost all x_0. Let now x_0 be a point where $N_1(x_0)$ and $N_2(x_0)$ are finite and in addition $f_{1a} \in T_0^p(x_0)$, $|a| = u$, and let $\mathcal{K}$ be an operator in $T_u(x_0)$. We will show that $\mathcal{K}f \in T_u^p(x_0)$ and that

$$(5.11)\qquad T_u^p(x_0, \mathcal{K}f) \leqslant C_{up} T_u(x_0, \mathcal{K}) \Big[T_u^p(x_0, f_2) + \sum_{|a|<u} \|f_{1a}\|_p + \sum_{|a|=u} T_0^p(x_0, f_{1a}) + N_1(x_0) + N_2(x_0) \Big].$$

For this purpose it will be enough to show that the sum of the norms in $T_u^p(x_0)$ of the terms of the series

$$\mathcal{K}f = \sum a_{lm}(x) \mathcal{R}_{lm} f_1 + \sum a_{lm}(x) \mathcal{R}_{lm} f_2$$

is finite.

We have

$$T_u^p(x_0, a_{lm}\mathcal{R}_{lm} f_2) \leqslant CT_u(x_0, a_{lm}) T_u^p(x_0, \mathcal{R}_{lm} f_2) \leqslant Cm^{n/2-2n-2[(u+1)/2]} T_u(x_0, \mathcal{K}) M_m [C_{up} T_u^p(x_0, f_2) + N_2(x_0)].$$

The first inequality follows from Lemma 2.4. The second inequality, in which M_m is the constant associated with $\mathcal{R}_{lm}$ as in Lemma 5.1, follows from part (i) of Theorem 14, and Lemma 5.2. We have

$$M_m \leqslant Cm^{(n-2)/2+[u+1]}$$

(see, e. g., [7], p. 904, formula (4)), and thus

$$\sum T_u^p(x_0, a_{lm}\mathcal{R}_{lm} f_2) \leqslant CT_u(x_0, \mathcal{K}) [C_{up} T_u^p(x_0, f_2) + N_2(x_0)].$$

On the other hand,

$$T_u^p(x_0, a_{lm}\mathcal{R}_{lm} f_1) \leqslant CT_u(x_0, a_{lm}) T_u^p(x_0, \mathcal{R}_{lm} f_1) \leqslant Cm^{n/2-2n-2[(u+1)/2]} T_u(x_0, \mathcal{K}) T_u^p(x_0, \mathcal{R}_{lm} f_1).$$

Now, by Lemma 5.4 we have

$$T_u^p(x_0, \mathcal{R}_{lm} f_1) \leqslant C_{up} \left[\sum_{|a|<u} \left\| \left(\frac{\partial}{\partial x}\right)^a \mathcal{R}_{lm} f_1 \right\|_p + \sum_{|a|=u} T_0^p\left(x_0, \left(\frac{\partial}{\partial x}\right)^a \mathcal{R}_{lm} f_1\right) \right],$$

and since $\dfrac{\partial}{\partial x_j} \mathcal{R}_{lm} = \mathcal{R}_{lm} \dfrac{\partial}{\partial x_j}$ (the proof is the same as in the case $m = 1$; see Lemma 4.3) it follows from Lemmas 5.5 and 5.3 that

$$T_u^p(x_0, \mathcal{R}_{lm} f_1) \leqslant C_{up} \Big[\sum_{|a|<u} \|f_{1a}\|_p + C_p M_m \sum_{|a|=u} T_0^p(x_0, f_{1a}) + C \sum_{|a|=u} (\mathcal{R}_{lm} f_{1a})^*(x_0) \Big],$$

where M_m is the same as before. Consequently,

$$\sum T_u^p(x_0, a_{lm}\mathcal{R}_{lm}f_1) \leqslant \sum C_{up} m^{n/2-2n-2[(u+1)/2]} T_u(x_0, \mathcal{K}) \Big[\sum_{|a|<u} \|f_{1a}\|_p +$$
$$+C_p M_m \sum_{|a|=u} T_0^p(x_0, f_{1a}) + C \sum_{|a|=u} (\mathcal{R}_{lm} f_{1a})^*(x_0)\Big]$$
$$\leqslant C_{up} T_u(x_0, \mathcal{K}) \Big[\sum_{|a|<u} \|f_{1a}\|_p + \sum_{|a|=u} T_0^p(x_0, f_{1a}) + N_1(x_0)\Big].$$

This completes the proof of the theorem if $u > 0$.

If $u = 0$ we set $f_2 = 0$ and argue with $f_1 = f$ as above, and (5.11) simplifies to

$$T_0^p(x_0, \mathcal{K}f) \leqslant C_p T_0(x_0, \mathcal{K})[T_0^p(x_0, f) + N_1(x_0)].$$

A special variant of Theorems 6 and 7 is the following

THEOREM 15. *The operators* $\dfrac{\partial}{\partial x_j} J$ *map* $T_u^p(x_0)$ *and* $t_u^p(x_0)$, $1 < p < \infty$, $-n/p \leqslant u \neq 0, 1, 2, \ldots$, *continuously into themselves. If, on the other hand, u is a non-negative integer and $f \in T_u^p(x_0)$ $1 < p < \infty$, for all x_0 in a set Q of positive measure, then* $\dfrac{\partial}{\partial x_j} Jf$ *belongs to* $T_u^p(x_0)$ *for almost all x_0 in Q.*

Proof. According to Lemma 4.3 we have

$$\frac{\partial}{\partial x_j} = -i\mathcal{R}_j \Lambda.$$

Consequently,

$$\frac{\partial}{\partial x_j} J = -i\mathcal{R}_j \Lambda J,$$

and an account of Theorems 6 and 7 it will be enough to show that ΛJ maps $T_u^p(x_0)$ and $t_u^p(x_0)$ continuously into themselves for $u \geqslant -n/p$.

