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Banach spaces of Lipschitz functions

by

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§ 1. Introduction. If $0<\alpha<1$, $\operatorname{Lip}\alpha$ is the space of all complex valued continuous functions on the real line R of period 1 with

$$\sup_{\sigma\in R}|f(\sigma+\tau)-f(\sigma)| = O(|\tau|^{\alpha}) \quad \text{as} \quad \tau\to 0.$$

$\operatorname{lip}\alpha$ is the subset of $\operatorname{Lip}\alpha$ consisting of those f with

$$\sup_{\sigma\in R}|f(\sigma+\tau)-f(\sigma)| = o(|\tau|^{\alpha}) \quad \text{as} \quad \tau\to 0.$$

Supplied with the norm $\|\cdot\|_{\alpha}$ defined by

$$\|f\|_{\alpha} = \sup_{\varrho,\sigma,\tau}\left\{|f(\varrho)|,\ \frac{|f(\sigma+\tau)-f(\sigma)|}{|\tau|^{\alpha}}\right\},$$

$\operatorname{Lip}\alpha$ is a Banach space and $\operatorname{lip}\alpha$ is a closed linear subspace (1).

We show in § 2 that the Banach space $\operatorname{Lip}\alpha$ is canonically isomorphic and isometric to the second dual space of the Banach space $\operatorname{lip}\alpha$. In § 3 we identify the extreme points of the unit sphere of the dual of $\operatorname{lip}\alpha$ and obtain as a consequence in § 4 the fact that $\operatorname{lip}\alpha$ has no isometries in addition to the expected ones.

§ 2. $\operatorname{Lip}\alpha$ is the second dual of $\operatorname{lip}\alpha$. Two definitions are necessary before we are able to state the main result of this section. For each σ in R, we define the functional Φ_{σ} in the dual space $(\operatorname{lip}\alpha)^*$ of $\operatorname{lip}\alpha$ by

$$\Phi_{\sigma}(f) = f(\sigma), \quad f\in\operatorname{lip}\alpha.$$

For each functional F in the dual space $(\operatorname{lip}\alpha)^{**}$ of $(\operatorname{lip}\alpha)^*$, we define the function $\hat{F}$ on R by

$$\hat{F}(\sigma) = F(\Phi_{\sigma}), \quad \sigma\in R.$$

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(1) In [3] it is shown that $\operatorname{lip}\alpha$ is the closed linear subspace of $\operatorname{Lip}\alpha$ spanned by trigonometric polynomials.

Note that if f is in $\operatorname{lip}\alpha$ and F_f is its image under the canonical imbedding of $\operatorname{lip}\alpha$ in $(\operatorname{lip}\alpha)^{**}$, the function $\hat{F}_f$ is simply f.

THEOREM 2.1. *The mapping $F \to \hat{F}$ is an isomorphism and isometry of $(\operatorname{lip}\alpha)^{**}$ onto $\operatorname{Lip}\alpha$.*

The proof proceeds by a sequence of lemmas. We shall denote by $\|\cdot\|_\alpha^*$ and $\|\cdot\|_\alpha^{**}$ the norms induced on $(\operatorname{lip}\alpha)^*$ and $(\operatorname{lip}\alpha)^{**}$ by the norm $\|\cdot\|_\alpha$ on $\operatorname{lip}\alpha$.

LEMMA 2.2. *If F is a functional in $(\operatorname{lip}\alpha)^{**}$, then the function $\hat{F}$ is in $\operatorname{Lip}\alpha$.*

Proof. If $\sigma\in R$, $\tau\in R$, and f in $\operatorname{lip}\alpha$ satisfies $\|f\|_\alpha \leqslant 1$, then

$$|\Phi_\sigma(f)-\Phi_\tau(f)| = |f(\sigma)-f(\tau)| \leqslant |\sigma-\tau|^\alpha .$$

Thus

$$\|\Phi_\sigma-\Phi_\tau\|_\alpha^* \leqslant |\sigma-\tau|^\alpha, \tag{2.1}$$

and as a consequence,

$$\begin{aligned}|\hat{F}(\sigma)-\hat{F}(\tau)| &= |F(\Phi_\sigma)-F(\Phi_\tau)| = |F(\Phi_\sigma-\Phi_\tau)|\\ &\leqslant \|F\|_\alpha^{**}\|\Phi_\sigma-\Phi_\tau\|_\alpha^* \leqslant \|F\|_\alpha^{**}|\sigma-\tau|^\alpha,\end{aligned}$$

and so $\hat{F}$ is in $\operatorname{Lip}\alpha$.

