where a_k is a sequence tending to infinity and $0 = m_0 < m_1 < m_2 < \ldots$ is a sequence of integers such that $m_k/k \to \infty$,

$$\frac{2}{3} < \frac{1}{m_k}\sum_{i=1}^{m_k}|x_i|^p < 1 \quad \text{and} \quad \frac{1}{m_k-1}\sum_{i=1}^{m_k-1}|x_i|^p < \frac{1}{3}$$

for $k = 1, 2, \ldots$ Then $x = \{x_n\}$ belongs to M_p and does not belong to M_p^b, although $x \in M_{p'}^c$ for every p' with $1 \leqslant p' < p$. Indeed, let us denote $l_k = m_{k+1} - m_k$; then

$$(\|x\|_p^M)^p = \sup_k \frac{|a_1|^p + \ldots + |a_k|^p}{l_1 + \ldots + l_k} \leqslant 1$$

and

$$\lim_{k\to\infty} \frac{|a_1|^{p'} + \ldots + |a_k|^{p'}}{l_1 + \ldots + l_k} \leqslant \lim_{k\to\infty} \frac{|a_1|^{p'} + \ldots + |a_k|^{p'}}{|a_1|^p + \ldots + |a_k|^p} = \lim_{k\to\infty} \frac{|a_k|^{p'}}{|a_k|^p} = 0$$

for $p' < p$. Hence

$$0 \leqslant \overline{\lim_{n\to\infty}} \frac{1}{n}\sum_{i=1}^{n}|x_i|^{p'} \leqslant \lim_{k\to\infty} \frac{|a_1|^{p'} + \ldots + |a_k|^{p'}}{l_1 + \ldots + l_k} = 0.$$

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STUDIA MATHEMATICA, T. XX. (1961)

On very strong Riesz-summability of orthogonal series

by

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1. Let $\{\lambda_n\}$ be a positive, strictly increasing, numerical sequence, with $\lambda_0 = 0$ and $\lambda_n \to \infty$.

A series

$$u_0 + u_1 + \ldots + u_n + \ldots, \tag{1.1}$$

with n-th partial sums s_n, is said to be *summable* $(R, \lambda_n, 1)$ *to the sum* s, if

$$\tau_n = \frac{1}{\lambda_{n+1}}\sum_{k=0}^{n}(\lambda_{k+1} - \lambda_k)s_k \to s, \quad \text{as} \quad n \to \infty.$$

Obviously, the Riesz-method of summation is a generalization of $(C, 1)$-method, which is obtained by putting $\lambda_n = n$.

Series (1.1) is said to be *very strongly summable* $(R, \lambda_n, 1)$ *to the sum* s, if

$$\sum_{k=0}^{n}(\lambda_{k+1} - \lambda_k)(s_{\nu_k} - s)^2 = o(\lambda_{n+1}), \quad \text{as} \quad n \to \infty,$$

for every strictly increasing sequence of indices $\{\nu_n\}$.

In particular, if $\nu_k = k$ $(k = 0, 1, 2, \ldots)$, we shall say that series (1.1) is *strongly summable* $(R, \lambda_n, 1)$ *to the sum* s.

Series (1.1) is said to be *strongly* (*very strongly*) *summable* $(C, 1)$, if it is strongly (very strongly) summable $(R, \lambda_n, 1)$ with $\lambda_n = n$.

2. Further, we shall consider the strong and the very strong Riesz-summability of orthogonal series.

Let $\mathrm{ON}\{\varphi_n(x)\}$ denote an orthonormal system defined in the interval $\langle 0, 1\rangle$ and $\{c_n\} \in l^2$, i. e.

$$\sum_{n=0}^{\infty} c_n^2 < \infty. \tag{2.1}$$

Further, let

$$\sum_{n=0}^{\infty} c_n \varphi_n(x) \tag{2.2}$$

denote orthogonal series being developments of functions $f(x) \in L^2$, i. e. integrable with the square in Lebesgue sense.

The strong summability $(C, 1)$ of orthogonal series, as well as that of Fourier series, has been investigated by several authors such as: A. Zygmund, S. Kaczmarz, S. Borgen, Z. Zalcwasser, A. Alexits, B. N. Prasad and N. N. Singh, and recently by K. Tandori ([5], Mitt. II, IV, VI). A. Zygmund [7] has proved the following theorem: If series (2.2) by condition (2.1) is summable $(C, 1)$ almost everywhere to a function $f(x)$, then it is strongly summable $(C, 1)$ almost everywhere to this function. K. Tandori showed ([5], IV Mitt., Satz II, p. 19) that under the same assumptions concerning the coefficients of series (2.2.) the very strong summability $(C, 1)$ almost everywhere of this series cannot be concluded. Moreover, he has proved ([5], Mitt. IV, Satz I, p. 19, and Mitt. VI, p. 14) the following two theorems:

I. Let $\{c_n\}$ denote a positive sequence such that $\{c_n\} \in l^2$ and $\sqrt{n}\cdot c_n \geqslant \sqrt{n+1}\cdot c_{n+1}$ $(n = 1, 2, \ldots)$. If series (2.2) with these coefficients is summable $(C, 1)$ almost everywhere to a function $f(x)$, then it is very strongly summable $(C, 1)$ almost everywhere to this function.

II. If $\sum_{n=2}^{\infty} c_n^2 (\log\log n)^2 < \infty$, then series (2.2) is very strongly summable $(C, 1)$ almost everywhere to a function $f(x) \in L^2$.

In the first part of this paper (Theorems 1-3) we generalize the above theorem of Zygmund and the last two theorems of Tandori, transfering them the more general Riesz-method of summation. In the second part (Th. 4) we give an example of orthogonal series (2.2) with coefficients satisfying condition (2.1) which, being summable $(R, 1)$ (i. e. by the first logarithmic means) almost everywhere to a function $f(x)$, is not very strongly summable $(R, 1)$ almost everywhere to this function.

