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On the form of pointwise continuous positive functionals and isomorphisms of function spaces

by

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In § 1 we are concerned with the form of pointwise continuous positive functionals defined on a linear space of continuous real-valued functions. In § 2 we give a sufficient condition under which the existence of an isomorphism of function spaces implies the existence of a homeomorphism of spaces of arguments. § 3 contains a generalization of those results to the case of linear spaces satisfying less restricted conditions than those assumed in §§ 1 and 2.

§ 1.

Given a topological space X (1), we denote by $C(X)$ ($C^*(X)$) the set of all continuous (all bounded continuous) real-valued functions defined on X. It is known that

(A) *A space X is a Q-space if and only if each non-trivial linear multiplicative functional φ (2) defined in $C(X)$ can be written in the form*

$$\varphi(f) = f(p_0) \tag{*}$$

where p_0 is a fixed point of X (3).

Now let R be any linear ring contained in $C(X)$ and satisfying the following conditions:

(α) *all constant functions belong to R;*

(β) *if $f_n \in R$, $0 \leqslant f_n(p) \leqslant 1$ ($n = 1, 2, \ldots,$), then there exists a sequence a_n of positive numbers such that $\sum_n a_n < +\infty$ and $\sum_n a_n \cdot f_n \in R$;*

(γ) *if $f \in R$ and $f(p) \neq 0$ for each p in X, then $1/f \in R$.*

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(1) All topological spaces under consideration are supposed to be Hausdorff completely regular.

(2) A functional is called *non-trivial* if it does not vanish identically.

(3) The original Hewitt definition of Q-spaces [2] is as follows: a maximal ideal $\mathfrak{M}$ in the ring $C(X)$ is said to be *hyper-real* provided that the quotient field $C(X)/\mathfrak{M}$ contains (within isomorphism) the field of real numbers as a proper subset; $\mathfrak{M}$ is called *free* if for each p in X there is an f in $\mathfrak{M}$ with $f(p) \neq 0$. Then a space X is said to be a *Q-space* if each maximal proper free ideal in $C(X)$ is hyper-real. Restating this definition in terms of functionals, we obtain (A).

In [6] the author proves following theorem (4):

(B) *Each non-trivial multiplicative linear functional φ defined on any linear ring $R \subset C(X)$ which satisfies conditions (α), (β) and (γ) can be written in the form $(*)$ if and only if X is a Lindelöf space* (5).

An analogous situation may be observed in connection with pointwise continuous positive linear functionals. Namely, in [5] the author proves that (6):

(C) *Each pointwise continuous positive linear functional φ* (7) *defined on $C^*(X)$ can be written in the form*

$$(**) \qquad \varphi(f) = a_1 \cdot f(p_1) + \ldots + a_k \cdot f(p_k),$$

where $p_1, \ldots, p_k$ are fixed points of X and $a_1, \ldots, a_k$ are fixed real numbers if and only if X is a Q-space.

Now let $E \subset C^*(X)$ be a linear space satisfying conditions (α) and (β) and the following one:

(δ) *if $f \in E$, then $|f| \in E$.*

Then the following analogue of statement (B) can be proved:

(D) *Each pointwise continuous positive linear functional φ defined on any linear space $E \subset C^*(X)$ which satisfies conditions (α), (β) and (δ) can be written in the form $(**)$ if and only if X is a Lindelöf space* (8).

We shall deduce statement (D) from a more general statement which will be given in the next section. Now we explain the role of condition (δ).

Notice that condition (δ) is satisfied if E is a linear subring of $C^*(X)$, which is closed with respect to uniform convergence and satisfies condition (α). Indeed, in this case, the function $|f|$ $(f \in E)$ can be written as the sum of a uniformly convergent series of members of E.

Condition (δ) is essential. We shall show that, even in the case when X is the unit interval [0,1], there exists a linear space $E \subset C^*(X)$ which

(4) This theorem is a generalization of an unpublished theorem of S. Mazur.

(5) A completely regular space is called *Lindelöf* if each open covering of the space contains an enumerable covering.

(6) This theorem is a generalization of a result of S. Mazur (see [7]).

(7) A functional φ is said to be *continuous with respect to pointwise convergence* (or, shortly, *pointwise continuous*) if the conditions $f, f_n \in$ domain φ, $f_n \to f$ imply the condition $\varphi(f_n) \to \varphi(f)$, where $f_n \to f$ means that $f_n(p) \to f(p)$ for each p in X; φ is said to be *positive* if $\varphi(f) \geqslant 0$ for each $f \geqslant 0$, where $f \geqslant 0$ means that $f(p) \geqslant 0$ for each p in X.

(8) The assumption of positivity of a functional in theorems (C) and (D) is not essential; indeed, every pointwise continuous linear functional φ can be written in the form $\varphi = \varphi_1 - \varphi_2$ where φ_1 and φ_2 are pointwise continuous positive functionals (see, for instance, [3]).

satisfies conditions (α) and (β), and a pointwise continuous positive linear functional φ defined on E which cannot be written in the form $(**)$. We shall use the following unpublished result of S. Mazur:

Suppose that p is an increasing sequence of positive integers with $\sum_i p_i^{-1} < +\infty$ and $f_n(x) = a_0^{(n)} + \sum_i a_i^{(n)} \cdot x^{p_i}$, where the series $\sum_i a_i^{(n)} x^{p_i}$ are convergent in the whole interval $[0,1]$. If $f_n(x) \to f(x)$ for each $0 \leqslant x \leqslant 1$; then $f(x)$ is also of the form $f(x) = a_0 + \sum_i a_i \cdot x^{p_i}$, where the series $\sum_i a_i \cdot x^{p_i}$ is convergent in the whole interval $[0,1]$; moreover, f_n converge to f almost uniformly in the interval $[0,1)$.