We next identify the continuous linear functionals of $\operatorname{lip}\alpha$ by constructing an isometric imbedding of $\operatorname{lip}\alpha$ into a space of continuous functions supplied with the sup norm.

Let W be the locally compact topological space $U\cup V$, where

$$U = \{\varrho: 0 \leqslant \varrho \leqslant 1)\}$$

and

$$V = \{(\sigma,\tau): 0\leqslant\sigma\leqslant 1,\ 0<\tau-\sigma\leqslant \tfrac{1}{2}\}.$$

We denote by $C_0(W)$ the Banach space of complex valued continuous functions on W that are zero at infinity, supplied with the norm $\|\cdot\|_W$ defined by

$$\|h\|_W = \sup_{x\in W}|h(x)| .$$

We denote the norm of the dual space $C_0(W)^*$ of $C_0(W)$ by $\|\cdot\|_W^*$. By the Riesz representation theorem, each element ψ of $C_0(W)^*$ is of the form

$$\psi(h) = \int_W h\,d\mu, \quad h\in C_0(W),$$

for a unique finite measure μ on W, and we define $\|\mu\|_W^*$ to be $\|\psi\|_W^*$.

For each function f in $\operatorname{lip}\alpha$, we denote by $\tilde{f}$ the function on W defined by

$$\tilde{f}(\varrho) = f(\varrho), \quad \varrho\in U,$$

$$\tilde{f}(\sigma,\tau) = \frac{f(\sigma)-f(\tau)}{|\sigma-\tau|^\alpha}, \quad (\sigma,\tau)\in V.$$

LEMMA 2.3. *The mapping $f\to\tilde{f}$ is a linear isometry of $\operatorname{lip}\alpha$, supplied with the norm $\|\cdot\|_\alpha$, into $C_0(W)$, supplied with the norm $\|\cdot\|_W$.*

Proof. It is clear that $f\to\tilde{f}$ is a linear mapping of $\operatorname{lip}\alpha$ into $C_0(W)$. If f is in $\operatorname{lip}\alpha$, f has period 1, so

$$\sup\{|f(\varrho)|: \varrho\in R\} = \sup\{|f(\varrho)|: \varrho\in U\}$$

and

$$\sup\left\{\frac{|f(\sigma)-f(\tau)|}{|\sigma-\tau|^\alpha}: \sigma,\tau\in R\right\} = \sup\left\{\frac{|f(\sigma)-f(\tau)|}{|\sigma-\tau|^\alpha}: (\sigma,\tau)\in V\right\},$$

and as a consequence, $\|f\|_\alpha = \|\tilde{f}\|_W$.

LEMMA 2.4. *Let Φ be a functional in $(\operatorname{lip}\alpha)^*$. Then there exists a measure μ on W with $\|\mu\|_W^* = \|\Phi\|_\alpha^*$ satisfying*

$$\Phi(f) = \int_U f(\varrho)\,d\mu(\varrho) + \int_V \frac{f(\sigma)-f(\tau)}{|\sigma-\tau|^\alpha}\,d\mu(\sigma,\tau) \tag{2.2}$$

for all f in $\operatorname{lip}\alpha$.

Proof. By Lemma 2.3, the linear functional ψ defined on the subspace

$$\{\tilde{f}: f\in\operatorname{lip}\alpha\}$$

of $C_0(W)$ by

$$\psi(\tilde{f}) = \Phi(f), \quad f\in\operatorname{lip}\alpha,$$

has its norm equal to $\|\Phi\|_\alpha^*$. ψ can be extended, by the Hahn-Banach theorem, to a linear functional of $C_0(W)$ having the same norm, and thus by the Riesz representation theorem there is a measure μ on W satisfying $\|\mu\|_W^* = \|\Phi\|_\alpha^*$ and

$$\Phi(f) = \int_W \tilde{f}\,d\mu \tag{2.3}$$

for all f in $\operatorname{lip}\alpha$. But (2.3) is simply another way of writing (2.2).

We shall denote by $(\operatorname{lip}\alpha)_m^*$ the subspace of $(\operatorname{lip}\alpha)^*$ consisting of all functionals Φ of the form

$$\Phi(f) = \int_U f\,d\lambda, \quad f\in\operatorname{lip}\alpha, \tag{2.4}$$

for λ a measure on U. The subset of $(\operatorname{lip}\alpha)^*_m$ consisting of all functionals of the form (2.4) for λ a measure concentrated at a finite number of points will be denoted by $(\operatorname{lip}\alpha)^*_p$. Equivalently, $(\operatorname{lip}\alpha)^*_p$ is the linear subspace of $(\operatorname{lip}\alpha)^*$ spanned by $\{\Phi_\sigma\colon \sigma\in R\}$.