By proving theorems given below we shall often refer to the following statements of A. Zygmund [6] (compare also G. G. Lorenz [3] and J. Meder [4], Th. 2 and Th. 3, p. 16-17):

Theorem A. *Let $\{p_n\}$ denote a strictly increasing sequence satisfying the condition*

$$1 < q \leqslant \frac{\lambda p_{n+1}}{\lambda p_n} \leqslant r,$$

where q and r are constants, independent of n.

In order that orthogonal series (2.2) with coefficients satisfying condition (2.1) should be summable $(R, \lambda_n, 1)$ in a set E almost everywhere, it is necessary and sufficient, that the sequence $\{s_{p_n}(x)\}$ of partial sums be convergent in E almost everywhere.

Theorem B. *The orthogonal series (2.2) is summable $(R, \lambda_n, 1)$ almost everywhere, if its coefficients satisfy the condition*

$$\sum_{n=N}^{\infty} c_n^2 (\log\log\lambda_n)^2 < \infty \text{ [1]}.$$

3. Theorem 1. *If orthogonal series (2.2) with coefficients satisfying condition (2.1) is summable $(R, \lambda_n, 1)$ almost everywhere to a function $f(x)$, then it is strongly summable $(R, \lambda_n, 1)$ almost everywhere to this function.*

Proof. Let us write

$$s_n(x) = c_0\varphi_0(x) + c_1\varphi_1(x) + \ldots + c_n\varphi_n(x),$$

$$\tau_n(x) = \frac{1}{\lambda_{n+1}} \sum_{k=0}^{n} (\lambda_{k+1} - \lambda_k) s_k(x).$$

We observe that for any n the following inequality is satisfied:

$$\frac{1}{\lambda_{n+1}} \sum_{k=0}^{n} (\lambda_{k+1} - \lambda_k) \big(s_k(x) - f(x)\big)^2$$

$$\leqslant \frac{2}{\lambda_{n+1}} \sum_{k=0}^{n} (\lambda_{k+1} - \lambda_k) \big(s_k(x) - \tau_k(x)\big)^2 + \frac{2}{\lambda_{n+1}} \sum_{k=0}^{n} (\lambda_{k+1} - \lambda_k) \big(\tau_k(x) - f(x)\big)^2.$$

Since series (2.2) is summable almost everywhere to a function $f(x)$, so the second term on the right side of the last inequality tends to zero almost everywhere. To prove that the first term on the right also tends to zero almost everywhere, we notice that

$$s_k(x) - \tau_k(x) = \frac{1}{\lambda_{k+1}} \sum_{v=0}^{k} (\lambda_{v+1} - \lambda_v) \sum_{i=v+1}^{k} c_i\varphi_i(x)$$

$$= \frac{1}{\lambda_{k+1}} \sum_{i=0}^{k} c_i\varphi_i(x) \sum_{v=0}^{i-1} (\lambda_{v+1} - \lambda_v)$$

$$= \frac{1}{\lambda_{k+1}} \sum_{i=0}^{k} c_i \lambda_i \varphi_i(x).$$

Therefore we have

$$s_k(x) - \tau_k(x) = \frac{1}{\lambda_{k+1}} \sum_{i=0}^{k} c_i \lambda_i \varphi_i(x).$$

[1] N denotes here the least positive integer such that $\log\log\lambda_N > 0$.

Hence

$$\sum_{k=0}^{\infty}\frac{\lambda_{k+1}-\lambda_k}{\lambda_{k+1}}\int_0^1\big(s_k(x)-\tau_k(x)\big)^2dx$$

$$=\sum_{k=0}^{\infty}\frac{\lambda_{k+1}-\lambda_k}{\lambda_{k+1}\lambda_{k+1}^2}\sum_{i=0}^{k}c_i^2\lambda_i^2=\sum_{i=1}^{\infty}c_i^2\lambda_i^2\sum_{k=i}^{\infty}\frac{\lambda_{k+1}-\lambda_k}{\lambda_{k+1}\lambda_{k+1}^2}$$

$$<\sum_{i=1}^{\infty}c_i^2\lambda_i^2\sum_{k=i}^{\infty}\left(\frac{1}{\lambda_k^2}-\frac{1}{\lambda_{k+1}^2}\right)=\sum_{i=1}^{\infty}c_i^2<\infty,$$

whence by Lévy theorem it follows that the series

$$\sum_{k=0}^{\infty}(\lambda_{k+1}-\lambda_k)\big(s_k(x)-\tau_k(x)\big)^2$$

converges almost everywhere. Applying to this series the well-known Kronecker theorem, we finally find that

$$\sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)\big(s_k(x)-\tau_k(x)\big)^2=o(\lambda_{n+1})$$

almost everywhere.

Thus we have shown that each of the terms on the right in the mentioned above inequality tends to zero almost everywhere. This completes the proof of Theorem 1.

THEOREM 2. *Let $\{c_n^*\}\in l^2$ be a numerical sequence of positive terms such that*

$$(3.1)\qquad \sqrt{\frac{\lambda_n}{\lambda_{n+1}-\lambda_n}}\,c_n^*\geqslant\sqrt{\frac{\lambda_{n+1}}{\lambda_{n+2}-\lambda_{n+1}}}\,c_{n+1}^*\qquad (n=1,2,\ldots).$$

Further let $\{c_n\}$ be an arbitrary sequence of real numbers satisfying the relation

$$(3.2)\qquad c_n=O(c_n^*).$$

If orthogonal series (2.2) consisting of coefficients $\{c_n\}$ is summable $(R,\lambda_n,1)$ to a certain function $f(x)$ almost everywhere, then it is very strongly summable $(R,\lambda_n,1)$ to this function almost everywhere.