Consider the following infinite matrix of positive integers:

$$\begin{array}{llllllll}
2\cdot 1, & 2\cdot 2, & & & & & & \\
2^2\cdot 1, & 2^2\cdot 2, & 2^2\cdot 3, & 2^2\cdot 4, & & & & \\
2^3\cdot 1, & 2^3\cdot 2, & 2^3\cdot 3, & 2^3\cdot 4, & 2^3\cdot 5, & 2^3\cdot 6, & & \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \\
2^k\cdot 1, & 2^k\cdot 2, & 2^k\cdot 3, & 2^k\cdot 4, & \ldots & \ldots & \ldots & 2^k\cdot 2^k, \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{array}$$

Of course, the sum of the inverses of numbers standing in the k-th row of the matrix does not exceed $2k/2^k$. Let p_i be the increasing sequence consisting of all terms of the matrix. If follows from the preceding remark that the series $\sum_i p_i^{-1}$ is convergent.

Let E be the set of all functions f on X which can be represented in the form $f(x) = a_0 + \sum_i a_i \cdot x^{p}$, where the series $\sum_i a_i \cdot x^{p_i}$ is convergent in the whole interval $[0,1]$. By Mazur's result quoted above, E is a linear subspace of $C^*(X)$ satisfying conditions (α) and (β).

Now let

$$\varphi(f) = \int_0^{1/2} f(x)\,dx \quad \text{for} \quad f \in E.$$

Of course, φ is a positive linear functional and it follows from Mazur's result quoted above that φ is pointwise continuous. But φ cannot be written in the form $(**)$. Indeed, let $x_1, \ldots, x_k$ be any points of the interval $[0,1]$ and let

$$f(x) = (x^{2^k} - x_1^{2^k})^2 \ldots (x^{2^k} - x_k^{2^k})^2.$$

Then $f \in E$ (indeed, exponents of the variable x occur in the k-th row of the matrix), $f(x_j) = 0$ $(j = 1, \ldots, k)$, $\varphi(f) \neq 0$.

I. The main result. Before formulating the main result of the present paragraph we recall some definitions.

A subset P of a topological space S is said to be *Q-closed in S* provided that for each p in $S\setminus P$ there exists a G_δ-set $G\subset S$ which contains p and is disjoint from P.

Of course we have

(i) *A set $P\subset S$ is Q-closed in S if and only if for each p in $S\setminus P$ there exists a continuous real-valued function f on S which is positive on P and zero at p.*

Moreover, we have (see [7])

(ii) *A space A is Q-closed in βX if and only if X is a Q-space.*

(iii) *A space X is Q-closed in each of its compactifications* [9] *if and only if X is a Lindelöf space.*

Suppose $\mathfrak{F}$ is a family of bounded continuous functions, each defined on a space X. Let I_f be the interval of values of a function $f\in\mathfrak{F}$ (i. e. I_f is the interval $[\inf_{p\in X} f(p), \sup_{p\in X} f(p)]$) and denote by $I^{\mathfrak{F}}$ the Cartesian product $\mathrm{P}_{f\in\mathfrak{F}} I_f$. Let $F_{\mathfrak{F}}$ be the mapping of X into $I^{\mathfrak{F}}$ which carries a point $p\in X$ into the point $\xi\in I^{\mathfrak{F}}$ whose f-th coordinate, is equal to $f(p)$. Clearly, $F_{\mathfrak{F}}$ is a continuous mapping.

Theorem 1. *Let $E\subset C^*(X)$ be a linear space satisfying conditions* (α), (β) and (δ). *Then each pointwise continuous positive linear functional φ defined on E can be written in the form* (**) *if and only if $F_E(X)$ is Q-closed in $\overline{F_E(X)}$ (the bar indicates the closure with respect to I^E).*

The proof of the above theorem will be given in the next sections. Now we shall show that statements (C) and (D) are immediate consequences of this theorem.

To begin with, if $E=C^*(X)$, then F_E is a homeomorphism and $\overline{F_E(X)}=\beta F_E(X)=\beta X$ [10]. Hence, each pointwise continuous positive linear functional φ defined on $C^*(X)$ can be written in the form (**) if and only if X is Q-closed in βX, i. e. if X is a Q-space. Thus (C) is proved. Now suppose that X is a Lindelöf space. If $E\subset C^*(X)$ is any linear space satisfying conditions (α), (β) and (δ), then $F_E(X)$, being the continuous image of X, is also a Lindelöf space, whence $F_E(X)$ is Q-closed in $\overline{F_E(X)}$. Conversely, suppose that X is not a Lindelöf space. Then, by (iii), there exists a space Y which is a compactification of X and is such that X is not Q-closed in Y. Denote by E the set of all functions on X which can be continuously extended over Y. Then E is a linear subspace

(9) By a *compactification* of a space X we understand any compact space which contains X as a dense subset.

(10) Theorem of E. Čech: see [1].

of $C^*(X)$ which satisfies conditions (α), (β) and (δ). Moreover, F_E is a homeomorphism and F_E can be extended to a homeomorphism which maps Y onto $\overline{F_E(X)}$. It follows that $F_E(X)$ is not Q-closed in $\overline{F_E(X)}$ and (D) is proved.

II. Auxiliary theorems. We say that a family F of functions, each defined on a space S, *distinguishes points of S* if for every $p, p'\in S$, $p\neq p'$, there is an f in F with $f(p)\neq f(p')$; we say that F *distinguishes points and closed sets of S* if for each $A=\bar{A}\subset S$ and $p_0\in S\setminus A$ there is an f in F with $f(p)=1$ for $p\in A$ and $f(p_0)=0$.