We first observe that J maps $T_u^p(x_0)$ continuously into itself. If u is not an integer, this is an immediate consequence of Theorem 4 and of Lemma 2.1. If u is an integer then for the same reasons $J^{1/2}$ maps $T_u^p(x_0)$ continuously into itself. Consequently $J = J^{1/2}J^{1/2}$ has the same property.

According to Lemma 4.6,

$$\Lambda J = I + a_1 J^2 + a_2 J^4 + \ldots + a_m J^{2m} - Q_m.$$

By what we have just shown, it only remains to prove that Q_m maps $T_u^p(x_0)$ continuously into $t_u^p(x_0)$. If m is large enough, Q_m is convolution with a bounded integrable kernel with bounded integrable derivatives up to order $[u]+1$. Hence, in the first place, $\|Q_m f\|_p \leqslant C_u \|f\|_p$ and $Q_m f$ has continuous derivatives up to order $[u]+1$ majorized by $C_u\|f\|_p$. Thus $Q_m f \in t_u^p(x_0)$ and $T_u^p(x_0, Q_m f) \leqslant C_u \|f\|_p \leqslant C_u T_u^p(x_0, f)$. This completes the proof of the theorem.

6. This section will be devoted to proving our theorems on differential equations.

LEMMA 6.1. *Let $\mathcal{L}$ be a system of differential operators with constant coefficients of homogeneous order m which is elliptic in the sense of Definition 5. Then*

$$\mathcal{L} = \mathcal{K}\Lambda^m, \tag{6.1}$$

where $\mathcal{K}$ is an $s \times r$, $s \geqslant r$, matrix of convolution singular integral operators, with the property that there exists an $r \times s$ matrix $\mathcal{H}$ of operators of the same kind which is a left inverse of $\mathcal{K}$, that is $\mathcal{H}\mathcal{K}$ is the identity operator. The norm of $\mathcal{H}$ in the various spaces L^p, $T_u^p(x_0)$ depends only on the space, the least upper bound of the absolute values of the coefficients of $\mathcal{L}$ and the constant of ellipticity μ of $\mathcal{L}$.

Proof. According to [7], Theorem 7, the operator $\mathcal{L}$ has the representation (6.1) and the matrix of symbols of operators in $\mathcal{K}$ is precisely the matrix

$$\sigma(\mathcal{K}) = (-i)^m \sum a_a z^a |z|^{-m},$$

where $\sum a_a \xi^a$ is the characteristic matrix of $\mathcal{L}$. The assumption of ellipticity implies that $\sigma(\mathcal{K})$ is of rank r for all $z \neq 0$; consequently if $\sigma(\mathcal{K})^*$ denotes the conjugate transposed matrix, then $\sigma(\mathcal{K})^*\sigma(\mathcal{K})$ is a positive self-adjoint $r \times r$ matrix, which consequently is invertible for all $z \neq 0$. Consider now the matrix $\sigma(\mathcal{H}) = [\sigma(\mathcal{K})^*\sigma(\mathcal{K})]^{-1}\sigma(\mathcal{K})^*$. The entries of this matrix are functions of z which are homogeneous of degree zero, and so according to Theorem 3 in [7], there exists an $r \times s$ matrix $\mathcal{H}$ of convolution singular integral operators whose symbols coincide with $\sigma(\mathcal{H})$. Thus, according to [7], Theorem 4, $\sigma(\mathcal{H}\mathcal{K}) = \sigma(\mathcal{H})\sigma(\mathcal{K}) = I$, where I is the identity $r \times r$ matrix, and consequently $\mathcal{H}\mathcal{K}$ is the identity operator. The norm of the operator $\mathcal{H}$ in L^p, $T_u^p(x_0)$ can be estimated using Theorem 14. In what follows we will deal with matrices of operators as a single operator acting on vector valued functions, and we shall apply to this case our previous results without further explanations. Here we only add that by the norm of a vector-valued function we mean the sum of the norms of its components.

