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Limit properties of ordered families of linear metric spaces (1)

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1. Contents

1. Introduction

The notion of convergence is fundamental in many branches of mathematics and is considered in various meanings. Besides the classical Cauchy definition of convergence of numerical sequences, a number of definitions of convergence of function sequences are in use. Moreover, various definitions of convergence of a sequence of sets are introduced. Thus, for example, in the measure theory, the notion of set-theoretical convergence and the convergence in the metric $\mu(A\setminus B)+\mu(B\setminus A)$ are applied. In topology, the following notions of convergence have been introduced: convergence of sequences of sets in the Hausdorff metric, convergence induced by the limit points (Li and Ls in the terminology of Kuratowski [9]), which may be considered as corresponding to the intuitive notion of convergence of sets and is applied for instance in differential geometry; finally, K. Borsuk [5] has introduced various notions of distance of closed subsets of a compact metric space assuming that sets lying in a small distance must possess similar topological properties.

(1) Some of these results were presented at the Conference on Functional Analysis at Zakopane (January 10, 1957) and at the Conference on Theory of Groups in Wrocław (October 9, 1959).

These examples point out the importance of the analysis of various notions of convergence. On the other hand, they show that notions of this kind may be introduced in various ways depending on the problem at hand and that each of those definitions may find its own applications.

The first problem considered in this paper is the investigation of limit properties of sequences X_n consisting of Banach spaces or, more generally, of linear metric spaces. The method consists here in considering certain spaces $\mathfrak{S}(X_n)$, $\overline{\mathfrak{S}}(X_n)$, $\mathfrak{S}_R(X_n)$ etc., which may be interpreted as limits of the sequence X_n.

So far these considerations have not led to results which could be applied to the solution of some problems of functional analysis. However, we obtain a method which makes it possible to find new relations between the properties of the spaces under consideration and, above all, sets of new problems, which have not been formulated explicitly.

The question of limit space of an increasing sequence of linear topological spaces has already been investigated by Köthe and by Dieudonné and Schwartz [6]. However, there is hardly any connection between the investigations presented in this paper and the inductive limits. The basis difference lies in the fact that in the Dieudonné-Schwartz theory the topology in the limit space depends on the topologies in spaces X_n, while the topology in the space $\mathfrak{S}(X_n)$ is determined by the metric properties of spaces X_n and our considerations regard almost exclusively metric properties. Especially, it is established in 4.3 that the definition of $\mathfrak{S}(X_n)$ cannot be generalized to linear topological spaces.

We shall point out that the space $\mathfrak{S}(X_n)$, similarly to other limits considered in this paper, is not an invariant of linear isometric operations; the notion of limit depends here essentially on the inclusions between the spaces. It may easily be stated that no notion of limit according to those inclusion requirements admits such an invariantness (cf. 4.2). Of course, this does not exclude the possibility of another treatment of the question; however, in that case some other advantages would be lost. After all, the inductive limit of a sequence of mutually isomorphic spaces may be non-isomorphic.

Considering spaces $\mathfrak{S}(X_n)$, let us point out two special cases. The first occurs if all spaces X_n are subsets of a fixed space with a fixed norm. Then the limit properties reduce to inclusion properties of subspaces and the treatment introduced in this paper does not give anything new. The second case concerns considerations of a fixed space X with a sequence of norms $\| \ \|_n$ defined in X; investigation of limit properties of the sequence $\langle X, \| \ \|_n\rangle$ is generally reduced to investigations of convergence and other properties of subadditive functionals defined in X and is a continuation of a number of papers concerning these problems, especially of the results of Mazur and Orlicz [10] obtained in 1933.

J. J. Schäffer [16] considers the class $\mathfrak{N}(L)$ of all linear subsets of a linear topological locally convex space L, provided with some homogeneous norms finer than the topology induced by L. $\mathfrak{N}(L)$ is a conditionally complete lattice with respect to the ordering:

$$\langle Y_1, \| \ \|_1\rangle \leqslant \langle Y_2, \| \ \|_2\rangle \quad \textit{if} \quad Y_1 \subset Y_2 \textit{ and } \|x\|_1 \geqslant \|x\|_2 \textit{ for } x \in Y_1.$$

One may verify that if $Y_1 \geqslant Y_2 \geqslant \ldots$, then $\bigwedge_{n=1}^{\infty} Y_n$ coincides with the space $\mathfrak{S}_R(Y_n)$ defined below, and if $Y_1 \leqslant Y_2 \leqslant \ldots$ then $\bigvee_{n=1}^{\infty} Y_n$ coincides with $\mathfrak{S}(Y_n)$.

The most interesting are monotone sequences of complete spaces, i. e. increasing sequences $X_1 \subset X_2 \subset \ldots$ with successively coarser norms and decreasing sequences $X_1 \supset X_2 \supset \ldots$ with successively finer norms. Excluding the trivial case where almost all norms are equivalent, it is proved that if $\{X_n\}$ is an increasing sequence of F-spaces, then $\bigcup_{n=1}^{\infty} X_n$ is always non-complete, and if $\{X_n\}$ is a decreasing sequence, then $\bigcap_{n=1}^{\infty} X_n$ cannot be a B^*-space, but (after a suitable modification of norms) may be a B_0-space. Thus, neither the union of an increasing sequence nor the intersection of a decreasing sequence of B-spaces can be a B-space.

This fact is the starting point of the second problem considered in this paper, being closely connected with the previous one. Suppose we are given a family $\{X_p\}$ of B-spaces with p taking values from a certain interval (a, b). Let $X_{p'} \subset X_p$ for $p' > p$. Moreover, we assume the norm $\| \ \|_{p'}$ in the space $X_{p'}$ to be finer than the norm $\| \ \|_p$ in the space X_p. The problem at hand is the following. How to introduce a *natural definition of continuous dependence of the family $\langle X_p, \| \ \|_p\rangle$ on the parameter p* in such a way that families which may be intuitively assigned to be dependent on p continuously (e. g. spaces L_p with $p \geqslant 1$) remain continuous in the sense of this definition and families intuitively discontinuous in p remain discontinuous? Finally, what are the properties of families depending on the parameter continuously?

A definition of continuous dependence on a parameter is given in section 9.

Some results presented in this paper are known in another aspect or in a more special case; they have been established by the examination of various problems and it seems impossible to give a full list of all positions where such contributions may be found. This paper contains a systematic treatment of these questions.

A number of further questions arise in a natural way; an investigation of them seems to be of importance.

Firstly, a precise classification of criteria which would allow us to deduce the convergence of the sequence of norms

$$\|U\|_m^n = \sup\{\|U(x)\|_m \colon \|x\|_n \leqslant 1\}$$

of a fixed linear operation U from X to Y, the convergence of the two sequences of norms

$$\|x\|_n \to \|x\|_0 \text{ for } x \in X \quad \text{ and } \quad \|y\|_m \to \|y\|_0 \text{ for } y \in Y$$

being assumed. This problem is solved in some special cases (e. g. it is easy to give some sufficient conditions); on the other hand, many questions in various branches of functional analysis may be reduced to this general one.

Secondly, investigations of limit properties of bases and unconditional bases, of the function $\delta(\varepsilon)$ defining uniform convexity, of many quantities connected with convergence of series, etc. Here the situation is similar to that in the former case.

Thirdly, a further study of families depending on a parameter continuously or semicontinuously, interpolation of properties which are valid in a dense set of values of the parameter and so on; an investigation of more special classes defined by means of some general schemes.

Finally, a classification of various pathological situations. The analysis of possible situations allows us to conclude that in many cases the only counter-examples of some theorems are ineffective examples defined under the axiom of choice. E. g. the starting point of many such definitions is the existence of discontinuous distributive functionals in an arbitrary infinite-dimensional space. On the other hand, it is obvious that such cases are unimportant from the point of view of applications and it is suitable to seek after assumptions which will eliminate the pathologic situations. Theorems of this type will be valid in every case which might occur in practice, being non-valid in general. Theorem 6.2 is a theorem of this type.

The starting point of such elimination is the notion of *conformable norms*, i. e. norms for which there exists a complete norm finer than all those norms (2). Hypothesis (H) introduced in section 3 excludes cases of non-conformable norms and linear subsets which do not satisfy the condition of Baire; consequently, it excludes some pathological situations. Obviously, this does not exhaust the problem. The purpose of this paper is to give some basis for further investigations.

(2) Besides this notion, the notion of *quasi-normal norms* is very useful; however, although almost all norms appearing in practice satisfy the condition of quasi-normality, there exist effective counter-examples (see [1], p. 120).

2. Terminology and notation

The term *norm* will mean a homogeneous or non-homogeneous norm; a homogeneous norm will be called a *B-norm*.

B-space will mean a Banach space, *B_0-space* — a locally convex metrisable complete space (see [1]) and *F-space* will mean a complete linear metric space with an invariant metric determined by a norm. Similarly, a space X will be called a *B^*-space*, a *B_0^*-space* or an *F^*-space* if its completion is a B-space, a B_0-space or an F-space, respectively.

$B_0^\#$-space will mean a B_0-space in which there exists a homogeneous norm continuous with respect to the topology of the space (this condition is equivalent to the existence of a sequence of norms determining the topology of the space; these spaces were considered by Bessaga and Pełczyński [4]).

The spaces L_p, l_p, c_0 will have their traditional meaning (as in [2] or [7]).

$\langle X, \| \|\rangle$ will denote a linear space X provided with the norm $\| \|$. Compl$\langle X, \| \|\rangle$ will denote the *Cantor completion* of $\langle X, \| \|\rangle$. X will be treated as a subset of Compl$\langle X, \| \|\rangle$.

Conj$\langle X, \| \|\rangle$ will denote the set of all linear functionals on $\langle X, \| \|\rangle$, i. e. distributive functionals on X continuous with respect to $\| \|$.

Given two spaces $\langle X_1, \| \|_1\rangle$ and $\langle X_2, \| \|_2\rangle$, $\| \|_1 \succeq \| \|_2$ will mean that the conditions $x_n \in X_1 \cap X_2$ $(n = 0, 1, 2, \ldots)$, $\|x_n - x_0\|_1 \to 0$ imply $\|x_n - x_0\|_2 \to 0$; in this case $\| \|_1$ will be called *finer* than $\| \|_2$, and $\| \|_2$ will be called *coarser* than $\| \|_1$. A sequence $\| \|_n$ will be called *uniformly coarser* than $\| \|$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x\| < \delta$ implies $\|x\|_n < \varepsilon$ for $n = 1, 2, \ldots$

$\| \|_1 \sim \| \|_2$ will mean the *equivalence* of norms $\| \|_1$ and $\| \|_2$ on $X_1 \cap X_2$, i. e. $\| \|_1 \succeq \| \|_2$ together with $\| \|_1 \preceq \| \|_2$. If neither $\| \|_1 \preceq \| \|_2$ nor $\| \|_1 \succeq \| \|_2$ holds, $\| \|_1$ and $\| \|_2$ will be called *incomparable*.