Lemma 1. *Suppose that S is a compact space and that E is a linear subspace of $C^*(S)$ satisfying conditions* (α), (β) *and* (δ) *and distinguishing points and closed sets of S. Then a set $P\subset S$ is Q-closed in S if and only if for each p_0 in $S\setminus P$ there is a function f in E and a sequence $f_1, f_2, \ldots$ ($f_n\in E$) such that $f_n(p)\to f(p)$ for each p in P and $f(p_0)\nrightarrow f(p_0)$.*

Proof. Let $P\subset S$ be a Q-closed set and let p_0 be any point in $S\setminus P$. Then there exists a sequence $G_1, G_2, \ldots$ of open sets such that $p_0\in G_n$ $(n=1,2,\ldots)$ and $P\cap\bigcap_n G_n=0$. Let $F_n=S\setminus G_n$. There is a function h_n in F such that $h_n(p_0)=0$ and $h_n(p)=1$ for $p\in F_n$. Setting $g_n(p)=\min\{1,\max\{0,h_n(p)\}\}$ we have $g_n\in E$, $g_n(p_0)=0$, $g_n(p)=1$ for $p\in F_n$ and $0\leqslant g_n(p)\leqslant 1$ for $p\in S$. By condition (β), there exists a sequence $a_1, a_2, \ldots$ of positive numbers such that $\sum_n a_n<+\infty$ and $g=\sum_n a_n\cdot g_n\in E$. Of course, $g(p_0)=0$ and $g(p)>0$ for $p\in P$. Setting $f_n=\min\{1, n\cdot g\}$, we have $f_n\in E, f_n(p)\to 1$, for $p\in P$ and $f_n(p_0)\nrightarrow 1$.

Conversely, suppose that P is not Q-closed in S. Then, by (i), there exists a point $p_0\in S\setminus P$ such that for each continuous function h on S the condition $h(p)>0$ for each $p\in P$ implies $h(p_0)>0$. Now let f be any function in E and let $f_1, f_2, \ldots$ ($f_n\in E$) be any sequence such that $f_n(p)\to f(p)$ for each p in P. Let

$$h(p)=|f(p)-f(p_0)|+\sum_n\frac{|f_n(p)-f_n(p_0)|}{2^n(1+|f_n(p)-f_n(p_0)|)}.$$

Then h is a continuous function on S and $h(p_0)=0$ and it follows that $f(p_1)=0$ for some p_1 in P. But $h(p_1)=0$ implies $f(p_0)=f(p_1)$ and $f_n(p_0)=f_n(p_1)$ for $n=1,2,\ldots$ Since $f_n(p_1)\to f(p_1), f_n(p_0)\to f(p_0)$ and the lemma follows.

Lemma 2. *Let S be a compact space and E a linear subspace of $C^*(S)$ $(=C(S))$ which satisfies conditions* (α), (β) *and* (δ). *Then if E distinguishes points, then E distinguishes points and closed sets.*

Proof. Suppose that A is a closed set in S and p_0 is a point in $S\setminus A$. For each p in A there is a function f_p in E such that $f_p(p_0)\neq f_p(p)$. Let g_p be defined by the equality

$$g_p(q)=\frac{2\,|f_p(q)-f_p(p_0)|}{|f_p(p)-f_p(p_0)|}.$$

Then $g_p\in E$ and $g_p(p_0)=0$, $g_p(p)=2$. Let $U_p=\{q\in S: g_p(q)>1\}$. Then the system $\{U_p\}_{p\in A}$ is an open covering of the set A, whence there exists a finite system $p_1,\ldots,p_k$ with $A\subset U_{p_1}\cup\ldots\cup U_{p_k}$. Let $g=\min\{1, g_{p_1}+\ldots+g_{p_k}\}$. Then $g\in E$, $g(p_0)=0$, and $g(p)=1$ for $p\in A$ and the lemma follows.

III. The case of a compact space. In this section we shall prove the following

THEOREM 1a. *Suppose that X is a compact space and that E is a linear subspace of $C^*(X)$ which satisfies conditions (α), (β) and (δ) and distinguishes points and closed sets. Then each pointwise continuous positive linear functional φ defined on E can be written in the form* (**).

Suppose that φ is any pointwise continuous positive non-trivial linear functional defined on E. For any f in E we denote by $Z(f)$ the set $\{p\in X: f(p)=0\}$ and let Z_φ be the intersection of all sets of the form $Z(f)$ where f is any non-negative function in E with $\varphi(f)=0$. The proof of Theorem 1a is based on the following lemmas:

LEMMA 3. *If f is a non-negative function in E and if $\varphi(f)=0$, then $Z(f)\neq 0$.*

Proof. Suppose that $Z(f)=0$. There is g in E with $\varphi(g)\neq 0$. Let $h=|g|$. Then $h\in E$ and $\varphi(h)\geqslant|\varphi(g)|>0$. Let $f_n=\min\{h, n\cdot f\}$. Then $f_n\in E$ and $f_n\to h$, whence $\varphi(f_n)\to\varphi(h)$. But $0\leqslant\varphi(f_n)\leqslant n\cdot\varphi(f)=0$, whence $\varphi(h)=0$, which leads to a contradiction.

LEMMA 4. *The set Z_φ is non-empty.*

Proof. If f and g are non-negative functions in E, then $Z(f)\cap Z(g)=Z(h)$, where $h=f+g$. It follows by Lemma 3 that Z_φ is the intersection of a centred system if closed non-empty subsets of X. By the compactness of X, Z_φ is non-empty.