Proof. Let $\{v_n\}$ be an arbitrary strictly increasing sequence of indices. We may suppose, without loss of generality of Theorem 2, that $v_1\geqslant p_1$.

Let $p_m\leqslant v_k<p_{m+1}$, where $\{p_m\}$ has the same meaning as in Theorem A. Let us assume that $\mu_k=p_{m+1}$. Since from the assumption $\{c_n^*\}\in l^2$ and orthogonal series (2.2) is summable $(R,\lambda_n,1)$ to a function $f(x)$ almost everywhere, so from Theorem A it follows that $\lim\limits_{m\to\infty}s_{p_m}(x)\doteq f(x)$ [2] and therefore

$$(3.3)\qquad \lim_{k\to\infty}s_{\mu_k}(x)\doteq f(x).$$

Let us observe that

$$(3.4)\qquad \frac{1}{\lambda_{n+1}}\sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)\big(s_{v_k}(x)-f(x)\big)^2$$
$$\leqslant\frac{2}{\lambda_{n+1}}\sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)\big(s_{\mu_k}(x)-s_{v_k}(x)\big)^2+$$
$$+\frac{2}{\lambda_{n+1}}\sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)\big(s_{\mu_k}(x)-f(x)\big)^2.$$

In virtue of (3.3) we have

$$(3.5)\qquad \sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)\big(s_{\mu_k}(x)-f(x)\big)^2\doteq o(\lambda_{n+1}),\quad\text{as}\quad n\to\infty.$$

Now, we show also that

$$(3.6)\qquad \sum_{k=0}^{n}(\lambda_{k+1}-\lambda_k)\big(s_{\mu_k}(x)-s_{v_k}(x)\big)^2\doteq o(\lambda_{n+1}),\quad\text{as}\quad n\to\infty.$$

Since

$$\frac{\lambda_{\mu_k+1}}{\lambda_{v_k+1}}<\frac{\lambda_{p_{m+2}}}{\lambda_{p_m}}\leqslant r^2\quad\text{or}\quad\lambda_{\mu_k+1}<r^2\lambda_{v_k+1},$$

hence we may write in view of (3.1) and (3.2)

$$\sum_{k=0}^{\infty}\frac{\lambda_{k+1}-\lambda_k}{\lambda_{k+1}}\int_0^1[s_{\mu_k}(x)-s_{v_k}(x)]^2dx=\sum_{k=0}^{\infty}\frac{\lambda_{k+1}-\lambda_k}{\lambda_{k+1}}\sum_{p=v_k+1}^{\mu_k}c_p^{*2}$$

$$\leqslant\sum_{k=0}^{\infty}\frac{\lambda_{k+1}-\lambda_k}{\lambda_{k+1}}\cdot\frac{\lambda_k c_k^{*2}}{\lambda_{k+1}-\lambda_k}\sum_{p=v_k+1}^{\mu_k}\frac{\lambda_{p+1}-\lambda_p}{\lambda_p}$$

$$<\sum_{k=0}^{\infty}\frac{c_k^{*2}}{\lambda_{v_k+1}}(\lambda_{\mu_k+1}-\lambda_{v_k+1})<r^2\sum_{k=0}^{\infty}c_k^{*2}<\infty.$$

According to the Lévy theorem the series

$$\sum_{k=0}^{\infty}\frac{\lambda_{k+1}-\lambda_k}{\lambda_{k+1}}\big(s_{\mu_k}(x)-s_{v_k}(x)\big)^2$$

[2] The symbol $\doteq$ denotes here that the corresponding equality is satisfied at almost any point of interval $\langle 0,1\rangle$.

is then convergent almost everywhere. Hence and from the Kronecker theorem follows (3.6), and (3.4) gives our theorem.

4. Let

$$\tau_n^{(v)}(x) = \frac{1}{\lambda_{n+1}} \sum_{k=0}^{n} (\lambda_{k+1} - \lambda_k) s_{v_k}(x),$$

where $\{v_n\}$ denotes the sequence defined above.

We shall prove the following two Lemmas:

LEMMA 1. *Let $\{c_n\} \epsilon l^2$. For the convergence almost everywhere of the sequence $\{\tau_n^{(v)}(x)\}$, it is necessary and sufficient that the sequence $\{s_{v_{p_n}}(x)\}$ be convergent almost everywhere.*

Proof. Necessity. Let $\{c_n\} \epsilon l^2$. Further, we assume that sequence $\{\tau_n^{(v)}(x)\}$, and therefore sequence $\{\tau_{p_n}^{(v)}(x)\}$ also, are convergent almost everywhere.

In order to show the necessity it suffices to prove that

$$\lim_{n\to\infty} [s_{v_{p_n}}(x) - \tau_{p_n}^{(v)}(x)] \doteq 0. \tag{4.1}$$

For this purpose we first notice that, applying to the expression $\{\tau_{p_n}^{(v)}(x)\}$ the Abel transformation, we may write

$$s_{v_{p_n}}(x) - \tau_{p_n}^{(v)}(x) = -\frac{1}{\lambda_{p_n+1}} \sum_{k=0}^{p_n-1} \lambda_{k+1} \sum_{v=v_k+1}^{v_{k+1}} c_v \varphi_v(x).$$

Hence and by (2.1) we have

$$\sum_{n=0}^{\infty} \int_0^1 [s_{v_{p_n}}(x) - \tau_{p_n}^{(v)}(x)]^2 dx < \sum_{n=0}^{\infty} \frac{1}{\lambda_{p_n}^2} \sum_{k=0}^{p_n-1} \lambda_{k+1}^2 \sum_{v=v_k+1}^{v_{k+1}} c_v^2$$

$$= \sum_{k=0}^{\infty} \lambda_{k+1}^2 \sum_{v=v_k+1}^{v_{k+1}} c_v^2 \sum_{p_n>k+1} \frac{1}{\lambda_{p_n}^2} < \frac{q^2}{q^2-1} \cdot \sum_{v=0}^{\infty} c_v^2 < \infty.$$

Basing upon the Lévy theorem we conclude that the series

$$\sum_{v=0}^{\infty} [s_{v_{p_n}}(x) - \tau_{p_n}^{(v)}(x)]^2$$

is convergent almost everywhere. Hence there follows formula (4.1), which proves, by virtue of convergence almost everywhere of the sequence $\{\tau_{p_n}^{(v)}(x)\}$, that the sequence $\{s_{v_{p_n}}(x)\}$ is convergent almost everywhere.