LEMMA 6.2. *Let $\mathcal{L}$ be a differential operator of order m which is elliptic at x_0 in the sense of Definition 5 and has coefficients in $T_u(x_0)$, $u > 0$. Let $\mathcal{L}_0$ be the operator obtained from $\mathcal{L}$ by evaluating its leading coefficients at x_0. Let $f \in L_m^p$ and let $h = (1-\Delta)^{m/2} f$ if m is even and $h = (1-\Delta)^{(m-1)/2}(i+\Lambda)f$ if m is odd, where Δ is the Laplacian operator. Then, if $\mathcal{K}\Lambda^m = \mathcal{L}_0$ is the representation of $\mathcal{L}_0$ described in Lemma 6.1, $\mathcal{H}$ is the left inverse of $\mathcal{K}$ and $\mathcal{L}f = g$, we have*

$$h = \mathcal{H}g + \mathcal{H}(\mathcal{L}_0 - \mathcal{L})f + (\mathcal{M}_1 + \Lambda\mathcal{M}_2)f, \tag{6.2}$$

where $\mathcal{M}_1$ and $\mathcal{M}_2$ are differential operators with constant coefficients of orders $m-1$ and $m-3$ respectively which depend only on m. When m is even, the order of $\mathcal{M}_1$ is $m-2$ and $\mathcal{M}_2 = 0$.

Proof. We have

$$\mathcal{L}f = \mathcal{L}_0 f + (\mathcal{L}-\mathcal{L}_0)f = g,$$

and representing $\mathcal{L}_0$ as in Lemma 6.1 and multiplying the equation by the left inverse $\mathcal{H}$ or $\mathcal{K}$ we obtain

$$\Lambda^m f = \mathcal{H}g + \mathcal{H}(\mathcal{L}_0-\mathcal{L})f.$$

Now it follows immediately from the definition of Λ that $\Lambda^2 = -\Delta$ (see also [7], p. 909, Corollary) and thus we have

$$h = (1-\Delta)^{m/2} f = \mathcal{H}g + \mathcal{H}(\mathcal{L}_0-\mathcal{L})f + [(1-\Delta)^{m/2} - (-\Delta)^{m/2}]f$$

if m is even; and if m is odd,

$$\begin{aligned} h &= (1-\Delta)^{(m-1)/2}(i+\Lambda)f \\ &= \mathcal{H}g + \mathcal{H}(\mathcal{L}_0-\mathcal{L})f + [(1-\Delta)^{(m-1)/2}(i+\Lambda) - (-\Delta)^{(m-1)/2}\Lambda]f. \end{aligned}$$

The lemma is thus established.

LEMMA 6.3. *Under the assumptions of Lemma 6.2, if $u \geqslant v \geqslant w \geqslant -n/p$, $v-w \leqslant \min(u, 1)$ and if in addition neither v nor w is an integer, then*

$$T_v^p(x_0, h) \leqslant T_v^p(x_0, \mathcal{H}g) + C_{pvwm}(1+MN)T_w^p(x_0, h),$$

where M is a bound for the norms in $T_u(x_0)$ of the coefficients of $\mathcal{L}$, and N is the norm of $\mathcal{H}$ as an operator on $T_v^p(x_0)$.

Proof. Referring to formula (6.2) we have

$$T_v^p(x_0, \mathcal{M}_1 f) \leqslant C_m \sum_{|\beta|\leqslant m-1} T_v^p(x_0, f_\beta), \quad \text{where} \quad f_\beta = \left(\frac{\partial}{\partial x}\right)^\beta f. \tag{6.3}$$

Since $\Lambda = i\sum \mathcal{R}_j \partial/\partial x_j$ it follows from Theorem 6 that

$$T_v^p(x_0, \Lambda\mathcal{M}_2 f) \leqslant C_{pvm} \sum_{|\beta|\leqslant m-2} T_v^p(x_0, f_\beta). \tag{6.4}$$

Similarly, if S_1 denotes the sum of terms of order $\leqslant m-1$ of $\mathcal{L}_0-\mathcal{L}$, then

$$T_v^p(x_0, \mathcal{H}S_1 f) \leqslant NT_v^p(x_0, S_1 f),$$

and, according to Lemma 2.4,

$$T_v^p(x_0, S_1 f) \leqslant CM \sum_{|\beta|\leqslant m-1} T_v^p(x_0, f_\beta).$$

Thus

$$T_v^p(x_0, \mathcal{H}S_1 f) \leqslant CMN \sum_{|\beta|\leqslant m-1} T_v^p(x_0, f_\beta). \tag{6.5}$$

Let now S_2 denote the sum of terms of order m of $\mathcal{L}_0-\mathcal{L}$. Since the coefficients of S_2 vanish at x_0 and belong to $T_u(x_0)$, Lemma 2.5 gives

$$T_v^p(x_0, S_2 f) \leqslant CM \sum_{|\beta|=m} T_w^p(x_0, f_\beta).$$

Thus

$$T_v^p(x_0, \mathcal{H}S_2 f) \leqslant NT_v^p(x_0, S_2 f) \leqslant CMN \sum_{|\beta|=m} T_w^p(x_0, f_\beta). \tag{6.6}$$

Combining (6.3) to (6.6) we obtain

$$\begin{aligned} &T_v^p(x_0, h) \\ &\leqslant T_v^p(x_0, \mathcal{H}g) + C_{vpm}\Big[(1+MN)\sum_{|\beta|\leqslant m-1} T_v^p(x_0, f_\beta)\Big] + CMN \sum_{|\beta|=m} T_w^p(x_0, f_\beta). \end{aligned} \tag{6.7}$$