The triple notation $\langle X, \| \|_1, \| \|_2\rangle$ will stand for the case where $\| \|_1$ and $\| \|_2$ are defined on X and $\| \|_1 \succeq \| \|_2$. Such a two-norm space $\langle X, \| \|_1, \| \|_2\rangle$ will be called *quasi-normal* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that the conditions

$$\|x_n - x\|_2 \to 0 \quad \text{ and } \quad \|x_n\|_1 \leqslant \delta \quad \text{ for } n = 1, 2, \ldots$$

imply $\|x\|_1 \leqslant \varepsilon$. In this case the norm $\| \|_2$ will be also called *quasi-normal with respect to* $\| \|_1$. If $\| \|_1$ and $\| \|_2$ are homogeneous, this condition may be replaced by the following:

$$\|x_n - x\|_2 \to 0 \quad \text{implies} \quad \|x\|_1 \leqslant K \varliminf \|x_n\|_1,$$

K being a constant (cf. [1]). $\langle X, \| \|_1, \| \|_2\rangle$ will be called *normal* if

$$\|x_n - x\|_2 \to 0 \quad \text{implies} \quad \|x\|_1 \leqslant \varliminf \|x_n\|_1.$$

$\langle X, \|\ \|_1, \|\ \|_2\rangle$ will be called *γ-complete* if the conditions $\|x_p-x_q\|_2\to 0$ as $p,q\to\infty$ and boundedness of (x_n) with respect to $\|\ \|_1$ imply the existence of an element $x_0\in X$ such that $\|x_n-x_0\|_2\to 0$.

A norm $\|\ \|$ in a linear space X will be called *complete* if $\langle X, \|\ \|\rangle$ is complete. A norm $\|\ \|$ in X will be called *a norm of the first kind* if there exists a complete norm in X finer than $\|\ \|$. In particular, any complete norm is of the first kind. A norm will be called *of the second kind* if it is not of the first kind. A norm of the first kind on X may be of the second kind on a linear subset of X.

Norms $\|\ \|_a$ in a linear space X will be called *conformable* if there exists a complete norm $\|\ \|$ in X such that $\|\ \|_a \dashv \|\ \|$ for all a. In particular, two norms $\|\ \|_1$ and $\|\ \|_2$ are conformable if and only if the norm $\|x\|_3 = \|x\|_1+\|x\|_2$ is of the first kind.

Let $\|\ \|_2 \dashv \|\ \|_1$. The norms $\|\ \|_1$ and $\|\ \|_2$ are called *compatible* if conditions $\|x_p-x_q\|_1\to 0$ as $p,q\to\infty$ and $\|x_p-x\|_2\to 0$ imply $\|x_p-x\|_1\to 0$ as $p\to\infty$.

In the sequel we shall consider the following hypothesis:

(H) *all the norms in question and those which can be constructed effectively by the notions considered are conformable whenever they are defined on the same linear space, and all linear sets which may be constructed effectively satisfy the condition of Baire.*

Finally, $\varphi(u)$ will denote the function $\varphi(u) = u/(1+u)$ for $u\geqslant 0$.

3. Definitions of convergences

In this section the limits $\mathfrak{S}(X_n)$, $\overline{\mathfrak{S}}(X_n)$, $\overline{\mathfrak{S}}^*(X_n)$, $\mathfrak{S}_R(X_n)$, $\overline{\mathfrak{S}}_R(X_n)$ and $\overline{\mathfrak{S}}^*_R(X_n)$ will be defined.

3.1. Definition. *A sequence $\langle X_n, \|\ \|_n\rangle$ of F^*-spaces will be termed $\mathfrak{S}$-convergent to an F^*-space $\langle X_0, \|\ \|_0\rangle$ if the following conditions are satisfied:*

(a_1) *X_0 and almost all X_n are contained in a linear space $\mathfrak{X}$ (considered without topology), and the addition and the multiplication by scalars are consistent. Thus, if $x,y\in X_i\cap X_j$, then*

$$\underset{X_i}{x+y} = \underset{X_j}{x+y} = \underset{\mathfrak{X}}{x+y},$$

(a_2) *$\{X_n\}$ converges to X_0 in the set-theoretical sense, i. e.*

$$X_0 = \bigcup_{m=1}^{\infty}\bigcap_{n=m}^{\infty} X_n = \bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty} X_n,$$

(a_3) *if $x\in X_0$, then* [3]

$$\|x\|_0 = \lim_{n\to\infty}\|x\|_n.$$

Conditions (a_1)-(a_3) being satisfied, we shall write

$$\langle X_0, \|\ \|_0\rangle = \mathop{\mathfrak{S}}_{n\to\infty}\langle X_n, \|\ \|_n\rangle \quad \text{or} \quad X_0 = \mathop{\mathfrak{S}}_{n\to\infty}(X_n).$$

The space $\mathfrak{S}(X_n)$ is defined uniquely. Obviously, the convergence $\mathfrak{S}$ is a convergence $\mathcal{L}^*$ of Fréchet (cf. [9], p. 83).

3.2. Definition. *A sequence $\langle X_n, \|\ \|_n\rangle$ of F-spaces will be termed $\overline{\mathfrak{S}}$-convergent to an F-space $\langle X_0, \|\ \|_0\rangle$, written*

$$\langle X_0, \|\ \|_0\rangle = \mathop{\overline{\mathfrak{S}}}_{n\to\infty}\langle X_n, \|\ \|_n\rangle \quad \textit{or} \quad X_0 = \overline{\mathfrak{S}}(X_n),$$

if there exists a linear dense subset Y of $\langle X_0, \|\ \|_0\rangle$ such that $\langle Y, \|\ \|_0\rangle = \mathop{\mathfrak{S}}_{n\to\infty}\langle X_n, \|\ \|_n\rangle$.

Thus, a sequence is $\overline{\mathfrak{S}}$-convergent if and only if it is $\mathfrak{S}$-convergent. The space $\overline{\mathfrak{S}}(X_n)$ is defined uniquely up to linear isometry. The convergence $\overline{\mathfrak{S}}$ is a convergence $\mathcal{L}$ of Fréchet in the domain of complete spaces, but it is not $\mathcal{L}^*$, as Example 4.4 shows.

3.3. Definition. *A sequence $\langle X_n, \|\ \|_n\rangle$ of F-spaces will be termed $\overline{\mathfrak{S}}^*$-convergent to an F-space $\langle X_0, \|\ \|_0\rangle$ if, for every sequence $n_1<n_2<\dots$ of indices, there exists a subsequence $n_{k_1}<n_{k_2}<\dots$ such that $\langle X_0, \|\ \|_0\rangle = \mathop{\overline{\mathfrak{S}}}_{l\to\infty}\langle X_{n_{k_l}}, \|\ \|_{n_{k_l}}\rangle$.*

The convergence $\overline{\mathfrak{S}}^*$ is a convergence $\mathcal{L}^*$ of Fréchet and is more general than the convergence $\overline{\mathfrak{S}}$.

3.4. Definition. *A sequence $\langle X_n, \|\ \|_n\rangle$ of F^*-spaces will be termed $\mathfrak{S}_R$-convergent to a space $\langle X_0, \|\ \|_0\rangle$ if condition (a_1) and the following conditions are satisfied:*

(a_2') *$\{X_n\}$ converges to a linear subset Y_0 of $\mathfrak{X}$ in the set-theoretical sense and*

$$X_0 = \{x : x\in Y_0,\ \sup_n\|x\|_n<\infty\},$$

(a_3') *if $x\in X_0$, then $\|x\|_0 = \lim_{n\to\infty}\|x\|_n$; if $x\in Y_0\setminus X_0$, then $\|x\|_n\to\infty$.*

The convergence $\mathfrak{S}_R$ is a convergence $\mathcal{L}^*$ and is more general than the convergence $\mathfrak{S}$.

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[3] By (a_2), the norms $\|x\|_n$ are defined for almost all n.

The convergences $\overline{\mathfrak{S}}_R$ and $\overline{\mathfrak{S}}^*_R$ are defined for complete spaces analogously. The following scheme establishes the relations between the convergences defined above:

$$\mathfrak{S} \subset \mathfrak{S}_R, \qquad \overline{\mathfrak{S}} \begin{array}{l} \subset \overline{\mathfrak{S}}^* \subset \\ \subset \overline{\mathfrak{S}}_R \subset \end{array} \overline{\mathfrak{S}}^*_R.$$

The left-hand convergences are defined for sequences of F^*-spaces, the right-hand ones — for sequences of F-spaces; $\mathfrak{S}_1 \subset \mathfrak{S}_2$ means that any $\mathfrak{S}_1$-convergent sequence is $\mathfrak{S}_2$-convergent and the limit spaces are identical.

3.5. Given a directed set $\langle X_\tau, \| \|_\tau\rangle$ of F^*-spaces or of F-spaces, respectively, we may define the Moore-Smith limits

$$\mathop{\mathfrak{S}}_\tau(X_\tau), \quad \mathop{\overline{\mathfrak{S}}}_\tau(X_\tau), \quad \mathop{\overline{\mathfrak{S}}^*}_\tau(X_\tau), \quad \mathop{\mathfrak{S}_R}_\tau(X_\tau), \quad \mathop{\overline{\mathfrak{S}}_R}_\tau(X_\tau), \quad \mathop{\overline{\mathfrak{S}}^*_R}_\tau(X_{\tau_0})$$

in a way analogous to the preceding. In particular, set theoretical convergence means that every element belonging to a cofinal subset X_{τ_σ} belongs to all spaces X_τ with $\tau \geqslant \tau_0$.

4. Examples and elementary properties

4.1. Let us consider the spaces L_p $(1 \leqslant p \leqslant \infty)$ with $\|x\|_p = \left(\int\limits_0^1 |x(t)|^p dt\right)^{1/p}$. In this case $\mathfrak{X}$ may denote the class of all measurable functions defined in $\langle 0, 1\rangle$, considered up to sets of measure 0.

If a sequence p_n possesses at least two accumulation points, then $\{L_{p_n}\}$ is not convergent with respect to any of the above definitions.

If $p_n \searrow p$, then $\mathop{\mathfrak{S}}\limits_{n\to\infty}(L_{p_n})$ is not complete and $\mathop{\overline{\mathfrak{S}}}\limits_{n\to\infty}(L_{p_n}) = L_p$.

If $p_n \nearrow p < \infty$, then $\mathfrak{S}(L_{p_n})$ does not exist, but $\overline{\mathfrak{S}}_R(L_{p_n}) = L_p$. However, this case may be treated differently. Let us write

$$\|x\|^*_n = \sum_{k=1}^{n} 2^{-k}\varphi(\|x\|_{p_k}), \qquad \|x\|^* = \sum_{n=1}^{\infty} 2^{-n}\varphi(\|x\|_{p_n}), \qquad \varphi(u) = \frac{u}{1+u}.$$

Then $\mathop{\mathfrak{S}}\limits_{n\to\infty}\langle L_{p_n}, \| \|^*_n\rangle$ is the class of all measurable functions such that $\int\limits_0^1 |x(t)|^r dt < \infty$ for every $r < p$, and is a B_0-space with the norm $\| \|^*$.

Finally, $\langle L_\infty, \| \|_\infty\rangle = \mathop{\mathfrak{S}_R}\limits_{p\to\infty}\langle L_p, \| \|_p\rangle$.