LEMMA 5. *If f is any non-negative function in E, then $\varphi(f)=0$ if and only if $Z_\varphi\subset Z(f)$.*

Proof. If $\varphi(f)=0$, then $Z_\varphi\subset Z(f)$ by the definition of Z_φ. Conversely, suppose that $Z_\varphi\subset Z(f)$ and let $M=\sup_{p\in X}f(p)$, $F_n=\{p\in X: f(p)\geqslant 1/n\}$. If $p\in F_n$, then $p\in Z_\varphi$, whence there exists a non-negative function h_p in E with $h_p(p)=1$ and $\varphi(h_p)=0$. Let $U_p=\{q\in X: h_p(q)>\frac{1}{2}\}$. Of course, the system $\{U_p\}_{p\in F_n}$ is an open covering of F_n, and thus

there exists a finite system $p_1,\ldots,p_k$ with $F_n\subset U_{p_1}\cup\ldots\cup U_{p_k}$. Let $g_n=h_{p_1}+\ldots+h_{p_k}$. Then $g_n\in E$ and $g_n(p)>\frac{1}{2}$ for $p\in F_n$. Let $f_n=\min\{f, 2Mg_n\}$. Then $f_n\in E$, $0\leqslant\varphi(f_n)\leqslant 2M\cdot\varphi(g_n)=0$ and $f_n(p)=f(p)$ for $p\in Z(f)\cup F_n$. Since $Z(f)\cup F_n\subset Z(f)\cup F_{n+1}$ and $\bigcup_n(Z(f)\cup F_n)=X$, $g_n(p)\to f(p)$ for each p in X. Thus $\varphi(f)=0$ and the lemma follows.

LEMMA 6. *If f is any function in E and $Z_\varphi\subset Z(f)$, then $\varphi(f)=0$.*

Proof. Let $h=|f|$. Then $h\in E$ and $Z(h)=Z(f)$, whence, by Lemma 5, $\varphi(h)=0$. But $-h\leqslant f\leqslant h$, whence $-\varphi(h)\leqslant\varphi(f)\leqslant\varphi(h)$, and $\varphi(f)=0$.

LEMMA 7. *The set Z_φ is finite.*

Proof. Suppose that Z_φ is infinite. Then one can select a sequence $G_1, G_2,\ldots$ of mutually disjoint open subsets of X such that $Z_\varphi\cap G_n\neq 0$ ($n=1,2,\ldots$) (see [7], Lemma 1). Let p_n be any point of $Z_\varphi\cap G_n$. Since E distinguishes points and closed sets, there is a non-negative function f_n in E such that $f_n(p_n)=0$ and $f_n(p)=1$ for $p\in X\setminus G_n$. Let $h_n=\max\{0, 1-f_n\}$. Then $h_n\in E$, $h_n(p_n)=1$, $h_n(p)=0$ for $p\in X\setminus G_n$. By Lemma 3, $\varphi(h_n)\neq 0$. Let $g_n=h_n/\varphi(h_n)$. Then $g_n\in E$, $g_n\to 0$, but $\varphi(g_n)\to 1$, which leads to a contradiction.

Proof of Theorem 1a. Let φ be any pointwise continuous positive linear functional defined on E. If φ is a trivial functional, then there is nothing to prove. Suppose that φ is a non-trivial functional. Then, by lemmas 4 and 7, the set Z_φ is non-empty and finite; let $Z_\varphi=\{p_1,\ldots,p_k\}$, where the points p_i are mutually distinct. Since E distinguishes points and closed sets, for each i there is a function $f_i\in E$ such that $f_i(p_i)=1$ and $f_i(p_j)=0$ for $j\neq i$. Let $a_i=\varphi(f_i)$. Let f be any function in E and let us set $h=f(p_1)\cdot f_1+\ldots+f(p_k)\cdot f_k-f$. Then $h\in E$ and $Z_\varphi\subset Z(h)$, whence $\varphi(h)=0$. But $\varphi(h)=f(p_1)\cdot\varphi(f_1)+\ldots+f(p_k)\cdot\varphi(f_k)-\varphi(f)=a_1\cdot f(p_1)+\ldots+a_kf(p_k)-\varphi(f)$ and it follows that $\varphi(f)=a_1\cdot f(p_1)+\ldots+a_k\cdot f(p_k)$. Thus Theorem 1 is proved.

IV. Proof of Theorem 1. Let E be any linear subspace of $C(X)$ satisfying conditions (α), (β) and (δ). We denote by E_1 the set of all continuous functions h defined on $Y=\overline{F_E(X)}$ for which there is a function f in E such that

$$f(p)=h(F_E(p))\quad\text{for each } p \text{ in } X. \tag{1}$$

Of course, for each f in E there exists exactly one function h in E_1 satisfying (1), namely the f-th coordinate of a point $y\in Y$. It follows that E_1 is a linear subspace if $C(X)$ satisfying conditions (α), (β) and (δ). Moreover, for each f in E, the function p_f belongs to E_1 (we recall that the coordinates of points of Y are enumerated by means of members of E and $p_f(y)$ denotes the f-th coordinate of a point $y\in Y$). It follows that E_1 distinguishes points of Y. Since Y is compact, E_1 distinguishes points and closed sets of Y.

Now assume that $F_E(X)$ is Q-closed in Y and let φ be any pointwise continuous positive linear functional defined on E. Let φ_1 be the functional defined on E_1 by the equality

$$(2) \qquad \varphi_1(h) = \varphi(f),$$

where f is a member of E satisfying (1). Of course, φ_1 is a pointwise continuous positive linear functional. By Theorem 1a, we have

$$(3) \qquad \varphi_1(h) = a_1\cdot h(y_1)+\ldots+a_k\cdot h(y_k) \quad \text{for each } h \text{ in } E_1,$$

where $a_1, \ldots, a_k$ are fixed real numbers and $y_1, \ldots, y_k$ are fixed points of Y.