Now, if m is even, then $h = (1-\Delta)^{m/2} f = J^{-m} f$ or $f = J^m h$. Thus

$$\left(\frac{\partial}{\partial x}\right)^\alpha f = f_\alpha = \left(\frac{\partial}{\partial x} J\right)^\alpha J^{m-|\alpha|} h,$$

whence by Theorem 15, Lemma 2.1 and Theorem 4 we have if $|\alpha| < m$,

$$\begin{aligned} T_v^p(x_0, f_\alpha) &\leqslant C_{pvm} T_v^p(x_0, J^{m-|\alpha|}h) \leqslant C_{pvm} T_{w+m-|\alpha|}^p(x_0, J^{m-|\alpha|}h) \\ &\leqslant C_{pvwm} T_w^p(x_0, h), \end{aligned} \tag{6.8}$$

and, if $|\alpha| = m$,

$$T_w^p(x_0, f_\alpha) \leqslant C_{pwm} T_w^p(x_0, h). \tag{6.9}$$

If on the other hand m is odd, then, since $\Lambda^2 = -\Delta$,

$$h = (1-\Delta)^{(m-1)/2}(i+\Lambda)f, \qquad J^{m-1}h = (i+\Lambda)f,$$

$$J^{m+1}h = J^2(i+\Lambda)f = (i+\Lambda)J^2 f, \qquad (-i+\Lambda)J^{m+1}h = (1-\Delta)J^2 f = f,$$

and on account of the definition of Λ and of Lemma 4.3 we have

$$\left(\frac{\partial}{\partial x}\right)^\alpha f = \left[i\sum_{j=1}^m \mathcal{R}_j\left(\frac{\partial}{\partial x_j} J\right) - iJ\right]\left(\frac{\partial}{\partial x} J\right)^\alpha J^{m-|\alpha|} h.$$

From this and Lemma 2.1 we find that the inequalities (6.8) and (6.9) also hold in this case.

Combining (6.7) with (6.8) and (6.9) the inequality of the lemma follows.

The next lemma states a known result.

LEMMA 6.4. *Let $\mathcal{L}$ be a uniformly elliptic differential operator of order m with bounded coefficients whose leading coefficients are uniformly continuous. Then there exists a constant A depending on $\mathcal{L}$ and p such that*

$$\|f\|_{p,m} \leqslant A\,[\|\mathcal{L}f\|_p + \|f\|_p]$$

for every f in L^p_m, $1 < p < \infty$.

Proof. We assume first that the leading coefficients of $\mathcal{L}$ have bounded continuous derivatives of the first two orders. Let now $\mathcal{L}_0$ denote the principal part of the operator $\mathcal{L}$, that is, the sum of its terms of order m. Then according to Theorem 7 of [7] we have $\mathcal{L}_0 = \mathcal{K}\Lambda^m$ where $\mathcal{K}$ is a singular integral operator whose symbol is given by

$$\sigma(\mathcal{K}) = (-i)^m \sum a_a(x) z^a |z|^{-m},$$

where $\sum a_a(x)\,\xi^a$ is the characteristic matrix of $\mathcal{L}$. The assumption of uniform ellipticity implies that the matrix $\sigma(\mathcal{K})$ has a left inverse $\sigma(\mathcal{H})$ which is the symbol of an operator of class C_2^∞ (see Definition 2 in [7]). Now we have

$$\mathcal{L}f = \mathcal{L}_0 f + (\mathcal{L}-\mathcal{L}_0)f = \mathcal{K}\Lambda^m f + (\mathcal{L}-\mathcal{L}_0)f = g,$$

and multiplying the equation on the left by $\mathcal{H}$ we get

$$\mathcal{H}\mathcal{K}\Lambda^m f = \mathcal{H}g - \mathcal{H}(\mathcal{L}-\mathcal{L}_0)f,$$
$$\Lambda^m f = \mathcal{H}g - \mathcal{H}(\mathcal{L}-\mathcal{L}_0)f + [(I-\mathcal{H}\mathcal{K})\Lambda]\Lambda^{m-1}f.$$

Since $\sigma(\mathcal{H})\sigma(\mathcal{K}) = I$ it follows from Theorem 5 of [7] that the operator in square brackets on the right is bounded on L^p. On the other hand, $\mathcal{H}$ is also bounded on L^p, and so taking norms in the last equation we find that

$$\|\Lambda^m f\|_p \leqslant C_{\mathcal{L},p}\Big(\sum_{|\beta|\leqslant m-1} \|f_\beta\|_p + \|\Lambda^{m-1}f\|_p\Big) + \|\mathcal{H}g\|_p.$$

Now, according to Lemma 4.3,

$$\left(\frac{\partial}{\partial x}\right)^a = (-i)^a \mathcal{R}^a \Lambda^a, \qquad \mathcal{R}^a = \mathcal{R}_1^{a_1}\mathcal{R}_2^{a_2}\ldots\mathcal{R}_n^{a_n},$$

which implies that for $|a| = m$ we have

$$\|f_a\|_p \leqslant C_{ap}\|\Lambda^m f\|_p.$$

On the other hand,

$$\Lambda = i\sum \mathcal{R}_j \frac{\partial}{\partial x_j},$$

and this implies that

$$\|\Lambda^{m-1}f\|_p \leqslant C_{p,m} \sum_{|\beta|=m-1} \|f\|_\beta.$$

This combined with the previous inequalities gives

$$\sum_{|a|=m} \|f_a\|_p \leqslant C_{\mathcal{L}p}\|f\|_{p,m-1} + \|\mathcal{H}g\|_p. \tag{6.10}$$