LEMMA 2.5. *$(\operatorname{lip}\alpha)^*_m$ in dense in $(\operatorname{lip}\alpha)^*$ in its norm topology.*

Proof. Let Φ be a functional in $(\operatorname{lip}\alpha)^*$. By Lemma 2.4 there is a measure μ on W that satisfies

$$\Phi(f) = \int_W \tilde{f}\,d\mu, \quad f\in \operatorname{lip}\alpha.$$

Let

$$W_1\subset W_2\subset \ldots \subset W_n\subset \ldots$$

be a sequence of compact subsets of W whose union is W. For each positive integer n, we define the functional Φ_n in $(\operatorname{lip}\alpha)^*$ by

$$\Phi_n(f) = \int_{W_n} \tilde{f}\,d\mu, \quad f\in \operatorname{lip}\alpha.$$

Because of Lemma 2.3.

$$\lim_{n\to\infty}\|\Phi_n-\Phi\|^*_\alpha = 0,$$

so it only remains to show that each Φ_n is in $(\operatorname{lip}\alpha)^*_m$. But since

$$\Phi_n(f) = \int_{U\cap W_n} f(\varrho)\,d\mu(\varrho)+\int_{V\cap W_n}\frac{f(\sigma)-f(\tau)}{|\sigma-\tau|^\alpha}\,d\mu(\sigma,\tau)$$

for all f in $\operatorname{lip}\alpha$, and $|\sigma-\tau|^\alpha$ is bounded away zero on $V\cap W_n$, this is indeed the case.

LEMMA 2.6. *$(\operatorname{lip}\alpha)^*_p$ is dense in $(\operatorname{lip}\alpha)^*$ in its norm topology.*

Proof. Let λ be a measure on U and Φ the functional in $(\operatorname{lip}\alpha)^*$ defined by (2.4). By Lemma 2.5, it suffices to show that Φ is in the closure in $(\operatorname{lip}\alpha)^*$ of $(\operatorname{lip}\alpha)^*_p$. Let $C(U)$ be the space of complex valued continuous functions on U. Using the Riesz representation theorem, we identify the space of measures on U with the dual space $C(U)^*$ and denote by $\|\cdot\|^*$ the norm on this space of measures induced by the sup norm on $C(U)$. Choose any $\varepsilon>0$. We shall denote by S the unit sphere

$$\{f\colon f\in\operatorname{lip}\alpha,\ \|f\|_\alpha\leqslant 1\}$$

of $\operatorname{lip}\alpha$. S is collection of functions having period 1 on R that is bounded by 1 and equicontinuous. Thus by Ascoli's theorem, S is conditionally compact in the topology of uniform convergence, so there is a finite subset T of S such that each function in S is uniformly within $\varepsilon(4\|\lambda\|^*)^{-1}$ of some function in T. It is well known (see [1], p. 75) that the subset of the sphere

$$\{\eta\colon \eta\in C(U)^*,\ \|\eta\|^*\leqslant\|\lambda\|^*\}$$

consisting of measures concentrated at a finite number of points of U is dense in this sphere in the weak* topology of $C(U)^*$. Thus there is a measure η concentrated at a finite number of points of U that satisfies $\|\eta\|^*\leqslant\|\lambda\|^*$ and

$$\Big|\int_U f\,d\lambda-\int_U f\,d\eta\Big|<\varepsilon/2,\quad f\in T.$$

Because of the choice of T,

$$\Big|\int_U f\,d\lambda-\int_U f\,d\eta\Big|<\varepsilon,\quad f\in S,$$

and as a consequence, the functional ψ in $(\operatorname{lip}\alpha)^*$ defined by

$$\psi(f) = \int_U f\,d\eta,\quad f\in\operatorname{lip}\alpha,$$

satisfies $\|\psi-\Phi\|<\varepsilon$. Since ε was arbitrary and ψ is in $(\operatorname{lip}\alpha)^*_p$, we have shown that Φ is in the closure of $(\operatorname{lip}\alpha)^*_p$, and the proof is complete.

COROLLARY 2.7. *The mapping $F\to\hat{F}$ of $(\operatorname{lip}\alpha)^{**}$ into $\operatorname{Lip}\alpha$ is one-one.*

Proof. It is clear that the mapping is linear. If F in $(\operatorname{lip}\alpha)^{**}$ is in the kernel of the mapping, $\hat{F}$ is the zero function, so

$$F(\Phi_\sigma) = \hat{F}(\sigma) = 0,\quad \sigma\in R.$$

But by Lemma 2.6, linear combinations of the Φ_σ are dense in $(\operatorname{lip}\alpha)^*$ in its norm topology. Thus F must be the zero functional and the mapping is one-one as claimed.