Sufficiency. To show this condition we first prove that the series

$$\sum_{n=0}^{\infty} \frac{\lambda_{n+1}}{\lambda_{n+2} - \lambda_{n+1}} [\tau_{n+1}^{(v)}(x) - \tau_n^{(v)}(x)]^2 \tag{4.2}$$

is convergent almost everywhere.

We observe that

$$\tau_{n+1}^{(v)}(x) - \tau_n^{(v)}(x) = \frac{1}{\lambda_{n+2}} \sum_{k=0}^{n+1} (\lambda_{k+1} - \lambda_k) s_{v_k}(x) - \frac{1}{\lambda_{n+1}} \sum_{k=0}^{n} (\lambda_{k+1} - \lambda_k) s_{v_k}(x)$$

$$= \frac{\lambda_{n+2} - \lambda_{n+1}}{\lambda_{n+2}} \sum_{v=v_n+1}^{v_{n+1}} c_v \varphi_v(x) - \frac{\lambda_{n+2} - \lambda_{n+1}}{\lambda_{n+1}\lambda_{n+2}} \sum_{k=0}^{n-1} \lambda_{k+1} \sum_{v=v_k+1}^{v_{k+1}} c_v \varphi_v(x).$$

Hence

$$\sum_{n=0}^{\infty} \frac{\lambda_{n+1}}{\lambda_{n+2} - \lambda_{n+1}} \int_0^1 [\tau_{n+1}^{(v)}(x) - \tau_n^{(v)}(x)]^2 dx$$

$$\leqslant 2 \sum_{n=0}^{\infty} \frac{(\lambda_{n+2} - \lambda_{n+1})\lambda_{n+1}}{\lambda_{n+2}^2} \sum_{v=v_n+1}^{v_{n+1}} c_v^2 + 2 \sum_{n=0}^{\infty} \frac{\lambda_{n+2} - \lambda_{n+1}}{\lambda_{n+1}\lambda_{n+2}^2} \sum_{k=0}^{n-1} \lambda_{k+1}^2 \sum_{v=v_k+1}^{v_{k+1}}.$$

Now we show that the two last series are convergent. We see at once that

$$\sum_{n=0}^{\infty} \frac{(\lambda_{n+2} - \lambda_{n+1})\lambda_{n+1}}{\lambda_{n+2}^2} \sum_{v=v_n+1}^{v_{n+1}} c_v^2$$

$$= \sum_{n=0}^{\infty} \sum_{v=v_n+1}^{v_{n+1}} c_v^2 \frac{\lambda_{n+1}}{\lambda_{n+2}} \left(1 - \frac{\lambda_{n+1}}{\lambda_{n+2}}\right) < \sum_{n=0}^{\infty} \sum_{v=v_n+1}^{v_{n+1}} c_v^2 < \infty,$$

which proves the convergence of the first investigated series. We may write

$$\sum_{n=0}^{\infty} \frac{\lambda_{n+2} - \lambda_{n+1}}{\lambda_{n+1}\lambda_{n+2}^2} \sum_{k=0}^{n-1} \lambda_{k+1}^2 \sum_{v=v_k+1}^{v_{k+1}} c_v^2 < \sum_{n=0}^{\infty} \left(\frac{1}{\lambda_{n+1}^2} - \frac{1}{\lambda_{n+2}^2}\right) \sum_{k=0}^{n} \lambda_{k+1}^2 \sum_{v=v_k+1}^{v_{k+1}} c_v^2$$

$$< \sum_{n=0}^{\infty} \lambda_{k+1}^2 \sum_{v=v_k+1}^{v_{k+1}} c_v^2 \sum_{n=k}^{\infty} \left(\frac{1}{\lambda_{n+1}^2} - \frac{1}{\lambda_{n+2}^2}\right) < \sum_{v=0}^{\infty} c_v^2 < \infty.$$

and this shows the convergence of the second series.

From the above convergence and the Lévy theorem it follows that series (4.2) is convergent almost everywhere.

To show the convergence of the sequence $\{\tau_n^{(v)}(x)\}$ let us assume that $p_n \leqslant k < p_{n+1}$ $(n = 1, 2, \ldots)$. Applying the Cauchy inequality we find that

$$[\tau_k^{(v)}(x) - \tau_{p_n}^{(v)}(x)]^2 = \Big\{\sum_{i=p_n+1}^{k} [\tau_i^{(v)}(x) - \tau_{i-1}^{(v)}(x)]\Big\}^2$$

$$\leqslant \sum_{i=p_n+1}^{p_{n+1}} \frac{\lambda_i}{\lambda_{i+1} - \lambda_i} [\tau_i^{(v)}(x) - \tau_{i-1}^{(v)}(x)]^2 \sum_{i=p_n+1}^{p_{n+1}} \frac{\lambda_{i+1} - \lambda_i}{\lambda_i}.$$

By (4.2), the first sum on the right side of the last inequality tends to zero almost everywhere and the second one is bounded, for

$$\sum_{i=p_n+1}^{p_{n+1}} \frac{\lambda_{i+1}-\lambda_i}{\lambda_i} \leqslant \frac{1}{\lambda_{p_n+1}} \sum_{i=p_n+1}^{p_{n+1}} (\lambda_{i+1}-\lambda_i) \leqslant r^2.$$

The sequence $\{s_{v_{p_n}}(x)\}$ and by (4.1) a fortiori the sequence $\{\tau_{p_n}^{(v)}(x)\}$ is convergent almost everywhere.