We shall show that $y_i \in F_E(X)$. Assume that $y_{i_0} \in Y \setminus F_E(X)$ and $a_{i_0} \neq 0$. Since $F_E(X)$ is Q-closed in Y by Lemma 1, there is a function $h \in E_1$ and a sequence $h_1, h_2, \ldots$ $(h_n \in E_1)$ such that $h_n(y) \to h(y)$ for each y in $F_E(X)$ and $h_n(y_{i_0}) \nrightarrow h(y_{i_0})$. Let $\bar{h}_n = |h_n - h|$. Then $\bar{h}_n \in E_1$, $\bar{h}_n(y) \to 0$ for each y in $F_E(X)$ and $\bar{h}_n(y_{i_0}) \nrightarrow 0$. Since E_1 distinguishes points and closed sets of Y, there exists a function g in E_1 such that $g(y_{i_0}) = 1$ and $g(y_i) = 0$ for $i \neq i_0$. Let $g_n = \max\{0, \min\{g, \bar{h}_n\}\}$. Then $g_n \in E_1$, $g_n(y) \to 0$ for each y in $F_E(X)$ and from (3) it follows that $\varphi_1(g_n) \nrightarrow 0$. But from (1) and (2) it follows that if $g_n(y) \to 0$ for each y in $F_E(X)$, then $\varphi_1(g_n) \to 0$, which leads to a contradiction. Thus $y_i \in F_E(X)$ for $i = 1, \ldots, k$.

Now let p_i be any point in X with $F_E(p_i) = y_i$ $(i = 1, \ldots, k)$. It follows from (1) and (2) that

$$\varphi(f) = a_1\cdot f(p_1)+\ldots+a_k\cdot f(p_k) \quad \text{for each } f \text{ in } E.$$

Conversely, suppose that $F_E(X)$ is not Q-closed in Y. Then, by Lemma 1, there is a point $y_0 \in Y \setminus F_E(X)$ such that for any function h in E_1 and each sequence $h_1, h_2, \ldots$ of members of E_1 the condition $h_n(y) \to h(y)$ for each y in $F_E(X)$ implies the condition $h_n(y_0) \to h(y_0)$. Let φ be the functional defined on E by the equality $\varphi(f) = h(y_0)$ where h is a function in E_1 satisfying (1). Then φ is a pointwise continuous positive linear functional which cannot be written in the form (**). Thus the proof of the theorem is complete.

§ 2.

In this paragraph we shall show that if $E \subset C^*(X)$ is a linear space which satisfies conditions (α), (β), (δ) and distinguishes points and closed sets of X and if $F_E(X)$ is Q-closed in $\overline{F_E(X)}$, then the topology of X is, in a certain sense, determined by E. Before an exact formulation of the theorem we give a definition.

Let E_1 and E_2 be linear spaces consisting of functions. A linear one-to-one mapping ξ of E_1 onto E_2 will be called an *isomorphism* if ξ satisfies the following conditions

(I_1) *if* $f \in E_1$, $g = \xi(f)$, *then* $f \geqslant 0$ *if and inly if* $g \geqslant 0$;

(I_2) *if* $f_n \in E_1$, $f_n \geqslant f_{n+1}$, $g_n = \xi(f_n)$, *then* $f_n \to 0$ *if and only if* $g_n \to 0$.

We shall prove the following

THEOREM 2. *Suppose* $E_1 \subset C^*(X_1)$, $E_2 \subset C^*(X_2)$, *are linear spaces satisfying conditions* (α), (β) *and* (δ) *and such that* E_i *distinguishes points and closed sets of* X_i *and* $F_{E_i}(X_i)$ *is* Q-*closed in* $\overline{F_{E_i}(X_i)}$ $(i = 1, 2)$. *If the spaces* E_1 *and* E_2 *are isomorphic, then the spaces* X_1 *and* X_2 *are homeomorphic.*

The proof of Theorem 2 will be given in section III. Now we shall show some elementary properties of isomorphisms (e_X denotes the function which is identically equal to 1 on X):

(iv) $\xi(|f|) = |\xi(f)|$ *for each* f *in* E_1.

We have $-|f| \leqslant f \leqslant |f|$, whence $-\xi(|f|) \leqslant \xi(f) \leqslant \xi(|f|)$ and $|\xi(f)| \leqslant \xi(|f|)$. On the other hand, $-|\xi(f)| \leqslant \xi(f) \leqslant |\xi(f)|$, whence $-\xi^{-1}(|\xi(f)|) \leqslant f \leqslant \xi^{-1}(|\xi(f)|)$ and $|f| \leqslant \xi^{-1}(|\xi(f)|)$, and thus $\xi(|f|) \leqslant |\xi(f)|$. Finally, $\xi(|f|) = |\xi(f)|$.

(v) *If* $f \in E_1$, $f \geqslant 0$, $g = \xi(f)$, *then* $Z(f) = 0$ *if and only if* $Z(g) = 0$.

Suppose that $Z(f) = 0$. Let $f_0 = e_{X_1} = \xi^{-1}(e_{X_2})$ and $f_n = \min\{f_0, n\cdot f\}$. Since $Z(f) = 0$, $f_n \nearrow f$ [11] we have $\xi(f_n) \nearrow \xi(f_0)$. Since $\xi(f_0) = \xi(e_{X_1}) = e_{X_2}$ is a strictly positive function on X_2 and $\xi(f_n) \leqslant n\cdot g$, $Z(g) = 0$. The converse can be shown in an analogous manner.