Now we use an inequality given in [11], p. 125, namely

$$\|f_\beta\|_p \leqslant C_{pm}\Big(\sum_{|a|=m}\|f_a\|_p\Big)^{\Theta}\|f\|_p^{1-\Theta} = C_{pm}\Big(\varepsilon\sum_{|a|=m}\|f_a\|_p\Big)^{\Theta}\big(\varepsilon^{-\Theta/(1-\Theta)}\|f\|_p\big)^{1-\Theta}$$
$$\leqslant C_{pm}\Big[\varepsilon\sum_{|a|=m}\|f_a\|_p + \varepsilon^{-\Theta/(1-\Theta)}\|f\|_p\Big],$$

where $\Theta = (m-|\beta|)/m$, $|\beta| < m$ and ε is an arbitrary positive number. Consequently we have

$$\|f\|_{p,m-1} \leqslant \varepsilon\sum_{|a|=m}\|f_a\|_p + C_{p,\varepsilon,m}\|f\|_p, \tag{6.11}$$

and from this and (6.10) we obtain

$$\sum_{|a|=m}\|f_a\|_p \leqslant C_{\mathcal{L}p}\|f\|_p + 2\|\mathcal{H}g\|_p. \tag{6.12}$$

In the general case, given an elliptic operator $\mathcal{L}$ whose leading coefficients are merely uniformly continuous, we approximate $\mathcal{L}$ by an operator $\bar{\mathcal{L}}$ which has the same lower order coefficients but whose leading coefficients have continuous bounded derivatives of the first two orders. Then Theorem 3 in [7] and the uniform ellipticity of $\mathcal{L}$ permit us to assert that it is possible to approximate the coefficients of $\mathcal{L}$ by those of $\bar{\mathcal{L}}$ uniformly, by less than any preassigned number ε, keeping at the same time the norm in L^p of the operator $\mathcal{H}$ associated with $\bar{\mathcal{L}}$ bounded, say, less than N. Thus from (6.12) we will have

$$\bar{\mathcal{L}}f = g + (\bar{\mathcal{L}}-\mathcal{L})f,$$

$$\sum_{|a|=m}\|f_a\|_p \leqslant C_{\bar{\mathcal{L}}p}\|f\|_p + 2\|\mathcal{H}g\|_p + 2\|\mathcal{H}(\bar{\mathcal{L}}-\mathcal{L})f\|_p$$
$$\leqslant C_{\bar{\mathcal{L}}p}\|f\|_p + 2N\|g\|_p + 2N\|(\bar{\mathcal{L}}-\mathcal{L})f\|_p$$
$$\leqslant C_{\bar{\mathcal{L}}p}\|f\|_p + 2N\|g\|_p + 2N\varepsilon\sum_{|a|=m}\|f_a\|_p,$$

and choosing ε so that $N\varepsilon < \frac{1}{4}$ we obtain finally

$$\sum_{|a|=m}\|f_a\|_p \leqslant C_{\mathcal{L}p}\|f\|_p + 4N\|g\|_p. \tag{6.12a}$$

This combined with (6.11) gives the desired result.

Remark. It may be worth noting that the coefficient $4N$ of $\|g\|_p$ in (6.12a) merely depends on the bounds for the leading coefficients of $\mathcal{L}$ and the bounds for uniform ellipticity.

Proof of Theorem 1. We refer to Lemmas 6.2 and 6.3. We will show first that if $g\in T_v^p(x_0)$, $1<p<\infty$, v not an integer and the coefficients of $\mathcal{L}$ are in $T_u(x_0)$, $u>0$, $u\geqslant v$, then $h\in T_v^p(x_0)$ and satisfies an appropriate inequality.

Since f is assumed to belong to L_m^p, the function h belongs to L^p, and thus also to $T_{-n/p}^p(x_0)$, and

$$T_{-n/p}^p(x_0,h)\leqslant 2\|h\|_p\leqslant C_{mp}\|f\|_{p,m}.$$

The last inequality is obvious from the definition of h if m is an even integer. If m is odd we use the fact stated in Lemma 4.3 that Λ maps L_m^p continuously into L_{m-1}^p. Let now k be an integer such that for $\nu=1,2,\ldots,k$

$$-\frac{n}{p}+\frac{\nu}{k}\left(v+\frac{n}{p}\right)=-\frac{n}{p}+\nu\delta$$

is never integral, and

$$\delta=\frac{1}{k}\left(v+\frac{n}{p}\right)\leqslant\min(u,1).$$

Then the inequality of Lemma 6.3 gives

$$T_{-n/p+(\nu+1)\delta}^p(x_0,h)\leqslant T_{-n/p+(\nu+1)\delta}^p(x_0,\mathcal{H}g)+C_{pvm}(1+NM)T_{-n/p+\nu\delta}^p(x_0,h),$$