Hence and from above inequalities we conclude that the sequence $\{\tau_n^{(v)}(x)\}$ is also convergent almost everywhere, which completes the proof of Lemma 1.

Remark. Theorem 2 seems to the author to be not true, if we do not introduce the additional assumptions (3.1) and (3.2) besides the assumptions of Theorem 1.

LEMMA 2. *Let $\{c_n\}\epsilon l^2$. For the convergence almost everywhere of the sequence $\{\tau_n^{(v)}(x)\}$ to a certain function $f(x)$, it is necessary and sufficient that series* (2.2) *be very strongly summable $(R, \lambda_n, 1)$ to this function.*

Proof. The sufficiency follows immediately from the inequality

$$|\tau_n^{(v)}(x)-f(x)| \leqslant \sqrt{\frac{1}{\lambda_{n+1}} \sum_{k=0}^{n} (\lambda_{k+1}-\lambda_k)[s_{v_k}(x)-f(x)]^2}. \tag{4.3}$$

To prove the necessity we assume that $\mu_n = v_{p_m}$ for $p_m \leqslant n < p_{m+1}$ $(m = 1, 2, \ldots)$.

Next, let us suppose the sequence $\{\tau_n^{(v)}(x)\}$ to be convergent almost everywhere to the function $f(x)$. Hence and from Lemma 1 we conclude that the sequence $\{s_{v_{p_n}}(x)\}$ is convergent almost everywhere to the function $f(x)$; thus, the second expression on the right side of inequality (3.4) tends to zero almost everywhere.

Now, let us observe that

$$\sum_{n=p_1+1}^{\infty} \frac{\lambda_{n+1}-\lambda_n}{\lambda_{n+1}} \int_0^1 [s_{v_n}(x)-s_{\mu_n}(x)]^2 dx$$
$$= \sum_{m=1}^{\infty} \sum_{n=p_m+1}^{p_{m+1}} \frac{\lambda_{n+1}-\lambda_n}{\lambda_{n+1}} (c_{\mu_n+1}^2 + c_{\mu_n+2}^2 + \ldots + c_{v_n}^2)$$
$$\leqslant \sum_{m=1}^{\infty} (c_{v_{p_m+1}}^2 + c_{v_{p_m+2}}^2 + \ldots + c_{v_{p_{m+1}}}^2) \sum_{n=p_m+1}^{p_{m+1}} \frac{\lambda_{n+1}-\lambda_n}{\lambda_{n+1}}$$
$$< r^2 \sum_{v=0}^{\infty} c_v^2 < \infty.$$

Hence and from the Levy theorem it follows that the series

$$\sum_{n=p_1+1}^{\infty} \frac{\lambda_{n+1}-\lambda_n}{\lambda_{n+1}} [s_{v_n}(x)-s_{\mu_n}(x)]^2$$

is convergent almost everywhere. Applying to the last series the Kronecker theorem, we conclude that the first expression on the right side of the inequality (3.4) converges almost everywhere also, which completes the proof of Lemma 2.

5. THEOREM 3. *If*

$$\sum_{v=N}^{\infty} c_v^2 (\log\log\lambda_v)^2 < \infty, \tag{5.1}$$

then series (2.2) *is very strongly summable $(R, \lambda_n, 1)$ to a function $f(x)$ almost everywhere.*

Proof. In order to prove Theorem 3 we extend the sequence $\{\lambda_n\}$ by linear interpolation to a strictly increasing function $\lambda(x)$, which for $x = n$ takes the value $\lambda(n) = \lambda_n$, and we denote by $l(x)$ the uniquely defined function inverse to $\lambda(x)$ (see [1], p. 127, 2.8.8.).

Since for $\lambda_n = 2^n$ the convergence of (2.2) holds, we may assume, without loss of generality, that there exist positive integers n such that

$$2^m < \lambda_n < 2^{m+1} \quad (m = 1, 2, \ldots).$$

Hence we may write

$$[l(2^m)] \leqslant l(2^m) < [l(2^m)]+1 \leqslant [l(2^{m+1})] \leqslant l(2^{m+1}) \text{ (}^3\text{)}.$$

Now, we put $p_m = [l(2^{2m-1})]$ $(m = 1, 2, \ldots)$.

It is easy to verify that the sequence $\{p_m\}$ is positive, strictly increasing to infinity, and furthermore,

$$2 < \frac{\lambda_{p_{m+1}}}{\lambda_{p_m}} < 8.$$

Hence by (5.1) and Theorems B and A, it follows that the sequence $\{s_{p_m}(x)\}$ is convergent almost everywhere to a function $f(x)$.

Further, let n be an arbitrary positive integer, for which the inequalities

$$p_m \leqslant v_{p_n} < p_{m+1}$$

are satisfied.

Let us denote by k and l the lower and the upper bound of the set of integers n fulfilling the above inequalities, respectively.

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(3) $[x]$ denotes the integral part of x.

We put

$$F_1(x) = s_{v_{p_k}}(x) - s_{p_m}(x),\ F_2(x) = s_{v_{p_{k+1}}}(x) - s_{v_{p_k}}(x),$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$F_{l-k+1}(x) = s_{v_{p_l}}(x) - s_{v_{p_{l-1}}}(x),\ F_{l-k+2}(x) = s_{p_{m+1}}(x) - s_{v_{p_l}}(x).$$

It is obvious that the system $\{F_r(x)\}$ $(r = 1, 2, \dots, l-k+2)$ is orthogonal in $\langle 0, 1\rangle$.