4.2. Now, we shall consider the spaces of continuous functions defined on various intervals. Since the definitions introduced above require the existence of common elements of the spaces considered, we cannot consider the spaces of functions defined on different sets; the functions considered should be extended on to a suitable common set. Accordingly, $C(a, b)$ will denote the space of all continuous functions $x(t)$ defined in $(-\infty, \infty)$ and such that

$$x(t) = x(a) \text{ for } t \leqslant a \quad \text{and} \quad x(t) = x(b) \text{ for } t \geqslant b,$$

provided with the norm $\|x\| = \sup\limits_t |x(t)|$. Let $X_n = C(a_n, b_n)$.

If $a_n \nearrow a$, $b_n \searrow b$ and $a \neq b$, then $\mathfrak{S}(X_n) = C(a, b)$.

If $a_n \searrow a$ and $b_n \nearrow b$, then $\mathfrak{S}(X_n)$ consists of all continuous functions $x(t)$ constant in the intervals $(-\infty, a+\delta)$ and $(b-\delta, \infty)$, δ being a positive number depending on x; $\overline{\mathfrak{S}}(X_n)$ is equal to $C(a, b)$.

If $a_n \nearrow a$, $b_n \searrow b$ and $a = b$, then $\mathfrak{S}(X_n)$ consists only of constant functions, i. e. $\mathfrak{S}(X_n)$ is equivalent to E^1.

4.3. Let E^m, l_1^m and l_∞^m be the spaces of sequences

$$x = \{t_1, t_2, \ldots, t_m, 0, 0, \ldots\}$$

provided with the norms

$$\|x\|_2^m = \Big(\sum_{k=1}^{m} |x_k|^2\Big)^{1/2}, \qquad \|x\|_1^m = \sum_{k=1}^{m} |x_k|, \qquad \|x\|_\infty^m = \max_{k\leqslant m} |x_k|,$$

respectively. Then

$$\mathop{\overline{\mathfrak{S}}}_{m\to\infty}(E^m) = l_2, \qquad \mathop{\overline{\mathfrak{S}}}_{m\to\infty}(l_1^m) = l_1, \qquad \mathop{\overline{\mathfrak{S}}}_{m\to\infty}(l_\infty^m) = c_0.$$

No space $\mathfrak{S}(E^m)$, $\mathfrak{S}(l_1^m)$, $\mathfrak{S}(l_\infty^m)$ is complete.

This example shows that the limits of sequences of isomorphic spaces may be non-isomorphic. However, if the relations $X_1 \subset X_2 \subset \ldots$, $\| \|_n' \sim \| \|_n''$, $\mathfrak{S}\langle X_n, \| \|_n'\rangle = \langle X_0, \| \|'\rangle$, $\mathfrak{S}\langle X_n, \| \|_n''\rangle = \langle X_0, \| \|''\rangle$ hold and if the identical isomorphisms $x \to x$ from $\langle X_n, \| \|_n'\rangle$ onto $\langle X_n, \| \|_n''\rangle$ are equicontinuous, then the limit spaces are isomorphic. The above inclusions are essential; example 4.2 shows that the limit of a sequence of mutually isometric infinite-dimensional B-spaces may be one-dimensional.

4.4. Let $X_{2n} = l_2$ and $X_{2n+1} = E^n$ (cf. 4.3). Then $\mathop{\overline{\mathfrak{S}}}\limits_{n\to\infty}(X_n)$ does not exist (since $\mathfrak{S}(X_n)$ does not exist), but $\overline{\mathfrak{S}}(X_n) = l_2$.

4.5. An F^*-space is the $\mathfrak{S}^*$-limit of a sequence of finite-dimensional spaces if and only if its Hamel basis is countable. An F-space is the $\overline{\mathfrak{S}}$-limit of a sequence of finite-dimensional spaces if and only if it is separable. Consequently, separable F-spaces are determined by sequences of finite-dimensional spaces, and the totality of all separable F-spaces (considered up to linear isometry) is of power $2^{\aleph_0}$.

4.6. Let α be a fixed limit ordinal number and let C_β ($\beta \leqslant \alpha$) be the space of all continuous real sequences $\{x_\xi\}_{\xi\leqslant\alpha}$ such that $x_\xi = x_\alpha$ for $\xi \geqslant \beta$, provided with the norm $\|x\| = \sup|x_\xi|$. Then $\overline{\mathfrak{S}}_{\beta\to\alpha}(C_\beta) = C_\alpha$.

4.7. If $\langle X_n, \|\;\|_n\rangle$ are B^*-spaces, so is $\mathfrak{S}\langle X_n, \|\;\|_n\rangle$. If $\langle X_n, \|\;\|_n\rangle$ are inner-product spaces, so is $\mathfrak{S}\langle X_n, \|\;\|_n\rangle$, because inner-product spaces are characterized by the identity $\|x+y\|^2+\|x-y\|^2 = 2\|x\|^2+2\|y\|^2$.

4.8. If $\mathfrak{S}(X_n) = X_0$, then $\mathfrak{S}(X_n \cap X_0) = X_0$, but not conversely.

4.9. Let $K_n = \{x\colon \|x\|_n < 1,\ x \in X_n\}$, $H_n = \{x\colon \|x\|_n \leqslant 1,\ x \in X_n\}$, $n = 0, 1, \ldots$ If $X_0 = \mathfrak{S}(X_n)$, then

$$K_0 \subset \operatorname{Limes} K_n \subset \operatorname{Limes} H_n \subset H_0.$$

4.10. Let $\langle X_0, \|\;\|_0\rangle = \mathfrak{S}\langle X_n, \|\;\|_n\rangle$ and let us consider the following condition of uniform convergence: for every $\varepsilon > 0$ and $r > 0$ there exists an N such that, for every $n > N$ and $x \in X_0$, the inequality $\|x\|_0 < r$ implies $\big|\|x\|_n - \|x\|_0\big| < \varepsilon$.

If this condition holds and if X_n are B^*-spaces, then $\|\;\|_n \sim \|\;\|_{n+1}$ for almost all n. Indeed, if $\|\;\|_1$ and $\|\;\|_2$ are homogeneous and if there exists an η such that $0 < \eta < 1$ and such that $\|x\|_1 \leqslant 1$ implies $\big|\|x\|_1 - \|x\|_2\big| < \eta$, then $(1-\eta)\|x\|_1 \leqslant \|x\|_2 \leqslant (1+\eta)\|x\|_1$.

5. Auxiliary general lemmas concerning various norms in linear spaces

Some statements established in this section are well-known; they are collected for the sake of completeness and for continuity of further considerations.

5.1. *Let $\|\;\|$ be a pseudonorm in X and let Y_0 be a linear subset of X such that*

$$x \in Y_0,\ y \in Y_0,\ x \neq y \ \textit{imply}\ \|x-y\| \neq 0.$$

Then there exists a linear set X_0 containing exactly one element of any coset of $X/\|\;\|$ and such that $Y_0 \subset X_0 \subset X$.

5.2. *Let $X_1 \subset X_2$, $\|\;\|_1 \succeq \|\;\|_2$, let Y_0 be linear and closed in $\langle X_2, \|\;\|_2\rangle$. Then $Y_0 \cap X_1$ is closed in $\langle X_1, \|\;\|_1\rangle$.*

5.3. *Let $\langle X, \|\;\|\rangle$ be any infinite-dimensional F-space and let ξ_0 be a non-trivial linear functional on $\langle X, \|\;\|\rangle$. There exists a sequence of norms $\|\;\|_n$ incomparable with $\|\;\|$ and such that $\|x\|_n \to \|x\|$ for all $x \in X$. If $\langle X, \|\;\|\rangle$ is complete, such a sequence may be chosen so that $\langle X, \|\;\|\rangle$ are also complete.*

Proof. Let $\xi_0(x_0) = 1$ and let $\{x_\alpha\}_{1\leqslant\alpha<\alpha_0}$ be a Hamel basis for $X_0 = \{x\colon x \in X,\ \xi_0(x) = 0\}$. Then $\{x_\alpha\}_{0\leqslant\alpha<\alpha_0}$ is a Hamel basis for X with biorthogonal distributive functionals $\{\xi_\alpha\}_{0\leqslant\alpha<\alpha_0}$. There exists an index $n_0 < \omega$ such that ξ_{n_0} is not continuous, since only a finite number of functionals ξ_α can be continuous (this follows by considering elements $\sum_{n=1}^{\infty} a_n x_n = \sum_{n=1}^{N} b_n x_{\beta_n}$ with $\sum_{n=1}^{\infty} \|a_n x_n\| < \infty$). We may assume $n_0 = 1$. The transformations

$$U_n(x) = U_n\Big(\sum_{0\leqslant\alpha<\alpha_0} \xi_\alpha(x)x_\alpha\Big) = \Big[\xi_0(x) + \frac{1}{n}\xi_1(x)\Big]x_0 + \sum_{1\leqslant\alpha<\alpha_0} \xi_\alpha(x)\cdot x_\alpha$$

are distributive and one-to-one, and the norms $\|x\|_n = \|U_n(x)\|$ are well defined. Since the functional $\xi_0 + \frac{1}{n}\xi_1$ is continuous with respect to $\|\;\|_n$ and not continuous with respect to $\|\;\|$ and since ξ_0 is continuous on $\langle X, \|\;\|\rangle$ and not on $\langle X, \|\;\|_n\rangle$, the norms $\|\;\|$ and $\|\;\|_n$ are incomparable for each n. Next, for every $x \in X$,

$$\big|\|x\| - \|x\|_n\big| \leqslant \Big\|\frac{1}{n}\xi_1(x)x_0\Big\| \to 0,$$

whence $\|x\|_n \to \|x\|$. Finally, if $\|\;\|$ is complete, so are the norms $\|\;\|_n$, $n = 1, 2, \ldots$

5.4. *Let $\langle X, \|\;\|\rangle$ be any infinite-dimensional F^*-space. There exists a non-equivalent finer norm $\|\;\|_1$ in X.*

Proof. ξ being a distributive non-continuous functional on $\langle X, \|\;\|\rangle$, the norm $\|x\|_1 = \|x\| + |\xi(x)|$ is the desired one.

If $\langle X, \|\;\|\rangle$ is complete, $\langle X, \|\;\|_1\rangle$ is not, by Banach's inversion theorem. This shows that *a coarser norm may determine a smaller Cantor completion.*

5.5. *Let $\langle X, \|\;\|\rangle$ be complete, let $\|\;\|_1$ be non-equivalent and coarser than $\|\;\|$ and let $\langle X_1, \|\;\|_1\rangle = \operatorname{Compl}\langle X, \|\;\|_1\rangle$. Then X is of the first Baire category in $\langle X_1, \|\;\|_1\rangle$. If $\langle X, \|\;\|\rangle$ is separable, X is a Borel set in $\langle X_1, \|\;\|_1\rangle$.*

The first part follows by a theorem of Banach ([2], p. 38), the second — by a theorem of Souslin (cf. [9], p. 396).

5.6. *In any infinite-dimensional B^*-space $\langle X, \|\;\|\rangle$ there exists a non-equivalent quasi-normal coarser B-norm.*

Proof. If $\langle X, \|\;\|\rangle$ is complete and not reflexive, this follows by Theorem 3.3 of [1]. The case of a reflexive space is similar; non-complete spaces can be treated as dense subsets of complete spaces.