LEMMA 2.8. *The mapping $F\to\hat{F}$ of $(\operatorname{lip}\alpha)^{**}$ into $\operatorname{Lip}\alpha$ is onto and norm preserving.*

Proof. Let h be a function in $\operatorname{Lip}\alpha$. We shall first construct a functional F in $(\operatorname{lip}\alpha)^{**}$ satisfying $\hat{F}=h$. For each positive integer n, the Féjer kernel K_n is defined by

$$K_n(\sigma) = \frac{2}{n+1}\left(\frac{\sin(n+1)\pi\sigma}{\sin\pi\sigma}\right)^2,\quad \sigma\in R.$$

(For the properties of the Féjer kernel that we shall use, see [4], Chap. 3). The convolution K_n*h is the n-th $(C,1)$ partial sum of the Fourier series of h. These $(C,1)$ sums converge uniformly to h, so

$$\lim_{n\to\infty}K_n*h(\sigma) = h(\sigma),\quad \sigma\in R. \tag{2.5}$$

Moreover, it is simple to check, using the fact that each K_n is positive and satisfies

$$\int_0^1 K_n(\sigma)\,d\sigma = 1,$$

that

(2.6) $$\|K_n * h\|_\alpha \leqslant \|h\|_\alpha .$$

$K_n * h$ is a trigonometric polynomial and thus in $\operatorname{lip}\alpha$. We shall denote by F_n the functional in $(\operatorname{lip}\alpha)^{**}$ corresponding to $K_n * h$ under the canonical imbedding of $\operatorname{lip}\alpha$ in $(\operatorname{lip}\alpha)^{**}$; i. e.

(2.7) $$F_n(\Phi) = \Phi(K_n * h), \quad \Phi \epsilon (\operatorname{lip}\alpha)^* .$$

Because of (2.6) and the fact that the imbedding of $\operatorname{lip}\alpha$ in $(\operatorname{lip}\alpha)^{**}$ is an isometry,

(2.8) $$\|F_n\|_\alpha^{**} \leqslant \|h\|_\alpha .$$

We define

$$F(\Phi) = \lim_{n\to\infty} F_n(\Phi)$$

for all Φ in $(\operatorname{lip}\alpha)^*$ for which the limit exists. By (2.5) and (2.7), $F(\Phi)$ exists for all Φ in $\{\Phi_\sigma : \sigma \epsilon R\}$, and thus by linearity exists for all Φ in $(\operatorname{lip}\alpha)_p^*$. But by Lemma 2.6, $(\operatorname{lip}\alpha)_p^*$ is dense in $(\operatorname{lip}\alpha)^*$ in its norm topology. As a consequence, because of (2.8), $F(\Phi)$ exists for all Φ in $(\operatorname{lip}\alpha)^*$ and F is a functional in $(\operatorname{lip}\alpha)^{**}$ satisfying

(2.9) $$\|F\|_\alpha^{**} \leqslant \|h\|_\alpha .$$

Furthermore, $\hat{F} = h$, since for each σ in R,

$$\hat{F}(\sigma) = F(\Phi_\sigma) = \lim_{n\to\infty} F_n(\Phi_\sigma) = \lim_{n\to\infty} K_n * h(\sigma) = h(\sigma).$$

By (2.9), $\|F\|_\alpha^{**} \leqslant \|\hat{F}\|_\alpha$, so to complete the proof of the lemma it remains only to demonstrate the reverse inequality. For each ϱ in R,

(2.10) $$|\hat{F}(\varrho)| = |F(\Phi_\varrho)| \leqslant \|F\|_\alpha^{**}\|\Phi_\varrho\|_\alpha^* \leqslant \|F\|_\alpha^{**} .$$

Furthermore, for each σ and τ in R,

(2.11) $$|\hat{F}(\sigma) - \hat{F}(\tau)| = |F(\Phi_\sigma - \Phi_\tau)| \leqslant \|F\|_\alpha^{**}\|\Phi_\sigma - \Phi_\tau\|_\alpha^* \leqslant \|F\|_\alpha^{**}|\sigma - \tau|^\alpha$$

by (2.1). (2.10) and (2.11) together show that $\|\hat{F}\|_\alpha \leqslant \|F\|_\alpha^{**}$ and the proof is complete.

Theorem 2.1 is now immediate consequence of Lemma 2.2, Corollary 2.7 and Lemma 2.8.