Let

$$\int_0^1 F_r^2(x)\,dx = A_r^2 \qquad (r = 1, 2, \dots, l-k+2).$$

From the Rademacher lemma ([2], p. 162) we conclude that there exists a function $0 \leqslant \delta_m(x) \in L^2$ such that

1° $$|s_{v_{p_n}}(x) - s_{p_m}(x)| = \Big|\sum_{r=1}^{n-k+1} F_r(x)\Big| \leqslant \delta_m(x),$$
$$\text{for}\quad 1 \leqslant n-k+1 < l-k+2,$$

2° $$\int_0^1 \delta_m^2(x)\,dx = O\big(\log^2(l-k+2)\big)\sum_{r=1}^{l-k+2} A_r^2.$$

Since $l-k+2 \leqslant m+1$ and $\log(m+1) = O\big(\log(2m-1)\big) = O\big(\log\log 2^{2m-1}\big) = O\big(\log\log\lambda l(2^{2m-1})\big)$, so by (5.1) and the definition of functions $F_r(x)$, we may write

$$\sum_{m=2}^{\infty}\int_0^1 \delta_m^2(x)\,dx = O(1)\sum_{m=2}^{\infty}\log^2(m+1)\sum_{v=p_m+1}^{p_{m+1}} c_v^2$$

$$= O(1)\sum_{m=2}^{\infty}\sum_{v=p_m+1}^{p_{m+1}} c_v^2(\log\log\lambda_v)^2 = O(1)\sum_{v=N}^{\infty} c_v^2(\log\log\lambda_v)^2 < \infty.$$

The series $\sum_{m=2}^{\infty}\delta_m^2(x)$ is then by the Lévy theorem convergent almost everywhere, whence it follows that $\delta_m(x) \to 0$ almost everywhere. Thus

$$s_{v_{p_n}}(x) - s_{p_m}(x) \to 0 \qquad (m \to \infty) \qquad \text{almost everywhere.}$$

The sequence $\{s_{v_{p_n}}(x)\}$ is then convergent almost everywhere to the function $f(x)$ and from Lemma 1 it follows that $\tau_n^{(v)}(x) \to f(x)$ almost everywhere, which finally, by virtue of Lemma 2, completes the proof of Theorem 3.

6. The purpose of this section is to study a special case which is in close connection with the previous results.

Namely, we shall prove an analogous theorem to that of K. Tandori ([5], Mitt. IV. p. 19), concerning the $(R, 1)$-method of first logarithmic means.

Series (1.1), with n-th partial sums s_n, is said to be *summable* $(R, 1)$ *to the sum* s, if

$$\tau_n = \frac{1}{\log(n+1)}\sum_{k=0}^{n}\frac{s_k}{k+1} \to s, \quad \text{as} \quad n \to \infty.$$

We shall now prove that the $(R, \lambda_n, 1)$-method is for $\lambda_n = \log(n+1)$ equivalent to the $(R, 1)$-method.

We put

$$\tau_n^* = \frac{1}{\log(n+1)}\sum_{k=0}^{n}\frac{s_k\eta_k}{k+1},$$

where $\{\eta_n\}$ denotes a sequence strictly increasing to 1, with $\eta_0 > 0$.

We verify easily that $\tau_n \to s$ involves $\tau_n^* \to s$ and conversely. In fact, putting $\eta_n = 1-\varepsilon_n$, where ε_n form a sequence decreasing monotonously to 0, we find

$$\tau_n^* = \tau_n - \frac{1}{\log(n+1)}\sum_{k=0}^{n}\frac{s_k\varepsilon_k}{k+1}.$$

Applying to the last sum the Abel transformation we may write

$$\frac{1}{\log(n+1)}\sum_{k=0}^{n}\frac{s_k\varepsilon_k}{k+1} = \varepsilon_n\tau_n + \frac{1}{\log(n+1)}\sum_{k=0}^{n-1}\log(k+1)\tau_k(\varepsilon_k - \varepsilon_{k+1}).$$

Assuming $\tau_n \to s$, and observing that the series $\sum_{k=0}^{\infty}(\varepsilon_k - \varepsilon_{k+1})$ converges, we conclude, in view of Kronecker theorem, that the last expression tends to zero, as $n \to \infty$. Thus $\tau_n^* \to s$.

Conversely, $\tau_n^* \to s$ involves $\tau_n \to s$. In order to show it, we proceed similarly as before.

We write

$$\tau_n = \frac{1}{\log(n+1)}\sum_{k=0}^{n}\frac{s_k\eta_k}{k+1} + \frac{1}{\log(n+1)}\sum_{k=0}^{n}\frac{s_k\varepsilon_k}{k+1}$$

$$= \tau_n^* + \frac{1}{\log(n+1)}\sum_{k=0}^{n}\frac{s_k\varepsilon_k\eta_k}{\eta_k(k+1)}$$

$$= \tau_n^* + \frac{\varepsilon_n}{\eta_n}\tau_n^* + \frac{1}{\log(n+1)}\sum_{k=0}^{n-1}\tau_k^*\log(k+1)\left(\frac{\varepsilon_k}{\eta_k} - \frac{\varepsilon_{k+1}}{\eta_{k+1}}\right).$$

Since $\tau_n^* \to s$ and the series

$$\sum_{k=0}^{\infty} \left(\frac{\varepsilon_k}{\eta_k} - \frac{\varepsilon_{k+1}}{\eta_{k+1}} \right)$$

converges, so arguing as before, we find also that $\tau_n \to s$.

This leads us to the conclusion that the $(R, 1)$-method of summation can be considered as a particular case of the $(R, \lambda_n, 1)$-method, with which it is that equivalent for $\lambda_n = \log(n+1)$. Therefore the results of preceding sections remain true for the $(R, 1)$-method also. To avoid repetition we take this for granted and the details of proofs we leave to the reader.