$\nu=0,1,\ldots,k-1$. On the other hand, we have

$$T_{-n/p+(\nu+1)\delta}^p(x_0,\mathcal{H}g)\leqslant CT_v^p(x_0,\mathcal{H}g)\leqslant CMT_v^p(x_0,g),$$

where M is, as in Lemma 6.3, the norm of the operator $\mathcal{H}$ in $T_v^p(x_0)$. Thus

$$T_{-n/p+(\nu+1)\delta}^p(x_0,h)\leqslant CMT_v^p(x_0,g)+C_{pvm}(1+NM)T_{-n/p+\nu\delta}^p(x_0,h).$$

If we write $a_\nu=T_{-n/p+\nu\delta}^p(x_0,h)$, $\beta=CMT_v^p(x_0,g)\gamma$ and $\gamma=[C_{pvm}(1+NM)]^{-1}$ the last inequality can be written as

$$\gamma a_{\nu+1}\leqslant\beta+a_\nu,\qquad \gamma^{\nu+1}a_{\nu+1}\leqslant\gamma^\nu\beta+\gamma^\nu a_\nu,$$

and summing from $\nu=0$ to $\nu=k-1$ we find

$$\gamma^k a_k\leqslant\beta\frac{1-\gamma^k}{1-\gamma}+a_0.$$

If we replace C_{pvm} by $2+C_{pvm}$ then $\gamma\leqslant\frac{1}{2}$ and

$$a_k\leqslant[2\beta+a_0]\gamma^{-k};$$

consequently,

$$\begin{aligned}T_v^p(x_0,h)&\leqslant C_{pvm}[MT_v^p(x_0,g)(1+NM)^{k-1}+T_{-n/p}^p(x_0,h)(1+NM)^k]\\&\leqslant C_{pvm}(1+NM)^k[T_v^p(x_0,g)+\|f\|_{p,m}].\end{aligned}$$

Now we use the identities in the proof of Lemma 6.3,

$$f_a=\left(\frac{\partial}{\partial x}\right)^a f=J^{m-|a|}\left(\frac{\partial}{\partial x}J\right)^a h \tag{6.13}$$

if m is even, and

$$f_a=\left(\frac{\partial}{\partial x}\right)^a f=J^{m-|a|}\left[i\sum_{j=1}^{n}\mathcal{R}_j\left(\frac{\partial}{\partial x_j}\right)-iJ\right]\left(\frac{\partial}{\partial x}J\right)^a h \tag{6.14}$$

if m is odd, and using Theorems 4, 6, 15 we finally obtain, with q as in Theorem 1,

$$T_{v+m-|a|}^q(x_0,f_a)\leqslant C_{pvm}[1+NM]^k[N^{-1}T_v^p(x_0,g)+\|f\|_{p,m}].$$

Combining these results with Lemma 6.4 we obtain parts (i) and (ii) of Theorem 1.

To show (iii) we return to equation (6.2) and show that under the present assumptions $h\in t_v^p(x_0)$. First we observe that the argument given above shows not only that $f_a\in T_{v+m-|a|}^q(x_0)$ but also $f_a\in T_{v+m-|a|}^p(x_0)\subset t_v^p(x_0)$ if $|a|<m$. On the other hand, on account of Theorem 6, $\mathcal{H}g\in t_v^p(x_0)$. Finally, the leading terms of $(\mathcal{L}_0-\mathcal{L})f$ have coefficients which vanish at x_0; since $f_a\in T_v^p(x_0)$ it follows from Lemma 2.5 that these terms represent functions in $t_v^p(x_0)$. Consequently $(\mathcal{L}_0-\mathcal{L})f\in t_u^p(x_0)$ and by Theorem 6 again $\mathcal{H}(\mathcal{L}_0-\mathcal{L})f\in t_u^p(x_0)$. Thus all terms on the right of equation (6.2) are functions in $t_u^p(x_0)$, and $h\in t_u^p(x_0)$. Using the representation of f_a in terms of h given above and applying the same theorems we conclude that $f_a\in t_{v+m-|a|}^q(x_0)$. Theorem 1 is thus established.

Proof of Theorem 2. We refer again to equation (6.2). We may suppose without loss of generality that Q is bounded and we will show that $h\in T_v^p(x_0)$ for almost all x_0 in Q. If $g\in T_v^p(x_0)$ and v is an integer, then also $g\in T_w^p(x_0)$ with w slightly smaller than v and non-integral and, as was seen in the proof of Theorem 1, $f_a\in T_{w+m-|a|}^p(x_0)\subset T_v^p(x_0)$ for $|a|<m$, and $f_a\in T_w^p(x_0)$ for all $|a|\leqslant m$. Let $P(x_0,x)=\sum_{|\beta|<w+m}a_\beta(x_0)(x-x_0)^\beta$ be the Taylor expansion of f at x_0. Then $(\partial/\partial x)^a P(x_0,x)$ is the Taylor expansion of $f_a(x_0)$ at every point x_0 where $g\in T_w^p(x_0)$ and where, consequently, also $f\in T_{w+m}^p(x_0)$ and $f_a\in T_{w+m-|a|}^p(x_0)$. To see this we merely have to consider the functions $f^\lambda=\lambda^n\varphi(\lambda x)*f$ where φ is a function of C_0^∞ of integral equal to 1; then, according to Lemma 2.3, f^λ converges to f in $T_{w+m-\varepsilon}^p(x_0)$ as λ tends to infinity and $(\partial/\partial x)^a f^\lambda=\lambda^n\varphi(\lambda x)*f_a$ converges to f_a in $T_{w+m-|a|-\varepsilon}^p(x_0)$. This clearly implies that the coefficients of $(\partial/\partial x)^a P(x_0,x)$ are the coefficients of the