5.7. *Let $\langle X, \|\;\|_1\rangle$ and $\langle X, \|\;\|_2\rangle$ be complete and let there exists a norm $\|\;\|_3$ such that $\|\;\|_3 \preceq \|\;\|_1$ and $\|\;\|_3 \preceq \|\;\|_2$. Then $\|\;\|_1 \sim \|\;\|_2$.*

Proof. By Banach's inversion theorem, we have to prove that the norm $\|x\| = \|x\|_1 + \|x\|_2$ is complete. Let $\|x_p - x_q\| \to 0$ as $p, q \to \infty$. Then $\|x_p - x_q\|_1 \to 0$, $\|x_p - x_q\|_2 \to 0$ whence there exist elements x', x'' such that

$\|x_p - x'\|_1 \to 0$ and $\|x_p - x''\|_2 \to 0$ as $p \to \infty$. Since $\|x_p - x'\|_3 \to 0$ and $\|x_p - x''\|_3 \to 0$, we have $x' = x'' = x$, whence $\|x_p - x\| \to 0$.

5.8. *Let $\langle X_1, \| \|_1\rangle$ and $\langle X_2, \| \|_2\rangle$ be complete, and let Γ be a total set of linear functionals continuous with respect to either norm. Then $\| \|_1 \sim \| \|_2$.*

5.9. Let us divide all the complete norms in a linear space X into classes $\mathfrak{N}_a$ so that two norms belong to the same class if and only if they are equivalent. Let $\mathfrak{M}_a$ be the class of all norms coarser than a norm of $\mathfrak{N}_a$. Then *two norms of the first kind belong to the same class $\mathfrak{M}_a$ if and only if they are conformable.*

This follows by 5.7, as $\mathfrak{M}_a \cap \mathfrak{M}_\beta = 0$ for $a \neq \beta$.

5.10. *Let $\| \|_2$ be a coarser and quasi-normal norm in $\langle X, \| \|_1\rangle$. Then $\| \|_1$ and $\| \|_2$ are compatible.*

Proof [4]. Let $\|x_p - x_q\|_1 < \delta$ for $p, q > N$. Let $q \to \infty$ with fixed p; then $\|(x_p - x_q) - (x_p - x)\|_2 \to 0$ whence $\|x_p - x\|_1 \leqslant \varepsilon = \varepsilon(\delta)$ for $p > N$.

5.11. *Let $\| \|_1$ be a finer norm in an F-space $\langle X, \| \|_2\rangle$ and let $\| \|_1$ and $\| \|_2$ be compatible. Then $\| \|_1 \sim \| \|_2$.*

5.12. *Let $\| \|_2$ be a coarser norm in an F^*-space $\langle X, \| \|_1\rangle$, and let $\| \|_1$ and $\| \|_2$ be compatible. Then* $\mathrm{Compl}\langle X, \| \|_1\rangle = \langle X_1, \| \|_1\rangle$ *may be treated as a subset of* $\mathrm{Compl}\langle X, \| \|_2\rangle = \langle X_2, \| \|_2\rangle$ in the following sense: there exists a linear set Y such that

1° $X \subset Y \subset X_2$,

2° for every $y \in Y \setminus X$ there exists a sequence x_n of elements of X satisfying the Cauchy condition with respect to either norm $\| \|_1$ and $\| \|_2$ and belonging to the coset y,

3° for every $z \in X_1 \setminus X$ there exists exactly one element $y \in Y \setminus X$ such that the Cauchy sequences corresponding to z (with respect to $\langle X_1, \| \|_1\rangle$) correspond to y (with respect to $\langle X_2, \| \|_2\rangle$).

5.13. *Let $\| \|_1$ and $\| \|_2$ be two compatible norms of the first kind in X. Then they are conformable.*

5.14. *There exist effective conformable norms $\| \|_1 \mathrel{\succ\!\!-} \| \|_2$ which are not compatible.*

Proof. Let $\langle X, \| \|\rangle$ be a B-space, let $\| \|_2$ be any coarser non-equivalent norm in it, let

$$\xi \in \mathrm{Conj}\langle X, \| \|\rangle \setminus \mathrm{Conj}\langle X, \| \|_2\rangle$$

(cf. [12], p. 138), and let $\|x\|_1 = \|x\|_2 + |\xi(x)|$. A sequence $x_n \in X$ such that $\|x_n\|_2 \to 0$ and $\xi(x_n) = 1$ contradicts compatibility.

[4] In this case $\| \|_2$ may be replaced by a convergence $\mathfrak{L}$ of Fréchet. Let us notice that almost all proofs of completeness of concrete linear spaces apply the above general scheme. Propositions 5.10 and 5.11 are close to a theorem established by W. Orlicz in section 3.1 of [14], p. 1, and to Theorems 2.1 and 2.2 of Schäffer's [16].

5.15. *If Y is a linear set dense in $\langle X, \| \|_1\rangle$ and if $\langle Y, \| \|_1, \| \|_2\rangle$ is quasi-normal, so is $\langle X, \| \|_1, \| \|_2\rangle$.*

More precisely: let $\varepsilon(\delta)$ be a continuous and strictly increasing function defined for $\delta > 0$ and let $\varepsilon(\delta) \to 0$ as $\delta \to 0$. Next, let Y be dense in X with respect to $\| \|_1$, and let $y_n \in Y$ $(n = 0, 1, 2, \ldots)$, $\|y_n - y_0\|_2 \to 0$, $\underline{\lim}\|y_n\|_1 \leqslant \delta$ imply $\|y_0\|_1 \leqslant \varepsilon(\delta)$. Then $x_n \in X$ $(n = 0, 1, 2, \ldots)$, $\|x_n - x_0\|_2 \to 0$, $\underline{\lim}\|x_n\|_1 \leqslant \delta$ imply $\|x_0\|_2 \leqslant \varepsilon(\delta)$.

Proof. Let us assume $\|x_n - x_0\|_2 \to 0$ and $\|x_n\|_1 \leqslant \delta$ for $n = 1, 2, \ldots$ Choose $\varepsilon' > 0$ arbitrarily and then a δ' such that $0 < \delta' < \varepsilon'$ and $\varepsilon(\delta + \delta') < \varepsilon(\delta) + \varepsilon' - \delta'$. There exist $y_n \in Y$ $(n = 0, 1, 2, \ldots)$ such that $\|y_0 - x_0\|_1 < \delta'$ and $\|y_n - (x_n + y_0 - x_0)\|_1 < 1/n$. Then

$$\underline{\lim}\|y_n\|_1 \leqslant \underline{\lim}\left[\|x_n\|_1 + \|y_0 - x_0\|_1 + \frac{1}{n}\right] \leqslant \delta + \delta'.$$

Hence $\|y_0\|_1 \leqslant \varepsilon(\delta + \delta') < \varepsilon(\delta) + \varepsilon' - \delta'$ and $\|x_0\|_1 < \varepsilon(\delta) + \varepsilon'$. Since ε' is arbitrary, we have $\|x_0\|_1 \leqslant \varepsilon(\delta)$.

5.16. *Let $\| \|_1$ and $\| \|_2$ be norms in X and let the functional $\Phi(x) = \|x\|_2$ be continuous on $\langle X, \| \|_1\rangle$, i. e. let $\|x_n - x\|_1 \to 0$ imply $\|x_n\|_2 \to \|x\|_2$. Then $\| \|_2 \mathrel{-\!\!\prec} \| \|_1$.*

6. B_0-spaces as limit spaces

We shall consider B_0^*-spaces provided with a monotone total sequence of homogeneous pseudonorms $\|x\|_1 \leqslant \|x\|_2 \leqslant \ldots$ and the norm

$$\|x\| = \sum_{n=1}^{\infty} 2^{-n}\varphi(\|x\|_n).$$

6.1. *Every B_0-space $\langle X, \| \|\rangle$ is the $\overline{\mathfrak{S}}$-limit of a sequence $\langle X_n, \| \|_n^*\rangle$ of spaces isomorphic to B^*-spaces and such that $\| \|_n^* \mathrel{-\!\!\prec} \| \|$.*

Proof. By 5.1. and by easy induction we can construct a sequence $X_1 \subset X_2 \subset \ldots$ of linear subsets of X such that X_n contains exactly one element of any coset of $X/\| \|_n$. Obviously

$$\|x\|_n^* = \sum_{k=1}^{n} 2^{-k}\varphi(\|x\|_k)$$

is a norm in X_n equivalent to the norm $\| \|_n$ and $\lim\|x\|_n^* = \|x\|$ for $x \in X$. Since $\bigcup_{n=1}^{\infty} X_n$ is dense in $\langle X, \| \|\rangle$, we have $\langle X, \| \|\rangle = \overline{\mathfrak{S}}\langle X_n, \| \|_n\rangle$.

6.2. *Let us assume hypothesis* (H). *A B_0-space $\langle X, \| \|\rangle$ is a $B_0^{\#}$-space if and only if it is the $\mathfrak{S}$-limit of a sequence $\langle X_n, \| \|_n\rangle$ of F^*-spaces isomorphic to B^*-spaces.*

Proof. Necessity. Let $\|x\|_1 \leqslant \|x\|_2 \leqslant \ldots$ be a sequence of B-norms determining the topology of $\langle X, \| \, \| \rangle$. First we shall prove the theorem under an additional hypothesis — we shall assume that each norm $\| \, \|_n$ is quasi-normal with respect to $\| \, \|_{n+1}$ (this proof is due to A. Pełczyński).

Let $\langle X_n, \| \, \|_n \rangle = \mathrm{Compl}\langle X, \| \, \|_n \rangle$. By 5.10 and 5.12, we may assume $X_1 \supset X_2 \supset \ldots$ Thus, we have to prove that $\bigcap_{n=1}^{\infty} X_n = X$. Inclusion $X \subset \bigcap X_n$ being obvious, let us consider any element $x_0 \in \bigcap X_n$. For each n there exists an element $x_n \in X$ such that $\|x_0 - x_n\|_n < 2^{-n-1}$. Hence $\|x_0 - x_n\|_m \leqslant \|x_0 - x_n\|_n \leqslant 2^{-n-1}$ for $m \leqslant n$ and, consequently, $\|x_n - x_m\|_m \leqslant 2^{-n-1} + 2^{-m-1} \leqslant 2^{-m}$. Thus, $\{x_n\}$ satisfies the Cauchy condition with respect to $\| \, \|$, whence it is convergent to a $y \in X$, $\langle X, \| \, \| \rangle$ being complete. Obviously, $x_0 = y$, whence $x_0 \in X$.

Considering norms $\| \, \|_n^*$ in X_n defined above we conclude that $\|x\|_n^* \to \|x\|$, whence $\langle X, \| \, \| \rangle = \mathfrak{S}\langle X_n, \| \, \|_n \rangle$.