Now we shall prove the following theorem:

THEOREM 4. *There exists such a system* $\mathrm{ON}\{\varphi_n(x)\}$ *in* $\langle 0, 1\rangle$, *sequence of coefficients* $\{c_n\} \epsilon l^2$ *and strictly increasing sequence* $\{v_n\}$ *of indices such that the series* $\sum_{n=0}^{\infty} c_n \varphi_n(x)$, *with n-th partial sums* $S_n(x)$, *is summable* $(R, 1)$ *almost everywhere to the function* $f(x)$, *and nevertheless the sequence*

$$\frac{1}{\log(n+1)} [S_{v_0}(x)/1 + S_{v_1}(x)/2 + \ldots + S_{v_n}(x)/(n+1)]$$

is divergent in $\langle 0, 1\rangle$ *almost everywhere.*

The proof of this Theorem is based on the following theorem of K. Tandori ([5], Mitt. II, Satz II, p. 151), see also ([1], Satz 2.9.1, p. 129):

Let $\{a_n\}$ denote a sequence satisfying the conditions:

(i) $$\sum_{n=0}^{\infty} a_n^2 < \infty,$$

(ii) $$\sqrt{n}\, a_n \geqslant \sqrt{n+1}\, a_{n+1} > 0 \qquad (n = 1, 2, \ldots),$$

(iii) $$\sum_{n=2}^{\infty} a_n^2 (\log\log n)^2 = \infty.$$

Then there exists a system $\mathrm{ON}\{\varphi_n(x)\}$ such that the series

$$\sum_{n=0}^{\infty} a_n \varphi_n(x)$$

consisting of these coefficients is non-summable $(C, 1)$ at any point of the interval of orthogonality.

Passing to the proof of our Theorem, we define the sequence $\{a_n\}$ as follows:

$$a_n = 1 \quad \text{for} \quad n = 1, 2, \ldots, [e^{e^e}],$$

$$a_n = 1/\sqrt{n \log n \log\log n}\, (\log\log\log n)^2 \quad \text{for} \quad n \geqslant [e^{e^e}].$$

It is easy to verify that the sequence $\{a_n\}$ satisfies assumptions (i), (ii) and (iii) of the Tandori theorem, and that

(6.1) $$\sum_{n=N}^{\infty} a_n^2 (\log\log\log n)^2 < \infty.$$

Hence and from Theorem B we conclude that there exists a system $\mathrm{ON}\{\psi_n(x)\}$ such that the series

(6.2) $$\sum_{n=0}^{\infty} a_n \psi_n(x)$$

is on the one hand non-summable $(C, 1)$ at any point of interval $\langle 0, 1\rangle$, and on the other hand summable $(R, 1)$ in $\langle 0, 1\rangle$ almost everywhere.

To continue the proof, we lean upon an inequality proved by K. Tandori ([5], Mitt. II, p. 164, 3.11):

Under the above assumptions the inequality

$$|a_{2N_m+1} \psi_{2N_m+1}(x) + \ldots + a_{2N_m+n_m(x)+1} \psi_{2N_m+n_m(x)+1}(x)| \geqslant B > 0$$

$$(B \text{ constant}, \ n_m(x) + 1 < 2^{m+2}, \qquad N_m = 2^{m+2} - 4)$$

is satisfied for every $x \epsilon E$, $|E| = 1$ and for an infinite number of indices m.

Hence we verify at once that, removing in series (6.2) the terms with indices $2^{N_{m+1}}$, the series obtained in this way will be yet non-summable $(C, 1)$ in set E, which is always possible, as $N_m + n_m(x) < N_{m+1}$. Moreover, by suitable change of the values of functions $\psi_n(x)$ in a set of measure zero, we can obtain such series, which will be non-summable $(C, 1)$ at any point of interval $\langle 0, 1\rangle$.

Let the new series be

(6.3) $$\sum_{n=0}^{\infty} \tilde{a}_n \tilde{\psi}_n(x).$$

Obviously, the coefficients $\{\tilde{a}_n\}$ of this series satisfy the condition (6.1), whence by Theorem B it follows that series (6.3) is summable $(R, 1)$ almost everywhere.

Now, we construct a strictly increasing sequence of indices $\{v_n\}$ and a strictly increasing sequence of positive integers $\{T_p\}$ satisfying the inequalities

(6.4) $$2^{2^{T_p}} \leqslant v_k < 2^{2^{T_{p+1}}} \qquad (2^{2^{2^p}} < k \leqslant 2^{2^{2^{p+1}}})$$

for every $p \geqslant 1$.

Putting $v_i = i$ for $i = 1, 2, \ldots, 2^{2^2}$, we denote by T_1 the least positive integer such that

$$v_{2^{2^2}} < 2^{2^{T_1}}.$$

T_1 being defined, we denote by T_2 the least positive integer such that

$$2^{T_2} > 2^{T_1} + 2^{2^2}.$$

Further, we put

$$v_{2^{2^2}+i} = 2^{2^{T_1}} \cdot i \quad \text{for} \quad i = 1, 2, \ldots, 2^{2^{2^2}} - 2^{2^2}.$$

It is obvious that for those values i the inequalities (6.4) are satisfied for $p = 1$.

Let us suppose that for an arbitrary positive integer $m \geqslant 2$, the indices $v_1 < v_2 < \ldots < v_{2^{2^m}}$ and positive integers $T_1 < T_2 < \ldots < T_m$ are already determined, so that inequalities (6.4) are satisfied for $p = 1, 2, \ldots, m-1$.