Taylor expansion of f_a. Let now ψ be a function in C_0^∞ which is equal to 1 on Q, and let us rewrite equation (6.2) as follows

$$h = \mathcal{H}g+\mathcal{H}(\mathcal{L}_0-\mathcal{L})[f-P(x_0, x)\psi(x)]+ +\mathcal{H}(\mathcal{L}_0-\mathcal{L})P(x_0, x)\psi(x)+(\mathcal{M}_1+\Lambda\mathcal{M}_2)f,$$

or, setting now $P(x_0, x) = \sum_{|\beta|<m+w} b_\beta(x_0)x^\beta$,

$$h = \mathcal{H}g+\mathcal{H}(\mathcal{L}_0-\mathcal{L})[f-P(x_0, x)\psi(x)]+ + \sum_{|\beta|<m+w} b_\beta(x_0)\mathcal{H}(\mathcal{L}_0-\mathcal{L})x^\beta\psi(x)+(\mathcal{M}_1+\Lambda\mathcal{M}_2)f. \tag{6.15}$$

Evidently $(\partial/\partial x)^a x^\beta\psi(x)\in T_v^p(x_0)$ for all x, a, and β, and $\mathcal{L}\{x^\beta\psi(x)\}\in T_v^p(x_0)$ for all $x_0\in Q$. Consequently, by Theorem 7, there exists a subset $\bar{Q}_1$ of Q of full measure such that $\mathcal{H}(\mathcal{L}_0-\mathcal{L})x^\beta\psi(x)\in T_v^p(x_0)$ for all $x_0\in\bar{Q}_1$, regardless of the choice of the operator $\mathcal{H}$, which in our case depends on x_0. Similarly, for x_0 in another subset $\bar{Q}_2$ of Q of full measure we have $\mathcal{H}g\in T_v^p(x_0)$ for all $x_0\in\bar{Q}_2$. On the other hand, since as we saw $f_a\in T_v^p(x_0)$ for $|a|<m$ and all x_0 in Q, we have $\mathcal{M}_1 f\in T_v^p(x_0)$ for $x_0\in Q$, and using the definition of Λ and Theorem 7 again we conclude that $\Lambda\mathcal{M}_2 f\in T_v^p(x_0)$ for almost all $x_0\in Q$.

There remains the second term on the right of (6.15). We note first that

$$(\mathcal{L}_0-\mathcal{L})[f-P(x_0, x)\psi(x)] = \sum_{|\beta|\leq m} c_\beta(x)\left(\frac{\partial}{\partial x}\right)^\beta[f-P(x_0, x)\psi(x)],$$

where the $c_\beta(x)$ are functions in $T_u(x_0)$ which *vanish* at x_0, and $(\partial/\partial x)^\beta[f-P(x_0, x)]$ are functions in $T_w^p(x_0)$ which also vanish at x_0, for every x_0 in Q. Consequently by Lemma 2.5, part (iii), these functions belong to $T_{w+1}^p(x_0)$ for all $x_0\in Q$. Now, since $w+1$ is not an integer, Theorem 6 guarantees that

$$\mathcal{H}(\mathcal{L}_0-\mathcal{L})[f-P(x_0, x)\psi(x)]$$

belongs to $T_{w+1}^p(x_0)\subset T_v^p(x_0)$ for all x_0 in Q. Summarizing we have found that $h\in T_v^p(x_0)$ for almost all x_0 in Q.

Using now the representation of f_a given in (6.13) and (6.14) and applying theorems 15, 7, 5 we obtain the desired result.

Proof of Theorem 3. This theorem is an immediate consequence of Theorem 1. For under the given assumptions, it follows that $f\in T_{v+m}^p(x_0)$ for all $x_0\in Q$, and that $T_{v+m}^p(x_0, f)$ is bounded on Q; and an application of Theorem 9 gives the desired result. If $g\in t_v^p(x_0)$ for all $x_0\in Q$, then $f\in t_{v+m}^p(x_0)$ for all $x_0\in Q$ and Theorem 9 asserts that $f\in b_{v+m}(Q)$.

7. In this section we make additional observations about the solutions of the equation $\mathcal{L}f = g$.

It is well known that Theorem 1 is not true when v is a non-negative integer (Theorem 2 is a substitute result), and the simplest illustration is the equation $\Delta f = g$: if g is merely continuous, f is not necessarily twice differentiable. The only thing we can then say is that f has continuous first order derivatives $f_j = \partial f/\partial x_j$ $(j = 1, 2, \ldots, n)$ which in turn satisfy a Hölder condition of any order less than 1, or more precisely, have a modulus of continuity $\omega(h) = o\left(|h|\log\frac{1}{|h|}\right)$.