Now, we shall prove the theorem without the assumption of quasi-normality. We shall define spaces $\langle X_n, \| \, \|_n \rangle$ by induction. Let $\langle X_1, \| \, \|_1 \rangle = \mathrm{Compl}\langle X, \| \, \|_1 \rangle$ and let us suppose that $\langle X_n, \| \, \|_n \rangle$ has just been defined. Let Z_n denote the set of all elements z of $X_n \setminus X$ for which there exist sequences $\{y_m\}$ of elements of X satisfying the Cauchy condition with respect to the norms $\| \, \|_n$ and $\| \, \|_{n+1}$, convergent to z with respect to the norm $\| \, \|_n$ in X_n. The norm $\| \, \|_{n+1}$ can be extended onto $Y_n = X \cup Z_n$ by continuity; let $\langle X_{n+1}, \| \, \|_{n+1} \rangle = \mathrm{Compl}\langle Y_n, \| \, \|_{n+1} \rangle$. Obviously, $\langle X_{n+1}, \| \, \|_{n+1} \rangle$ is a completion of $\langle X, \| \, \|_{n+1} \rangle$ and $X_{n+1} \cap X_n = Y_n$. The set $X_{n+1} \setminus Y_n$ consists of elements z such that a sequence $z_m \in X$ converges to z with respect to $\| \, \|_{n+1}$ and converges to another element of X_n with respect to $\| \, \|_n$. Since $\|x\|_n \leqslant \|x\|_{n+1}$ for $x \in X$, the limit $\lim_{m \to \infty} \|z_m\|_n$ exists for every sequence $\{z_m\}$ satisfying the Cauchy condition with respect to $\| \, \|_{n+1}$ and we can define $\|z\|_n$ for $z \in X_{n+1} \setminus X$. Obviously, $\| \, \|_n$ is a homogeneous pseudonorm in X_{n+1} and is a norm if and only if $X_n \supset X_{n+1}$, i. e. if $X_{n+1} = Y_n$. We define $\| \, \|_k$ on X_{n+1} similarly.

We have to prove that $X = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} X_k = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} X_k$. The inclusions $X \subset \bigcup \bigcap X_k \subset \bigcap \bigcup X_k$ are evident. Thus let us consider any element x_0 belonging to infinitely many spaces $X_{k_1}, X_{k_2}, \ldots$ where $k_1 < k_2 < \ldots$ There exist $x_n \in X$ such that $\|x_0 - x_n\|_{k_n} \leqslant 2^{-n}$, $n = 1, 2, \ldots$ Since $\{x_m\}$ satisfies the Cauchy condition with respect to each norm $\| \, \|_n$, there exists $y_0 \in X$ such that $\|x_n - y_0\|_n \to 0$ for $n = 1, 2, \ldots$ The next argument is analogous to that in the preceding case.

Sufficiency. Let $\langle X, \| \, \| \rangle = \mathfrak{S}\langle X_n, \| \, \|_n \rangle$, where $\langle X_n, \| \, \|_n \rangle$ are isomorphic to B^*-spaces. By axiom (H), all sets $X \cap X_n$ satisfy the condition of Baire in $\langle X, \| \, \| \rangle$ whence, by the theorem on the category of subgroups ([2], p. 22), there exists n_0 such that $X = X_{n_0} \cap X$, i.e. $X \subset X_{n_0}$. Hence $\| \, \|_{n_0}$ is defined on X and, by axiom (H) again, we have $\| \, \| \succeq \| \, \|_{n_0}$, which means that X is a $B_0^{\#}$-space.

The postulate (H) can be replaced by an additional hypothesis in the theorem, e. g. the above proof is also valid for the following statement:

6.3. *A B_0-space $\langle X, \| \, \| \rangle$ is a $B_0^{\#}$-space if and only if there exist F^*-spaces $\langle X_n, \| \, \|_n \rangle$ such that $X \subset X_n$, $\| \, \| \succeq \| \, \|_n$ and such that every space $\langle X_n, \| \, \|_n \rangle$ is isomorphic to a B^*-space.*

A comparison of this theorem with the former one shows the importance of hypothesis (H) or of a similar axiom in such cases. The next statement shows that (H) is essential in 6.2.

6.4. *Any B_0^*-space is the $\mathfrak{S}$-limit of a sequence of F^*-spaces isomorphic to B^*-spaces.*

Proof. Let $\| \, \|_1, \| \, \|_2, \ldots$ be a sequence of pseudonorms determining the topology on X and let $\| \, \|^*$ be any homogeneous norm in X; such a norm always exists, it may be constructed for example by an embedding of X into a Hilbert space with a suitable power of Hamel basis. Writing

$$\|x\|_n^* = \sum_{k=1}^{n} 2^{-k} \varphi(\|x\|_k) + \frac{1}{n} \|x\|^*$$

we obtain $\|x\| = \lim \|x\|_n^*$ for $x \in X$.

6.5. *Neither $\mathfrak{S}$-limit nor $\overline{\mathfrak{S}}$-limit of a sequence of locally convex F^*-spaces need be locally convex.*

Example. The space X of all real sequences $x = \{t_1, t_2, \ldots\}$ such that $\|x\| = \sum_{k=1}^{\infty} |t_n|^{1/2} < \infty$, X_n — the subspace consisting of the sequences such that $t_m = 0$ for $m > n$, and $X_0 = \bigcup_{n=1}^{\infty} X_n$.

7. Decreasing sequences ($X_n \supset X_{n+1}$, $\| \, \|_n \preceq \| \, \|_{n+1}$)

In typical cases finer norms correspond to smaller spaces, although $\| \, \|_1 \preceq \| \, \|_2$ does not imply $\mathrm{Compl}\langle X, \| \, \|_1 \rangle \supset \mathrm{Compl}\langle X, \| \, \|_2 \rangle$, in general (compare 5.4, 5.10, 5.12). Therefore, when we consider decreasing sequences

$$X_1 \supset X_2 \supset X_3 \supset \ldots,$$

we shall assume that the relations between the norms are converse:

$$\| \, \|_1 \preceq \| \, \|_2 \preceq \| \, \|_3 \preceq \ldots$$

Then $\mathfrak{S}(X_n) = \bigcap_{n=1}^{\infty} X_n$; moreover, $\| \, \|_0 \succeq \| \, \|_n$ is often valid for $n = 1, 2, \ldots$

The following question, related to the preceding, arises naturally. Given a linear space X and a sequence of norms in X such that $\|x\|_1 \leqslant \|x\|_2 \leqslant \ldots$ for all $x \in X$, let $\langle X_n, \| \|_n\rangle = \mathrm{Compl}\langle X, \| \|_n\rangle$. What are the properties of $\mathfrak{S}_{n\to\infty}\langle X_n, \| \|_n\rangle$?

7.1. *Let* $\langle X_n, \| \|_n\rangle$ $(n = 1, 2, \ldots)$ *be* F*-spaces such that*

(i) $X_n \supset X_{n+1}$, $\| \|_n \precsim \| \|_{n+1}$ *for* $n = 1, 2, \ldots$,

(ii) $X_0 = \bigcap_{n=1}^{\infty} X_n$, $\|x\|_0 = \lim_{n\to\infty} \|x\|_n < \infty$ *for* $x \in X_0$,

(iii) $\| \|_0 \succsim \| \|_n$ *for* $n = 1, 2, \ldots$

and satisfying at least one of the following conditions:

(u_1) $\| \|_n$ *are homogeneous,*

(u_2) $\langle X_0, \| \|_0\rangle$ *is complete,*

(u_3) $\|x\|_n \leqslant \|x\|_0$ *for* $x \in X_0$,

(u_4) *there exists a dense subset* Y *of* $\langle X_0, \| \|_0\rangle$ *such that, for every* $x \in Y$ *and* $\varepsilon > 0$, *there exists a sphere* $K_x = \{y \in X_0 : \|x - y\|_0 < \delta_x\}$ *and an indice* N *with the following property* (⁵): *if* $z \in K_x$ *and* $n > N$, *then*

$$\big|\|z\|_0 - \|z\|_n\big| < \varepsilon.$$

Then the norm

$$\|x\|^* = \sum_{n=1}^{\infty} 2^{-n} \varphi(\|x\|_n)$$

is equivalent to the norm $\| \|_0$ *on* X_0 (⁶).

Proof. First, we shall prove that the space $\langle X_0, \| \|^*\rangle$ is complete. Let $x_n \in X_0$ and $\|x_n - x_m\|^* \to 0$ as $n, m \to \infty$. Then $x_n \in X_1$ for $n = 1, 2, \ldots$ and $\|x_n - x_m\|_1 \to 0$ as $n, m \to \infty$. Consequently, by the completeness of $\langle X_1, \| \|_1\rangle$, there exists a $y_1 \in X_1$ such that $\|x_n - y_1\|_1 \to 0$ as $n \to \infty$. Next, $\langle X_2, \| \|_2\rangle$ being complete, there exists a $y_2 \in X_2$ such that $\|x_n - y_2\|_2 \to 0$ as $n \to \infty$. Since $X_1 \supset X_2$ and $\| \|_1 \precsim \| \|_2$, y_2 belongs to X_1 and $\|x_n - y_2\|_2 \to 0$, whence $y_1 = y_2$. Arguing similarly we prove that $y_1 \in X_k$ and $\lim_{n\to\infty} \|x_n - y_1\|_k = 0$ for $k = 1, 2, \ldots$ Hence $y_1 \in X_0$ and $\|x_n - y_1\|^* \to 0$ as $n \to \infty$. By (iii), the norm $\| \|^*$ is coarser than $\| \|_0$.

Now, let us consider conditions (u_1)-(u_4) separately.

(⁵) In other words: there exists a dense collection of spheres in $\langle X_0, \| \|_0\rangle$ such that the convergence $\|x\|_n \to \|x\|_0$ is uniform in each sphere separately.

(⁶) This theorem establishes that in typical cases the intersection of a decreasing sequence of B-spaces is a B_0-space, and it is not a B^*-space excepting trivial cases (cf. Mazur and Orlicz [11], p. 189). It is close to a theorem on decreasing fields of summability of a sequence of matrix methods (Zeller [18], p. 52). Some results of this type have been proved by Gorin and Mitiagin [8].

If (u_1) holds, then the functional

$$U_0(x) = \|x\|_0 = \lim_{n\to\infty} \|x\|_n = \lim_{n\to\infty} U_n(x)$$

is the limit of a sequence of convex continuous functionals and is continuous on $\langle X_0, \| \|^*\rangle$, by a theorem of Mazur and Orlicz [10], p. 157, th. 6.1; see also [15]. This means that $\| \|_0 \sim \| \|^*$ (cf. 5.16).

If $\langle X_0, \| \|_0\rangle$ is complete, $\| \|_0 \sim \| \|^*$ follows by Banach's inversion theorem.

Next, we shall prove that (u_3) implies normality of $\langle X_0, \| \|_0, \| \|^*\rangle$. Let $x_m \in X_0$ for $m = 0, 1, 2, \ldots$, let $\|x_m - x_0\|^* \to 0$ and $\|x_m\|_0 \leqslant 1$ for $m = 1, 2, \ldots$ Then $\|x_m\|_n \leqslant \|x_m\|_0 \leqslant 1$ for $m, n = 1, 2, \ldots$, whence $\|x_0\|_n \leqslant 1$ for $n = 1, 2, \ldots$ and $\|x_0\|_0 = \lim_{n\to\infty} \|x_0\|_n \leqslant 1$. Thus, $\| \|_0 \sim \| \|^*$ by 5.10 and 5.11.

Finally, condition (u_4) implies normality for a dense set of limits, whence, by 5.15, $\| \|_0 \sim \| \|^*$ as in the preceding case.