Remark 1. It can be shown that condition (v) actually characterizes isomorphisms among linear one-to-one mappings satisfying condition (I_1). In fact, suppose that ξ is such a mapping of $E_1 \subset C^*(X_1)$ onto $E_2 \subset C^*(X_2)$ and let $f_n \in E_1$, $f_n \geqslant f_{n+1}$, $g_n = \xi(f_n)$, $g_0 = \xi(e_{X_1})$. Suppose that $g_n \to 0$ and let p_0 be any point of X_1. Let $\bar{f}_n = \min\{e_{X_1}, |f_n - f_n(p_0)|\}$, $\bar{g}_n = \xi(\bar{f}_n)$. By condition ($\delta$) there exists a sequence a_n of positive numbers such that $\sum\limits_n a_n < +\infty$ and $\bar{f} = \sum\limits_n a_n\cdot \bar{f}_n \in E_1$. Let $\bar{g} = \xi(\bar{f})$. Of course, $\bar{f} \geqslant 0$ and $f(p_0) = 0$, and it follows by condition (v) that $g(q_0) = 0$ for some $q_0 \in X_2$. On the other hand, let $r_n = \sum\limits_{m=n+1}^{\infty} a_m\cdot \bar{f}_m$. Of course, $0 \leqslant r_n \leqslant \sum\limits_{m=n+1}^{\infty} a_m$ whence $0 \leqslant \xi(r_n) \leqslant \sum\limits_{m=n+1}^{\infty} a_m\cdot \xi(e_{X_1})$, and thus $\xi(r_n) \to 0$; consequently $\bar{g} = \xi(\bar{f}) = \sum\limits_n a_n\cdot \bar{g}_n$ and it follows that $\bar{g}_n(q_0) = 0$ for

[11] $f_n \searrow f$ means that f_n is a decreasing sequence and $f_n \to f$; $f_n \nearrow f$ has a similar meaning.

$n = 1, 2, \ldots$ But by condition (iv) (its proof does not depend on the condition (I_2)) $\bar{g}_n = \xi(\bar{f}_n) = \min\{g_0, |g_n - f_n(p_0)\cdot g_0|\}$. Since $g_0(q_0) > 0$ (here we use again condition (v)), $g_n(q_0) = f_n(p_0)\cdot g_0(q_0)$. Since $g_n(q_0) \to 0$, $f_n(p_0) \to 0$. Since p_0 is an arbitrary point of X_1, $f_n \to 0$. In an analogous manner one can show that the assumption $f_n \to 0$ implies $g_n \to 0$.

II. Given a linear space $E \subset C^*(X)$, we denote by $\Phi(E)$ the class of all non-trivial positive linear functionals φ on E which satisfy the following conditions:

$\varphi(|f|) = |\varphi(f)|$ *for each* f *in* E;

if $f_n \in E$, $f_n \searrow 0$, *then* $\varphi(f_n) \to 0$.

Lemma 1. *If E is a linear space satisfying the conditions* (α), (β) *and* (δ) *and $\varphi \in \Phi(E)$, then φ is a pointwise continuous functional.*

Proof. Suppose that $\varphi \in \Phi(E)$. We can assume, without loss of generality, that $\varphi(e_X) = 1$. First let us notice that if $f \in E$, $f \geqslant 0$ and $\varphi(f) = 0$, then $Z(f) \neq 0$ (the proof of Lemma 3 of § 1 applies to this case). Now let $f_n \to 0$ ($f_n \in E$). Let us set $g_n = \min\{1, |f_n - \varphi(f_n)e_X|\}$. By condition (δ) there exists a sequence $a_1, a_2, \ldots$ of positive numbers such that $\sum_n a_n < +\infty$ and $g = \sum_n a_n\cdot g_n \in E$. Let $r_n = \sum_{m=n+1}^{\infty} a_m\cdot g_m$. Of course, $r_n \in E$ and $r_n \searrow 0$, whence $\varphi(r_n) \to 0$ and it follows that $\varphi(g) = \sum_n a_n\cdot\varphi(g_n)$. But $\varphi(g_n) = \min\{1, |\varphi(f_n) - \varphi(f_n)|\} = 0$, whence $\varphi(g) = 0$. On the other hand, $g \geqslant 0$ and it follows that $g(p_0) = 0$ for some p_0 in X. But $g(p_0) = 0$ implies $\varphi(f_n) = f_n(p_0)$ for $n = 1, 2, \ldots$ Since $f_n(p_0) \to 0$, $\varphi(f_n) \to 0$ and the lemma follows.

Lemma 2. *If $E \subset C^*(X)$ is a linear space which satisfies conditions* (α), (β), (δ) *and distinguishes points and closed sets of X, and if $F_E(X)$ is Q-closed in $F_E(X)$, then each functional $\varphi \in \Phi(E)$ is of the form $\varphi(f) = a\cdot f(p_0)$, where $a > 0$ and p_0 is a fixed point of X which is uniquely determined by the functional* φ [12].

Proof. By the preceding lemma and Theorem 1, φ is of the form

$$\varphi(f) = a_1 f(p_1) + \ldots + a_k f(p_k),$$

where $a_i \geqslant 0$ and the points $p_1, \ldots, p_k$ are mutually distinct. We shall show that at most one a_i is different from 0. Assume for instance that $a_1, a_2 \neq 0$. Since E distinguishes points and closed sets of X, there is a function $f_0 \in E$ such that $f_0(p_1) = 1/a_1$, $f_0(p_2) = -1/a_2$ and $f_0(p_i) = 0$ for $i = 3, \ldots, k$. Of course, $\varphi(f_0) = 0$ and $\varphi(|f_0|) = 2$, which leads to a contradiction. Thus we have $\varphi(f) = a\cdot f(p_0)$ for some $p_0 \in X$ and for each f in E. Since φ is non-trivial, $a > 0$. Since E distinguishes points and closed sets of X, the point φ_0 is uniquely determined by the functional φ.

[12] We say that p_0 *corresponds* to the functional φ.

III. Proof of Theorem 2. Let ξ be an isomorphism of E_1 onto E_2. For each p in X_1, we denote by φ_p the functional defined by the equality $\varphi_p(f) = f(p)$. Of course, $\varphi_p \in \Phi(E_1)$ and it follows by (iv) that the functional $\varphi_p(\xi^{-1}(g))$ $(g \in E_2)$ belongs to $\Phi(E_2)$; denote by $h(p)$ the point which corresponds to the functional $\varphi_p(\xi^{-1}(g))$. We see that the following relation is satisfied:

$$\varphi_p(\xi^{-1}(g)) = a_p\cdot g(h(p)) \quad \text{for each } g \text{ in } E_2, \tag{1}$$

where a_p is a positive number which depends only upon the point p. We shall show that h is a one-to-one mapping. If $p, q \in X_1$, $p \neq q$, then there is a function $f \in E_1$ with $f(p) = 1$, $f(q) = 0$. Let $g = \xi(f)$. Then, by (1), $a_p\cdot g(h(p)) = 1$, $a_q\cdot g(h(q)) = 0$ and it follows that $h(p) \neq h(q)$.