It is however not difficult to see, and this result is a special case of Theorem 16 below, that if g is continuous at a given point x_0, then any solution f of $\Delta f = g$ has continuous derivatives f_j near x_0 and the f_j satisfy at that point the condition of "smoothness":

$$f_j(x_0+h)+f_j(x_0-h)-2f_j(x_0) = o(|h|), \quad |h|\to 0. \tag{7.1}$$

(this notion was first introduced by Riemann).

This result can be interpreted as the differentiability at $h = 0$ of the "even part" $\frac{1}{2}[f_j(x_0+h)+f_j(x_0-h)]$ of $f_j(x_0+h)$, and from this it is easy to deduce that the "odd part" $\frac{1}{2}[f(x_0+h)-f(x_0-h)]$ of $f(x_0+h)$ has a second differential at $h = 0$. This fact gives a clue to the situation in the general case. We may add, and the result is familiar enough (see e. g., 15_{I}, p. 44), that if a continuous function $g(x)$ satisfies the condition $g(x+h)+g(x-h)-2g(x) = o(|h|)$ uniformly in a domain, then in every compact subdomain g has modulus of continuity $o\left(|h|\log\frac{1}{|h|}\right)$, so that the result about the modulus of continuity of the derivatives f_j in the case of the equation $\Delta f = g$ is a corollary of 7.1.

Definition 10. We will say that $f(x)$ *belongs to the class* $\Lambda_u^p(x_0)$, where u is a non-negative integer, if $\frac{1}{2}[f(x_0+h)-(-1)^u f(x_0-h)]$ belongs to $T_u^p(0)$ as a function of h. We will say that $f(x)$ *belongs to* $M_u^p(x_0)$, where u is again a non-negative integer, if $\frac{1}{2}[f(x_0+h)+(-1)^u f(x_0-h)]$ belongs to $T_u^p(0)$ as a function of h. By replacing $T_u^p(0)$ by $t_u^p(0)$ we obtain the definitions of $\lambda_u^p(x_0)$ and $\mu_u^p(x_0)$.

THEOREM 16. *Let $\mathcal{L}f = g$ be an equation of order m with coefficients in $T_v(x_0)$, which is elliptic at x_0 in the sense of Definition 5. Let u, $u<v$, be a non-negative integer. Then, if m is even and $g\in T_w^p(x_0)$, $1<p<\infty$, $w>u-1$, and $g\in\Lambda_u^p(x_0)$, the function $f_a = (\partial/\partial x)^a f$ belongs to $\Lambda_{u+m-|a|}^q(x_0)$, where p and q are related as in Theorem 1. If m is odd the same conclusion holds with the classes $\Lambda_u^p(x_0)$ replaced by $M_u^p(x_0)$. Analogous conclusions hold with classes Λ replaced by λ, or M by μ, as the case may be.*

We will omit proving in detail the preceding statement, and will confine ourselves to merely sketching the main line of the argument. Referring to Lemma 6.2, one shows first that the function h on the left-hand side of formula (6.2) belongs to $\Lambda_u^p(x_0)$ or $M_u^p(x_0)$ as the case may be. According to Theorem 1, and the hypothesis made above, f and its derivatives of orders less than m belong to $T_{w+1}^p(x_0)$ and the derivatives of order m belong to $T_w^p(x_0)$. Since the leading coefficients of $\mathcal{L}-\mathcal{L}_0$ vanish at x_0 and belong to $T_v(x_0)$ where $v>u$, it follows that $(\mathcal{L}_0-\mathcal{L})f$ belongs to a class $T_r^p(x_0)$, with $r>u$ (see Lemma 2.5) and by Theorem 6 the same is true of $\mathcal{H}(\mathcal{L}_0-\mathcal{L})f$. Consequently all terms on the right of (6.2) except $\mathcal{H}g$ belong to $T_u^p(x_0)$ and a fortiori, also to $\Lambda_u^p(x_0)$, or $M_u^p(x_0)$.

Consider now the term $\mathcal{H}g$. If m is even then the symbol of the operator $\mathcal{H}$ is an even function and thus also the kernel of the operator is an even function. Let O_uf be the function of x given by $\frac{1}{2}[f(x_0+x)-(-1)^uf(x_0-x)]$ and thus, since the kernel of $\mathcal{H}$ is even we find that $O_u(\mathcal{H}g)=\mathcal{H}(O_ug)$ and applying Lemma 5.2 we conclude that $O_u(\mathcal{H}g)$ belongs to $T_u^p(x_0)$, that is $\mathcal{H}g\in\Lambda_u^p(x_0)$. A similar argument shows that $\mathcal{H}g\in M_u^p(x_0)$ if m is odd. Consequently h belongs to $\Lambda_u^p(x_0)$ or $M_u^p(x_0)$ according to the parity of m. Once this has been established we obtain f and its derivatives f_α from h by means of the identities (6.13) and (6.14). In case m is even we obtain

$$O_{m+u}f=O_{m+u}J^mh=J^mO_{m+u}h,$$

and taking into account the remark to the proof of Theorem 4 (p. 198), we find that $O_{m+u}f\in T_{u+m}^q(x_0)$, that is $f\in\Lambda_{u+m}^q(x_0)$. A similar argument employing Theorem 15 (for which a remark analogous to that to Theorem 4 holds) gives the desired result for the derivatives of f. The case of odd m is treated similarly.

Inequalities for the norms can also be obtained by this argument.

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