7.2. *If conditions* (i), (ii), (iii), (u_1) *hold, then* $\| \|_n \sim \| \|_0$ *for almost all* n.

In other words, these conditions can be satisfied only if $X_n = X_{n+1}$ for $n > N$ or if X_n are decreasing subspaces of a fixed space, admitting some change of norms to equivalent ones.

This follows from 7.1. (cf. [11], p. 194).

7.3. *Let* $\langle X_n, \| \|_n\rangle$ *be* B*-spaces,* $X_n \supset X_{n+1}$ *and* $\|x\|_n \leqslant \|x\|_{n+1}$ *for* $x \in X_{n+1}$ *and for* $n = 1, 2, \ldots$, *and let* $\bigcap_{n=1}^{\infty} X_n$ *be dense in each space* $\langle X_n, \| \|_n\rangle$. *Then either* $X_n = X_{n+1}$ *and* $\| \|_n \sim \| \|_{n+1}$ *for almost all* n *or there exists an element* $x_0 \in \bigcap_{n=1}^{\infty} X_n$ *such that* $\lim_{n\to\infty} \|x_0\|_n = \infty$.

Proof. Let $\|x\|_0 = \lim \|x\|_n < \infty$ for all $x \in X_0 = \bigcap X_n$. Then $\langle X_0, \| \|_0\rangle = \mathfrak{S}\langle X_n, \| \|_n\rangle$ and, by 7.2, $\| \|_n \sim \| \|_{n+1}$ and $X_n = X_{n+1}$ for $n > N$, since the spaces under consideration are complete and contain a common dense subset.

7.4. *Under the hypotheses of* 7.1 *we have*

$$\langle Y_0, \| \|_0\rangle = \mathfrak{S}\langle Y_n, \| \|_n\rangle,$$

Y *being any linear subset of* X_0 *and* Y_n *being its closure in* $\langle X_n, \| \|_n\rangle$, $n = 0, 1, 2, \ldots$ *In particular,* $\langle X_0, \| \|_0\rangle = \mathfrak{S}\langle Z_n, \| \|_n\rangle$ *where* Z_n *is the closure of* X_0 *in* $\langle X_n, \| \|_n\rangle$.

Proof. Obviously, $Y_0 \subset Y_n \subset Y_{n-1}$ and $Y_0 \subset \bigcap Y_n$. To prove $Y_0 = \bigcap Y_n$ let us consider any element $x \in \bigcap Y_n$. For every m there exists an element $x_m \in Y$ such that $\|x_m - x\|_k < 1/m$ for $k = 1, 2, \ldots, m$. Consequently, $\|x_m - x\|^* \to 0$ as $m \to \infty$ whence $\|x_m - x\|_0 \to 0$ by 7.1, and $x \in Y_0$.

7.5. The completeness of the spaces $\langle X_n, \| \|_n\rangle$ is essential in 5.1, even if $\langle X_0, \| \|_0\rangle$ is complete.

Example. $\langle L_\infty, \| \|_0\rangle = \underset{p\to\infty}{\mathfrak{S}} \langle L_\infty, \| \|_p\rangle$.

7.6. The following generalization of Theorem 7.1 to the case of groups G_n provided with metrics $\varrho_n(x, y)$ (non necessarily invariant) may be proved. *If the groups $\langle G_n, \varrho_n\rangle$ are complete, $G_n \supset G_{n+1}$ and $\varrho_n \dashv \varrho_{n+1}$, and if the group $G_0 = \bigcap_{n=1}^{\infty} G_n$ is complete and separable with respect to the metrics*

$$\varrho_0(x, y) = \lim_{n\to\infty} \varrho_n(x, y)$$

or if $\langle G_0, \varrho_0\rangle$ is separable and $\varrho_n(x, y) \leqslant \varrho_{n+1}(x, y)$ for $x, y \in G_0$, then the metrics ϱ_0 and

$$\varrho^*(x, y) = \sum_{n=1}^{\infty} 2^{-n}\varphi\big(\varrho_n(x, y)\big)$$

are equivalent.

The proof is analogous to that in 7.1; the assumption of separability enables us to apply Banach's theorem on the inversion of one-to-one homomorphisms (cf. [3] and [9], p. 399).

7.7. *Let $X_n \supset X_0$ for $n = 1, 2, \ldots$, let Y be a set dense in $\langle X_0, \| \|_0\rangle$, let $\|x\|_0 = \lim_{n\to\infty} \|x\|_n$ for $x \in Y$ and let $\| \|_n$ be uniformly coarser than $\| \|_0$. Then $\|x\|_0 = \lim_{n\to\infty} \|x\|_n$ holds for every $x \in X_0$.*

7.8. *Let $\langle X_0, \| \|_0\rangle = \mathfrak{S}\langle X_n, \| \|_n\rangle$, $X_0 = \bigcap X_n$ and let $\langle X_0, \| \|_0\rangle$ be complete. Then condition (u_4) is satisfied.*

Proof. Writing

$$E_m = \{x \in X_0 \colon |\|x\|_0 - \|x\|_m| \leqslant \varepsilon\} \quad \text{and} \quad F_m = \bigcap_{k\geqslant m} E_k$$

we apply the Baire category theorem.

7.9. *Let $\langle X_0, \| \|_0\rangle = \mathfrak{S}\langle X_n, \| \|_n\rangle$, $X_0 = \bigcap X_n$, $\| \|_n \dashv \| \|_0$, let $\langle X_0, \| \|_0\rangle$ be complete and let $\| \|_n$ be homogeneous. Then there exists a constant K such that*

$$\|x\|_n \leqslant K\|x\|_0.$$

Proof. This is a consequence of a theorem of Mazur and Orlicz ([10], p. 157) as well as of 7.8.

7.10. The assumption of completeness of $\langle X_0, \| \|_0\rangle$ in 7.9 and that of the uniformity of relation $\| \|_n \dashv \| \|_0$ in 7.7 are essential.

Example. Let X be the class of null-convergent sequences $x = \{x_n\}$ and let X_0 the subset of all sequences with almost all x_m equal to 0, and let

$$\|x\|_0 = \sup_m |x_m|, \qquad \|x\|_n = \max(\|x\|_0, n|x_n|).$$

Then $\| \|_n \sim \| \|_0$ and $\|x\|_0 \leqslant \|x\|_n \leqslant n\|x\|_0$ $(n = 1, 2, \ldots)$. For every $x \in X_0$ the relation $\lim \|x\|_n = \|x\|_0$ is fulfilled, but $\lim \|x\|_n = \infty$ for some x in X.

7.11. It may happen that any two norms $\| \|_n$, $\| \|_m$ are uncomparable $(n \neq m,\ n \neq 0,\ m \neq 0)$ although

$$X_0 = \mathfrak{S}(X_n), \quad X_n \supset X_{n+1} \supset X_0 \quad \text{and} \quad \| \|_n \dashv \| \|_0 \text{ for } n = 1, 2, \ldots$$

Example. $X = X_n = C\langle 0, 1\rangle$, $\|x\|_0 = \max_{0\leqslant t\leqslant 1} |x(t)|$,

$$\|x\|_n = \int_{(n-1)/n}^{n/(n+1)} |x(t)|\,dt + \max\left\{|x(t)| \colon t \in \left\langle 0, \frac{n-1}{n}\right\rangle \cup \left\langle \frac{n}{n+1}, 1\right\rangle\right\}.$$

7.12. The following example shows that $\mathfrak{S}(X_n) = X_0$, $\| \|_n \dashv \| \|_0$, $\| \|_n \sim \| \|_{n+1}$, $X_n \supset X_{n+1} \supset X_0$ do not imply $\| \|_n \sim \| \|_0$ for any n.

Example. $X = l^1$, $\|x\|_0 = \sum_{n=1}^{\infty} |x_n|$, $\|x\|_n = \sum_{k=1}^{n} |t_k| + \big(\sum_{k=n+1}^{\infty} |t_k|^2\big)^{1/2}$.

7.13. *Let $X_1 \supset X_2 \supset \ldots$ be F-spaces such that $\|x\|_n \leqslant \|x\|_{n+1}$ for $x \in X_{n+1}$. Then the space*

$$\mathfrak{S}_R(X_n) = \{x \colon x \in \bigcap_{n=1}^{\infty} X_n,\ \sup_n \|x\|_n < \infty\}$$

is complete with respect to the norm $\|x\|_0 = \sup_n \|x\|_n$.

7.14. *If $\| \|_n$ $(n = 0, 1, 2, \ldots)$ are norms defined in a linear space X and $\|x\|_n \leqslant \|x\|_{n+1}$ and $\|x\|_0 = \lim \|x\|_n$, then the sequence of completions*

$$\langle X_n, \| \|_n\rangle = \mathrm{Compl}\langle X, \| \|_n\rangle$$

is not $\mathfrak{S}$-convergent to $\langle X_0, \| \|_0\rangle = \mathrm{Compl}\langle X, \| \|_0\rangle$ even if $X_n \supset X_{n+1} \supset X_0$, unless the norm $\| \|_0$ is equivalent to the norm $\| \|^*$ defined in 7.1. However, by 7.7, the convergence of norms in X implies the convergence of extended norms in the completion of the limit space $\langle X, \| \|_0\rangle$.

8. Increasing sequences ($X_n \subset X_{n+1}$, $\| \|_n \vdash \| \|_{n+1}$)

In this section we shall consider sequences $\{X_n\}$ with $X_n \subset X_{n+1}$ whence $\underset{n\to\infty}{\mathfrak{S}}(X_n) = \bigcup_{n=1}^{\infty} X_n$. In typical cases the condition $\| \|_n \vdash \| \|_{n+1}$ is also satisfied, and often we have $\| \|_n \vdash \| \|_0$ for $n = 1, 2, \ldots$ However, just as in preceding section, in some cases we may assume $X_n \subset X_0$ and $\| \|_n \vdash \| \|_0$ for $n = 1, 2, \ldots$

The following theorem is of importance in further considerations (7).

(7) This theorem was formulated in various special cases. E. g. it is closely connected with a theorem stating that the union of the sequence of fields of summability of a sequence of methods cannot be the field of summability of a method (cf. [18], p. 30 and 51).

8.1. *Let* $\langle X_n, \| \|_n \rangle$ $(n = 0, 1, 2, \ldots)$ *be F-spaces satisfying the following conditions*:

(i) $X_n \subset X_0$ *and* $\| \|_n \succeq \| \|_0$ *for* $n = 1, 2, \ldots$,

(ii) $\langle X_0, \| \|_0 \rangle = \mathfrak{S}_{n\to\infty} \langle X_n, \| \|_n \rangle$.

Then there exists N *such that* $X_n = X_0$ *and* $\| \|_n \sim \| \|_0$ *for* $n > N$. *Thus, if* $X_n \neq X_{n+1}$ *for almost all* n, *then* $\mathfrak{S}(X_n)$ *is necessarily non-complete.*

Proof. Given fixed n $(n \geqslant 1)$, let us consider the identical operation $U(x) = x$ as an operation from $\langle X_n, \| \|_n \rangle$ into $\langle X_0, \| \|_0 \rangle$. U is linear, so either $X_n = X_0$ or X_n is of the first category in $\langle X_0, \| \|_0 \rangle$, by a theorem of Banach ([2], p. 38).