We shall show that h maps X_1 onto X_2. Let $g_0 = \xi(e_{X_1})$ and let s be any point of X_2. By (v), $g_0(s) > 0$. Let φ be a functional defined on E_1 by the equality

$$\varphi(f) = \frac{g(s)}{g_0(s)}, \quad \text{where} \quad g = \xi(f).$$

Of course, $\varphi \in \Phi(E_1)$; let p be a point which corresponds to φ. We have $\varphi(e_{X_1}) = 1$ and it follows that $\varphi = \varphi_p$. We have

$$\varphi_p(\xi^{-1}(g)) = \frac{1}{g_0(s)}\cdot g(s) \quad \text{for each } g \text{ in } E_2,$$

whence, by (1) $s = h(p)$.

It remains to show that h and h^{-1} are continuous. Let V be any neighbourhood of a point $h(p) \in X_2$. Since E_2 distinguishes the points and closed sets of X_2, there is a function $g_1 \in E_2$ which is 1 at $h(p)$ and 0 on $X \setminus V$. Let $f_1 = \xi^{-1}(g_1)$. By (1), $f_1(p) > 0$, whence $U = \{q \in X_1 : f_1(q) > 0\}$ is a neighbourhood of p. Moreover, if $q \in U$, then, by (1), $0 < \varphi_q(f_1) = \varphi_q(\xi^{-1}(g_1)) = a_q\cdot g_1(h(q))$, whence $g_1(h(q)) > 0$, and thus $h(q) \in V$ and $h(U) \subset V$. Thus h is continuous, and in the same manner one can show that h^{-1} is also continuous. Finally, h is a homeomorphism of X_1 onto X_2 and the theorem follows.

§ 3.

In this paragraph we give a generalization of Theorems 1 and 2 to the case of linear spaces which satisfy a weaker condition than condition (δ), namely the following one:

(δ') *for each f in E and each $\varepsilon > 0$ there is a g in E with $\| |f| - g \| < \varepsilon$, where $\|f\|$ denotes, as usual, the number* $\sup\limits_{p \in X} |f(p)|$.

This generalization is based on the following lemma ($\tilde{E}$ denotes the set of all functions which are limits of uniformly convergent sequences of members of E, $f_n \rightrightarrows f$ means that the sequence f_n is uniformly convergent to f. Moreover, we say that $E \subset C^*(X)$ *separates* points and closed sets of X provided that for any $p_0 \notin A = \bar{A} \subset C$ there is an f in E such that $|f(p_0) - f(p)| \geqslant 1$ for $p \in A$):

LEMMA. *Suppose that $E \subset C^*(X)$ ($E_1 \subset C^*(X_1)$, $E_2 \subset C^*(X_2)$) is a linear space satisfying conditions (α) and (δ'). Then*

1° *$\tilde{E}$ is a linear space satisfying conditions (α), (β) and (δ);*

2° *if E separates points and closed sets of X, then $\tilde{E}$ distinguishes points and closed sets of X;*

3° *each pointwise continuous positive functional φ defined on E admits an extension to a pointwise continuous positive functional $\tilde{\varphi}$ defined on $\tilde{E}$;*

4° *each isomorphism ξ between E_1 and E_2 admits an extension to an isomorphism $\tilde{\xi}$ between $\tilde{E}_1$ and $\tilde{E}_2$.*

Proof. Part 1° is obvious.

Part 2°. Let $p_0 \notin A = \bar{A} \subset X$. There is an $f \in E$ such that $|f(p_0) - f(p)| \geqslant 1$ for $p \in A$. Setting $g(p) = f(p) - f(p_0)$, we have $g \in E$, $g(p_0) = 0$ and $|g(p)| \geqslant 1$ for $p \in A$. Since $\tilde{E}$ satisfies condition (δ), $h = \min\{1, |g|\} \in \tilde{E}$. But $h(p_0) = 0$ and $h(p) = 1$ for $p \in A$, whence $\tilde{E}$ distinguishes points and closed sets of X.

Part 3°. If $f_n \rightrightarrows f (f_n \in E)$, then, by the inequality $|\varphi(f_n) - \varphi(f_m)| \leqslant \varphi(e_X) \cdot \|f_n - f_m\|$, we infer that the sequence $\varphi(f_n)$ is convergent; let

$$\tilde{\varphi}(f) = \lim_n \varphi(f_n). \tag{1}$$

One can easily verify that formula (1) defines a linear functional $\tilde{\varphi}$ on $\tilde{E}$ which is an extension of φ. Of course, $\tilde{\varphi}$ is a positive functional. To prove that $\tilde{\varphi}$ is pointwise continuous, assume that $f_n \to 0$ $(f_n \in \tilde{E})$. Let g_n be a member of E with $\|f_n - g_n\| < 1/n$. Of course, $g_n \to 0$, whence $\tilde{\varphi}(g_n) = \varphi(g_n) \to 0$. But $|\tilde{\varphi}(f_n) - \tilde{\varphi}(g_n)| \leqslant \tilde{\varphi}(e_X) \cdot n^{-1}$ and it follows that $\tilde{\varphi}(f_n) \to 0$.