Further, let T_{m+1} denote the least positive integer satisfying the conditions

$$2^{T_{m+1}} > 2^{T_m} + 2^{2^{m+1}}, \quad v_{2^{2^{2^m}}+i} = 2^{2^{T_m}} \cdot i \quad \text{for} \quad i = 1, 2, \ldots, 2^{2^{2^{m+1}}} - 2^{2^{2^m}}.$$

We may easily verify that

$$v_1 < v_2 < \ldots < v_{2^{2^{2^{m+1}}}}; \ T_1 < T_2 < \ldots < T_{m+1}$$

and that the condition (6.4) is satisfied for $p = m$.

Hence, in virtue of complete induction we conclude that these inequalities are satisfied for every positive integer p.

Next, we construct the series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x) \tag{6.5}$$

which is mentioned in Theorem 4.

Thus, let

$$\varphi_{v_{2^n}}(x) = \tilde{\psi}_n(x) \quad \text{for} \quad n \neq 2^{N_{m+1}} \quad (m = 1, 2, \ldots),$$

$$\varphi_v(x) = \tilde{\psi}_{2^{N_v+1}}(x) \quad \text{for} \quad v = 1, 2, \ldots; \quad v \neq v_{2^n} \quad (n = 0, 1, \ldots),$$

$$c_{v_{2^n}} = \tilde{a}_n \quad \text{for} \quad n \neq 2^{N_{m+1}} \quad (n = 0, 1, \ldots; \ m = 1, 2, \ldots),$$

$$c_v = 0 \quad \text{for} \quad v \neq v_{2^n} \quad (n = 0, 1, 2, \ldots).$$

The system $\{\varphi_v(x)\}$ is, of course, orthonormal in $\langle 0, 1\rangle$. By (6.4) and the above construction we have

$$S_{2^{2^r}}(x) = \tilde{S}_{2^{2^{p+1}}}(x) \tag{6.6}$$

for $T_p \leqslant r < T_{p+1}$ $(p = 1, 2, \ldots)$.

$S_n(x)$ and $\tilde{S}_n(x)$ denote here n-th partial sums of series (6.5) and (6.3), respectively.

Since the sequence $\{\tilde{a}_n\}$ satisfies the condition (6.1), so according to Theorems B and A

$$\lim_{p \to \infty} \tilde{S}_{2^{2^{p+1}}}(x) = f(x) \tag{6.7}$$

almost everywhere in $\langle 0, 1\rangle$.

From (6.6) and (6.7) it follows that

$$\lim_{r \to \infty} S_{2^{2^r}}(x) = f(x)$$

almost everywhere in $\langle 0, 1\rangle$.

In virtue of Theorem A and $\{c_n\} \in l^2$, series (6.5) is summable $(R, 1)$ almost everywhere to the function $f(x)$.

Taking into account the above construction we obtain

$$S_{v_{2^{2^n}}}(x) = \tilde{S}_{2^n}(x) \quad (n = 0, 1, 2, \ldots).$$

Since the sequence $\{\tilde{S}_{2^n}(x)\}$ is divergent in $\langle 0, 1\rangle$ almost everywhere so from the last equality it follows that the sequence $\{S_{v_{2^{2^n}}}(x)\}$ and by Lemma 1 the sequence

$$\tau_n^{(v)}(x) = \frac{1}{\log(n+1)} [S_{v_0}(x)/1 + S_{v_1}(x)/2 + \ldots + S_{v_n}(x)/(n+1)] \tag{6.8}$$

is divergent in $\langle 0, 1\rangle$ almost everywhere also.

Replacing in the inequality (4.3) λ_n by $\log(n+1)$, we conclude hence that

$$\sum_{k=0}^{n} \frac{1}{k+1} [S_{v_k}(x) - f(x)]^2 \neq o\big(\log(n+1)\big)$$

in $\langle 0, 1\rangle$ almost everywhere. This, together with a result obtained by the author ([4], see the proof of Th. 1, p. 15), or with Theorem 1 with $\lambda_n = \log(n+1)$, leads us to the following conclusion: If orthogonal series (2.2), with coefficients satisfying condition (2.1), is summable $(R, 1)$ almost everywhere, then it is strongly summable $(R, 1)$. However, there exist orthogonal series (2.2), with coefficients satisfying condition (2.1,) which are not very strongly summable $(R, 1)$ almost everywhere.

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Convolution of functions of several variables

by

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Introduction. In this paper we give a new proof of a theorem on convolution which is an extension to several variables of the well known Titchmarsh theorem [5]. The first proof, due to Lions [1], has been based on the Fourier Transform. Another proof, due to Mikusiński and Ryll-Nardzewski [4], has been based upon a geometrical method. The proof of this paper is based on the concept of Banach algebra.

We give several equivalent formulations of the theorem (Theorems VIII-VIII d).

1. Let $\mathcal{A}$ be a commutative Banach algebra over the field of complex numbers, and let $\mathcal{A}_1$ be its least extension with unity.

Let $E(t)$ $(t \geqslant 0)$ be an *exponential operator*, i. e. an operator such that $E(t)x \in \mathcal{A}$ for $x \in \mathcal{A}$, $x \neq 0$, and moreover

1° $E(0) = 1$;

2° $E(t)(xy) = \big(E(t)x\big)y$;

3° For every $x \in \mathcal{A}$, the function $E(t)x$ is continuous;

4° There exists an element $l \in \mathcal{A}_1$, non divisor of zero, such that

$$\frac{d}{dt}E(t)l = E(t)\ (^1).$$

Letting $y = 0$ in 2° we obtain

$$E(t)0 = 0.$$

It is also easy to verify that, by 2°,

$$E(t_1)x \cdot E(t_2)y = E(t_1)y \cdot E(t_2)x = E(t_1)E(t_2)xy.$$

We shall prove that

$$E(t_1)E(t_2) = E(t_1+t_2). \tag{1}$$

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(1) This means that $\frac{d}{dt}E(t)lx = E(t)x$ for every $x \in \mathcal{A}$.