Now we shall show that the asumption $X_{n_k} \neq X_0$ $(n_1 < n_2 < \ldots)$ leads to a contradiction. Indeed, since

$$X_0 = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} X_n = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} X_n,$$

every element of X_0 belongs to almost all X_n, whence $X_0 \subset \bigcup_{k=1}^{\infty} X_{n_k}$. Consequently, if X_{n_k} were of the first category, X_0 would also be such in spite of the completeness of X_0.

If $X_n = X_0$, then $\| \|_n \sim \| \|_0$ by Banach's inversion theorem.

8.2. The preceding theorem may be generalized as follows:

Let $\langle G_n, \varrho_n \rangle$ $(n = 0, 1, \ldots)$ *be a sequence of complete separable metric groups such that*

$$G_0 \supset G_n \quad \text{and} \quad \varrho_0 \preceq \varrho_n \quad \text{for} \quad n = 1, 2, \ldots$$

and $\langle G_0, \varrho_0 \rangle = \mathfrak{S}_{n\to\infty} \langle G_n, \varrho_n \rangle$. *Then* $G_n = G_0$ *and* $\varrho_n \sim \varrho_n$ *for* $n > N$, *unless* G_n *are open-and-closed in* $\langle G_0, \varrho_0 \rangle$ *for* $n = 1, 2, \ldots$

The proof is analogous to that of 8.1. It is based on the following theorem of Souslin (cf. [9], p. 396): *if* Φ *is a one-to-one continuous map from a complete separable metric space* X *into any metric space* Y, *then* Φ *transforms* X *onto a Borel set in* Y; moreover, it is based on the theorem on the category of subgroups and on Banach's theorem on the inversion of homomorphism in separable metric groups (see [2], p. 22, and [3]).

8.3. *Let* $\langle X_n, \| \|_n \rangle$ $(n = 0, 1, \ldots)$ *be* F^**-spaces such that* $X_n \subset X_0$ *and* $\| \|_n \succeq \| \|_0$ *for* $n = 1, 2, \ldots$ *and* $\langle X_0, \| \|_0 \rangle = \mathfrak{S}_{n\to\infty} \langle X_n, \| \|_n \rangle$. *Next, let* Z *be a subset of* X_0 *dense in infinitely many spaces* $\langle X_{n_k}, \| \|_{n_k} \rangle$. *Then* Z *is dense in* $\langle X_0, \| \|_0 \rangle$.

In particular, if $\langle X_{n_k}, \| \|_{n_k} \rangle$ are separable, so is $\langle X_0, \| \|_0 \rangle$.

8.4. *Let* $\langle X_n, \| \|_n \rangle$ $(n = 0, 1, 2, \ldots)$ *be* F^**-spaces such that*

$$X_n \subset X_0 = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} X_k = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} X_k$$

and $\| \|_n \succeq \| \|_0$ *for* $n = 1, 2, \ldots$ *and such that* $\| \|_n$ *is finer than* $\| \|_{n+m}$ *uniformly with respect to* m, *i. e.*

$$\prod_n \prod_{\varepsilon > 0} \sum_{\delta > 0} \prod_m \prod_{x \in X_n \cap X_{n+m}} \|x\|_n < \delta \Rightarrow \|x\|_{n+m} < \varepsilon.$$

In particular, this last condition is satisfied if $\|x\|_n \geqslant \|x\|_{n+1}$ for $x \in X_n \cap X_{n+1}$, $n = 1, 2, \ldots$

Next, *let* Y *be a subset of* X_0 *dense in infinitely many spaces* $\langle X_{n_k}, \| \|_{n_k} \rangle$ *and such that*

$$\|y\|_0 = \lim_{n\to\infty} \|y\|_n \quad \text{for all} \quad y \in Y.$$

Then $\|x\|_0 = \lim_{n\to\infty} \|x\|_n$ *for all* $x \in X_0$.

Proof. Let $x \in X_0$; then $x \in X_{n_k}$ for a certain k. Choose an arbitrary $\varepsilon > 0$ and then a $\delta > 0$ so that $z \in X_{n_k} \cap X_m$, $m > n_k$, $\|z\| < \delta$ imply $\|z\|_m < \varepsilon/3$ and $\|z\|_0 < \varepsilon/3$. Choose $y \in Y$ so that $\|y - x\|_{n_k} < \delta$; then $y - x \in X_m$ for $m > m_0$, $\|y - x\|_m < \varepsilon/3$ for $m \geqslant \max(m_0, n_k)$ and $\|y - x\|_0 < \varepsilon/3$. Finally, choose N so that $N \geqslant n_k$ and $|\|y\|_0 - \|y\|_m| < \varepsilon/3$ for $m > N$. Then

$$\|x\|_0 \leqslant \|x - y\|_0 + \|y\|_0 \quad \text{and} \quad -\|x\|_m \leqslant \|x - y\|_m - \|y\|_m,$$

whence $\|x\|_0 - \|x\|_m \leqslant \|x - y\|_0 + \|x - y\|_m + |\|y\|_0 - \|y\|_m| < \varepsilon$ for $m > N$. Similarly $\|x\|_m - \|x\|_0 < \varepsilon$; consequently, $\|x\|_m \to \|x\|_0$ as $m \to \infty$.

A theorem analogous to 7.9 is not valid in the case $\| \|_n \succeq \| \|_0$, namely:

8.5. *There exists a Banach space* $\langle X, \| \|_0 \rangle$ *and a sequence* $\| \|_n$ *of B-norms in* X *such that* $\| \|_n \sim \| \|_0$ *for* $n = 1, 2, \ldots$, $\|x\|_n \leqslant \|x\|_0$ *for* $x \in X$ *and* $n = 1, 2, \ldots$, $\lim \|x\|_n = \|x\|_0$ *for* $x \in X$ *and such that the condition*

$$\|x\|_0 \leqslant K \|x\|_n \quad \text{for all} \quad x \in X \text{ and } n = 1, 2, \ldots$$

is satisfied for no constant K (*independent of* x *and* n).

Example. $X = C\langle 0, 1 \rangle$, $\|x\|_0 = \max_{\langle 0,1 \rangle} |x(t)|$,

$$a_k = \tfrac{1}{3} \min \{|w_i - w_j| \colon 1 \leqslant i < j \leqslant k\},$$

where $w_1, w_2, \ldots$ is a sequence of all rational numbers of $\langle 0, 1 \rangle$,

$$\|x\|_n = \frac{1}{n+1} \|x\|_0 + \frac{n}{n+1} \sup_{k \leqslant n} \frac{1}{a_k} \int_{w_k}^{w_k + a_k} |x(t)|\, dt.$$

9. Monotone one-parameter classes of Banach spaces

This section is devoted to the study of the following question: given a family $\{X_p\}$ of B-spaces depending on a parameter p, let us establish a natural definition of continuity of this family with respect to p. The usual Heine criterion *$p_n \to p$ implies the convergence of X_{p_n} to X_p in a suitable sense* is not appriopriate, since the definition of convergence should be different for $p_n \nearrow p$ and for $p_n \searrow p$.

Such a definition of continuity should be based on the statements of the preceding sections, especially on 7.1 and 8.1; a requirement of too restrictive conditions would reduce us to the trivial case of isomorphic spaces. That being excluded, neither the union of an increasing sequence nor the intersection of a decreasing sequence of B-spaces is a B-space. In the first case there is a non-complete space, in the second — a B_0-space non-isomorphic to a B^*-space.

Such considerations and an examination of a number of natural examples enable us to state that the following definition of continuity is the most appropriate ([8]).

Let $\{X_p\}_{\alpha<p<\beta}$ be a decreasing family of Banach spaces. Through this section we shall assume the following conditions:

$$\text{(i)} \qquad X_{p'} \subset X_p \quad \text{for} \quad p' > p,$$

$$\text{(ii)} \qquad \| \; \|_{p'} \succcurlyeq \| \; \|_p \quad \text{for} \quad p' > p.$$

We shall say that the family $\{X_p\}$ depends on p continuously at a point p_0 ($\alpha < p_0 < \beta$) if the following conditions are satisfied:

(c_1) $\|x\|_{p_0+\varepsilon} \to \|x\|_{p_0}$ *(as $\varepsilon \searrow 0$) for* $x \in \bigcup_{\varepsilon>0} X_{p_0+\varepsilon}$,

(c_2) $\|x\|_{p_0-\varepsilon} \to \|x\|_{p_0}$ *(as $\varepsilon \searrow 0$) for* $x \in \bigcap_{\varepsilon>0} X_{p_0-\varepsilon}$,

(c_3) $\bigcup_{\varepsilon>0} X_{p_0+\varepsilon}$ *is dense in* $\langle X_{p_0}, \| \; \|_{p_0}\rangle$,

(c_4) X_{p_0} *coincides with* $\{x\colon x \in \bigcap_{\varepsilon>0} X_{p_0-\varepsilon},\ \overline{\lim_{\varepsilon\searrow 0}} \|x\|_{p_0-\varepsilon} < \infty\}$,

(c_5) X_{p_0} *is dense in* $\bigcap_{\varepsilon>0} X_{p_0-\varepsilon}$ *with respect to the norm*

$$\|x\|^* = \sum_{n=1}^{\infty} 2^{-n} \varphi(\|x\|_{p_0-\varepsilon_n}),$$

where $\alpha < p_0 - \varepsilon < p_0 < \beta$ and $\varepsilon_n \searrow 0$.

([8]) This definition is formulated only for monotone classes; the case of an increasing family reduces to the case in question by the substitution of parameter $-p$ for p. On may introduce some generalizations, but the general definition cannot be formulated in such a simple and convincing manner.

If conditions (c_1), (c_2) and (c_3) are satisfied, $\{X_p\}$ will be called *semicontinuous from below at p_0*; if (c_1), (c_2), (c_4) and (c_5) are satisfied, $\{X_p\}$ will be called *semicontinuous from above at p_0* ([9]).

Continuity in a closed interval $\langle \alpha_1, \beta_1 \rangle$ means continuity at every point of the interior, semicontinuity from above at β_1 and semicontinuity from below at α_1. E. g. the family $\{L_p\}$ is continuous for $1 \leqslant p \leqslant \infty$. Continuity from below does not imply continuity from above, and conversely.

That definition retains traditional properties of continuity. E. g. if $\{X_p\}_{-1\leqslant p\leqslant 1}$ is a decreasing family and if $\{X_p\}_{-1\leqslant p\leqslant 0}$ and $\{X_p\}_{0\leqslant p\leqslant 1}$ are continuous on the intervals $\langle -1, 0\rangle$ and $\langle 0, 1\rangle$, separately, then $\{X_p\}_{-1\leqslant p\leqslant 1}$ is continuous on $\langle -1, 1\rangle$. Moreover, a semicontinuous family is determined by the spaces $\langle X_p, \| \; \|_p\rangle$ defined for a dense set of values of p.

9.1. *Let Y be a subset of $\bigcap_p X_p$ dense in each space $\langle X_p, \| \; \|_p\rangle$ and such that*

$$\text{(v)} \qquad \|x\|_p = \lim_{q\to p} \|x\|_q \quad \textit{for all } x \in Y \textit{ and } \alpha < p < \beta.$$

Next, let us assume $\|x\|_p \leqslant \|x\|_{p'}$ for $p < p'$. Then $\{X_p\}$ depends on p continuously whenever (c_4) *holds.*

Proof. Theorems 7.7 and 8.4 state that convergence of norms in a dense subset implies convergence everywhere; this yields conditions (c_1) and (c_2). Conditions (c_3) and (c_5) are evident.