Part 4°. We have $|f| \leqslant e_{X_1} \cdot \|f\|$, whence $\xi(|f|) \leqslant \xi(e_{X_1}) \cdot \|f\|$; but $\xi(|f|) = |\xi(f)|$, whence $|\xi(f)| \leqslant \xi(e_{X_1}) \cdot \|f\|$, and thus

$$\|\xi(f)\| \leqslant \|\xi(e_{X_1})\| \cdot \|f\| \quad \text{for each } f \text{ in } E.$$

Analogously

$$\|\xi^{-1}(g)\| \leqslant \|\xi^{-1}(e_{X_2})\| \cdot \|g\| \quad \text{for each } g \text{ in } E_2.$$

It follows that a sequence f_n of members of E_1 is uniformly convergent if and only if the sequence $g_n = \xi(f_n)$ is uniformly convergent. Hence ξ can be extended to an isomorphism $\tilde{\xi}$ between $\tilde{E}_1$ and $\tilde{E}_2$.

From Theorems 1 and 2 we obtain by the foregoing lemma (we recall that a continuous image of a Lindelöf space is again a Lindelöf space and a Lindelöf space is Q-closed in any of its compactifications):

THEOREM 1'. *If $E \subset C^*(X)$ is a linear space satisfying conditions (α) and (δ') and X is a Lindelöf space, then each pointwise continuous positive functional φ on E can be written in the form* (**).

THEOREM 2'. *Suppose that $E_1 \subset C^*(X_1)$ and $E_2 \subset C^*(X_2)$ are linear spaces satisfying conditions (α) and (δ') and such that E_i separates points and closed sets of X_i, and X_i is a Lindelöf space ($i = 1, 2$). If the space E_1 and E_2 are isomorphic, then the spaces X_1 and X_2 are homeomorphic.*

Remark 2. In section I we have shown that if spaces $E_1 \to C^*(X_1)$, $E_2 \subset C^*(X_2)$ satisfy conditions (α), (β) and (δ) and ξ is a linear one-to-ons mappin g of E_1 onto E_2 which satisfies conditions (I_1) and (v), then ξ y an isomorphism. This is not true if the spaces E_1 and E_2 satisfy onle conditions (α) and (δ'); moreover, in this case the existence of such a mapping of E_1 onto E_2 does not imply the existence of a homeomorphism between X_1 and X_2 even if they are Lindelöf spaces. Let us consider the following example:

Let $X_1 = [0, 1]$ and $X_2 = [0, 1)$. Let $E_1 \subset C^*(X_1)$ be the space of all continuous functions f defined on X_1 for which there exists a positive number δ_f such that f is constant in the interval $[1 - \delta_f, 1]$, let $E_2 \subset C^*(X_2)$ consist of all functions of the form $f|X_2$ [13] where $f \in E_1$. Then E_1 and E_2 are linear spaces satisfying conditions (α) and (δ) and E_i distinguishes points and closed sets of X_i ($i = 1, 2$). Moreover, the mapping $\xi(f) = f|X_2$ is a linear one-to-one mapping of E_1 onto E_2 which satisfies conditions (I_1) and (v); nevertheless the spaces X_1 and X_2 are not homeomorphic.

Remark 3. Theorems 1 and 2 can be applied to spaces of differentiable functions. Indeed, if $\mathfrak{M}$ is a manifold of the class C^n ($n = 1, 2, \ldots, \infty$) (or an analytic manifold), then the space E of all bounded functions on $\mathfrak{M}$ being of the class C^n (or analytic functions) is a linear space which separates points and closed sets of $\mathfrak{M}$ and satisfies conditions (α) and (δ') (the last follows from the fact that, for each member f of E, $|f|$ can be uniformly approximated by means of polynomials with respect to f).

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[13] $f|X_2$ denotes the function f restricted to the set X_2.

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On quasi-modular spaces

by

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§ 1. Introduction. Let R be a *universally continuous semi-ordered linear space* (i. e. a lattice ordered linear space in which there exists $\bigcap_{\lambda\in\Lambda} a_\lambda$ for every system of positive elements $\{a_\lambda;\ \lambda\in\Lambda\}$ of R).

H. Nakano has considered a kind of functional on R which is called a *modular* (¹), and constructed the most important parts of the theory of modular spaces (i. e. spaces on which modulars are defined).

In this paper we shall consider a functional ϱ on R which satisfies the following conditions, weaker than those of modulars:

(ϱ.1) $0 \leqslant \varrho(x) = \varrho(-x) \leqslant +\infty$ for all $x\in R$;

(ϱ.2) $\varrho(x+y) = \varrho(x)+\varrho(y)$ for every $x, y\in R$ with $|x|\frown|y| = 0$;

(ϱ.3) for any system $\{x_\lambda;\ \lambda\in\Lambda\}$ such that $|x_\lambda|\frown|x_\gamma| = 0$ for $\lambda\neq\gamma$, $\lambda,\gamma\in\Lambda$ and $\sum_{\lambda\in\Lambda}\varrho(x_\lambda) < +\infty$, there exists $x_0\in R$ with $\sum_{\lambda\in\Lambda} x_\lambda = x_0$ and $\sum_{\lambda\in\Lambda}\varrho(x_\lambda) = \varrho(x_0)$;

(ϱ.4) $\overline{\lim_{\alpha\to 0}}\,\varrho(\alpha x) < +\infty$ for all $x\in R$.

R is called a *quasi-modular space* if the above ϱ is defined on R and ϱ is called a *quasi-modular*. This quasi-modular is considered as a generalization of a Nakano's monotone complete modular or of a concave modular [4 and 6].

Recently, J. Musielak and W. Orlicz considered the *pseudo-modular* on a linear space in [8]. If we add the order structure to linear spaces and additive conditions: (ϱ.2) and (ϱ.3) to those of a pseudo-modular, then a quasi-modular can be considered as a pseudo-modular in the case of semi-ordered linear spaces.

Some of the examples of a pseudo-modular established in [8] are regarded as those of a quasi-modular.

(¹) For the definition of a modular see § 2.