The family $\{H_p^0\}$ (cf. [13]) shows that the hypotheses of 9.1 do not imply (c_4).

9.2. *The continuity of $\{X_p\}$ does not imply the existence of a common dense subset of all spaces $\langle X_p, \| \; \|_p\rangle$.*

Example. Let Z_p ($1 \leqslant p < 2$) be the space of all measurable functions in $(-\infty, \infty)$ vanishing outside the interval $(0, 1/p)$ and such that

$$\|x\|_p = \Big(\int_{-\infty}^{\infty} |x(t)|^p dt\Big)^{1/p} < \infty.$$

Obviously, conditions (i)-(ii) and (c_1)-(c_5) are satisfied although each space Z_p is nowhere dense in $\langle Z_{p'}, \| \; \|_{p'}\rangle$ for $p' < p$. The existence of a common dense subset is very useful in many cases. One may also require a more restrictive condition: for every $x \in \bigcap_{\alpha<p<\beta} X_p$ there exists a sequence $x_n \in Y$ such that $\|x_n - x\|_p \to 0$ for all p ($\alpha < p < \beta$). This con-

([9]) The terms *left-continuous* and *right-continuous* are not convenient, for they are not invariant with respect to the replacing of p by $-p$.

dition is satisfied, for example, for the set of step-functions in the spaces L_p $(1 \leqslant p < \infty)$, namely the Haar orthogonal expansion of any function of L_p is convergent to the function in the metric $\| \|_p$. Similarly, this condition is satisfied by each of the classes $\{S_p\}$, $\{W_p\}$, $\{B_p\}$ $(1 \leqslant p < \infty)$ of almost periodic functions in the sense of Stiepanoff, Weyl and Besicovitch respectively (the Féjer-Bochner trigonometrical polynomials of an almost periodic function are convergent in any metric $\| \|_p^S, \| \|_p^W, \| \|_p^B$ for which the function can be approximated by trigonometric polynomials).

9.3. *Let* $\langle X, \| \|_1, \| \|_0 \rangle$ *be a normal* γ*-complete* (cf. [1]) *two-norm space (with both norms homogeneous). There exists a continuous family of* B*-spaces* $\langle X_p, \| \|_p \rangle$, $0 \leqslant p \leqslant 1$, *such that* $\langle X_0, \| \|_0 \rangle = \operatorname{Compl}\langle X, \| \|_0 \rangle$ *and* $X_1 = X$.

Proof. Let $\langle X_0, \| \|_0 \rangle = \operatorname{Compl}\langle X, \| \|_0 \rangle$ and let $\langle X_p, \| \|_p \rangle$ be the space X_0 with

$$\|x\|_p = \inf\left\{\frac{1}{1-p}\|y\|_0 + \|x+y\|_1 :\ y \in X\right\} \quad \text{for} \quad 0 \leqslant p < 1,$$

X with $\|x\|_p = \|x\|_1$ for $p = 1$.

Then $\| \|_p$ are mutually equivalent for $0 \leqslant p < 1$ and $\|x\|_p \nearrow \|x\|_1$ as $p \nearrow 1$ (see [17]). Let $\langle Z, \| \|_1 \rangle = \underset{p \nearrow 1}{\mathfrak{S}} \langle X_p, \| \|_p \rangle$. One may easily verify that $\langle Z, \| \|_1, \| \|_0 \rangle$ is normal and, for every $z \in Z$, there exists a sequence $x_n \in X$ such that $\|x_n - z\|_0 \to 0$ and $\|x_n\|_1 \to \|z\|_1$. Hence $\bar{Z} = X$, which establishes the continuity of $\{X_p\}$ at $p = 1$.

9.4. *Let* $\{X_p\}$ *be semicontinuous from below in* (α, β). *If for every* $p \in (\alpha, \beta)$ *there exist* p' *such that* $p > p' > \alpha$ *and* X_p *is dense in* $\langle X_{p'}, \| \|_{p'} \rangle$, *then every space* $X_{q''}$ *is dense in* $\langle X_{q'}, \| \|_{q'} \rangle$ *for any* $q' < q''$.

Proof. If it were not so, there would exist numbers q' and q'' such that $\alpha < q' < q'' < \beta$ and $X_{q''}$ is not dense in $\langle X_{q'}, \| \|_{q'} \rangle$. Obviously, if $\alpha < p_1 < p_2 < p_3 < \beta$, if X_{p_3} is dense in $\langle X_{p_2}, \| \|_{p_2} \rangle$ and if X_{p_2} is dense in $\langle X_{p_1}, \| \|_{p_1} \rangle$, then X_{p_3} is dense in $\langle X_{p_1}, \| \|_{p_1} \rangle$, for $\| \|_{p_1} \precsim \| \|_{p_2}$. Thus we may assume q'' to be fixed and q' to be the g. l. b. of numbers q such that $X_{q''}$ is dense in $\langle X_q, \| \|_q \rangle$, which contradicts assumption (c_3).

9.5. *Let* $p_n \searrow p_0$. *If* X_{p_0} *is dense in each space* $\langle X_{p_n}, \| \|_{p_n} \rangle$, *then it is dense in* $\bigcap_n X_{p_n}$ *with respect to norm* $\| \|^*$ *defined above.*

9.6. Let $\{X_p\}$ satisfy conditions (c_1), (c_2) and

$$\text{(w)} \qquad \|x\|_p \leqslant \|x\|_{p'} \quad \text{for} \quad p < p'.$$

We shall consider the following spaces: the space X_p^+ of all elements of $\bigcap_{\varepsilon>0} X_{p-\varepsilon}$ such that $\sup_{\varepsilon>0} \|x\|_{p-\varepsilon} < \infty$ and the closure X_p^- of $\bigcup_{\varepsilon>0} X_{p+\varepsilon}$ in $\langle X_p, \| \|_p \rangle$. X_p^+ is a Banach space with the norm

$$\|x\|_p = \sup_{\varepsilon>0} \|x\|_{p-\varepsilon} = \lim_{\varepsilon \searrow 0} \|x\|_{p-\varepsilon};$$

X_p and X_p^- are closed subspaces of $\langle X_p^+, \| \|_p \rangle$. Moreover,

$$(X_p^+)^- = X_p^- \quad \text{and} \quad (X_p^-)^+ = X_p^+.$$

So either class determines the other uniquely.

In [13] two typical examples of such spaces are considered:

$$H_p = (H_p^0)^+, \quad H_p^0 = (H_p)^- \quad \text{and} \quad AC_p = (CV_p)^-, \quad CV_p = (AC_p)^+.$$

If there exists a countable set dense in each space $\langle X_p, \| \|_p \rangle$ and if (w) holds, then (c_1) and (c_2) are satisfied except for a countable set of values of p. However, it may happen that (c_4) is not satisfied by any p, as examples $\{H_p^0\}$ and $\{AC_p\}$ show.

9.7. *Let* X *be a linear set,* $\| \|_p$ $(\alpha < p < \beta)$ *a family of norms defined in* X *and satisfying* (c_1), (c_2) *and* (w), *and let* $\| \|_p$ *be quasi-normal with respect to* $\| \|_{p'}$ *for* $p < p'$. *Then the family*

$$X_p^+ = \{x: x \in \bigcap_{\varepsilon>0} \operatorname{Compl}\langle X, \| \|_{p-\varepsilon} \rangle,\ \sup_{\varepsilon>0} \|x\|_{p-\varepsilon} < \infty\}$$

is semicontinuous from above and the family $\{X_p^-\}$ *of closures of the spaces*

$$\{x: x \in \bigcup_{\varepsilon>0} \operatorname{Compl}\langle X, \| \|_{p-\varepsilon} \rangle\}$$

is semicontinuous from below.

9.8. *If families* $\{X_p^{(n)}\}_{\alpha<p<\beta}$, $n = 1, 2, \ldots$, *depend on* p *continuously, so does the family*

$$(X_p^{(1)} \times X_p^{(2)} \times \ldots)_m$$

consisting of all sequences $x = (x_1, x_2, \ldots)$ *with* $x_n \in X_p^{(n)}$ *and* $\|x\|_p = \sup_n \|x_n\|_p^{(n)} < \infty$.

9.9. Given a family $\{X_p\}_{\alpha<p<\beta}$ depending on p continuously and a linear set $Y \subset \bigcap_{\alpha<p<\beta} X_p$, it may happen that the family $\{Y_p\}$ of closures of Y in the spaces $\langle X_p, \| \|_p \rangle$ does not satisfy (c_4) and is only semicontinuous from below.

Example. The set Y of all continuous functions in $\langle 0, 1 \rangle$ and the spaces L^p, $1 \leqslant p \leqslant \infty$, the family $\{Y_p\}$ is not continuous at $p = \infty$.

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Some classes of Banach spaces depending on a parameter

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In this paper we shall consider the following classes of Banach spaces:

H_p —functions satisfying Hölder condition with an exponent p,

CV_p —continuous functions with finite p-th variation,

AC_p —absolutely continuous functions of order p,

S_p and B_p —almost periodic functions in the sense of Stepanoff and Besicovitch, respectively,

M_p —strongly p-summable sequences.

These classes may be treated as families of Banach spaces X_p depending on a parameter p. In each of these classes there are known inclusions between spaces X_p, $X_{p'}$ and inequalities between norms $\| \|_p$, $\| \|_{p'}$ for $p < p'$. We shall consider the following problem: given a sequence p_n convergent to p_0, establish connections between the corresponding spaces X_{p_n} and X_{p_0}. This problem is closely related to the problem of the continuity (suitably defined) of the spaces X_p with respect to the parameter p.

These problems are considered from a general point of view in paper [8], where, in the following definition, the limit $\mathop{\mathfrak{S}}\limits_{n\to\infty}(X_n)$ of a sequence X_n of linear metric spaces is introduced. $\langle X_0, \| \|_0\rangle$ is termed $\mathfrak{S}$*-limit of* $\langle X_n, \| \|_n\rangle$ (written $\langle X_0, \| \|_0\rangle = \mathop{\mathfrak{S}}\limits_{n\to\infty}\langle X_n, \| \|_n\rangle$) if the following conditions are satisfied:

1° X_0 and almost all X_n are subspaces of a linear space,

2° X_n converges to X_0 in the sense of the theory of sets (i. e. $X_0 = \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} X_n = \bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty} X_n$),

3° $\|x\|_0 = \lim_{n\to\infty}\|x\|_n$ for all $x \in X_0$.

Next, we write $\langle X_0, \| \|_0\rangle = \mathop{\overline{\mathfrak{S}}}\limits_{n\to\infty}\langle X_n, \| \|_n\rangle$ if $\mathop{\mathfrak{S}}\limits_{n\to\infty}(X_n)$ is dense in $\langle X_0, \| \|_0\rangle$.

Let $\{X_p\}_{a<p<\beta}$ be a family of Banach spaces $\langle X_p, \| \|_p\rangle$ such that