§ 3. Extreme points in $(\operatorname{lip}\alpha)^*$. Our aim in this section is the identification of the extreme points (²) of the unit sphere of the dual of $\operatorname{lip}\alpha$. Because of Lemma 2.3 it suffices to consider the corresponding problem for a linear space of continuous functions under the sup norm.

(²) φ is an extreme point of a convex set if it is not the mid-point of any segment lying in the set.

Let X be a locally compact topological space and $C_0(X)$ the space of complex valued continuous functions on X that are zero at infinity. Suppose that A is a closed linear subspace of $C_0(X)$. A is a Banach space under the sup norm and we shall denote its dual by A^*.

The following result is contained in Lemma V.8.6 of [2]:

LEMMA 3.1. *Each extreme point of the unit sphere of A^* is of the form*

$$\Phi(g) = \lambda g(x), \quad g \epsilon A,$$

for some x in X and some complex number λ with $|\lambda| = 1$.

One further definition is necessary before we are able to state a partial converse to Lemma 3.1. Let x be a point of X. A function h in A is said to *peak at x relative to A* if $h(x) = 1$ and

$$|h(y)| \leqslant 1, \quad y \epsilon X, y \neq x,$$

with equality holding only for those y in X that satisfy either

$$g(y) = g(x), \quad \text{all } g \epsilon A,$$

or

$$g(y) = -g(x), \quad \text{all } g \epsilon A.$$

LEMMA 3.2. *Let x be a point of X. Suppose that there is a function in A that peaks at x relative to A. Then the functional Φ in A^* defined by*

$$\Phi(g) = g(x), \quad g \epsilon A,$$

is an extreme point of the unit sphere of A^*.

Proof. It is clear that Φ is in the unit sphere of A^*. Suppose that $\Phi = \frac{1}{2}(\psi_1 + \psi_2)$, where ψ_1 and ψ_2 are also in the unit sphere. We must show that $\psi_1 = \psi_2 = \varphi$. By the Hahn-Banach theorem, the functionals ψ_1 and ψ_2 can be extended in a norm preserving manner to $C_0(X)$ and thus by the Riesz representation theorem there are measures μ_1 and μ_2 in the unit sphere of $C_0(X)^*$ satisfying

$$\psi_i(g) = \int_X g\, d\mu_i, \quad g \epsilon A, i = 1, 2.$$

Let h be a function in A that peaks at x relative to A. Since μ_1 and μ_2 are in the unit sphere of $C_0(X)^*$,

$$\Big|\int_X h\, d\mu_i\Big| \leqslant \sup_{y \epsilon X} |h(y)| = 1, \quad i = 1, 2.$$

Thus, because

$$1 = h(x) = \Phi(h) = \tfrac{1}{2}(\psi_1(h) + \psi_2(h)) = \tfrac{1}{2}\Big(\int_X h\, d\mu_1 + \int_X h\, d\mu_2\Big),$$

we must have

$$\int_X h\,d\mu_1 = \int_X h\,d\mu_2 = 1. \tag{3.1}$$

We define the subsets Y_+, Y_- and Y_0 of X by

$$Y_+ = \{y : h(y) = 1\} = \{y : g(y) = g(x),\ \text{all } g \in A\},$$

$$Y_- = \{y : h(y) = -1\} = \{y : g(y) = -g(x),\ \text{all } g \in A\},$$

$$Y_0 = \{y : |h(y)| < 1\} = \{y : y \in' Y_+, y \in' Y_-\}.$$

Since (3.1) holds and the μ_i are in the unit sphere of $C_0(X)^*$, we must have

$$\mu_i(Y_+) - \mu_i(Y_-) = 1, \qquad \mu_i(Y_0) = 0, \qquad i = 1, 2.$$

Thus for each g in A,

$$\psi_i(g) = \int_X g\,d\mu_i = \int_{Y_+} g\,d\mu_i + \int_{Y_-} g\,d\mu_i + \int_{Y_0} g\,d\mu_i$$

$$= g(x)\mu_i(Y_+) - g(x)\mu_i(Y_-) = g(x) = \Phi(g), \qquad i = 1, 2.$$

As a consequence, $\psi_1 = \psi_2 = \Phi$ and Φ is extreme as claimed.

THEOREM 3.3. *A functional Φ in $(\operatorname{lip}\alpha)^*$ is an extreme point of the unit sphere of $(\operatorname{lip}\alpha)^*$ if and only if it is either of the form*

$$\Phi(f) = \lambda f(\varrho), \qquad f \in \operatorname{lip}\alpha, \tag{3.2}$$

for ϱ in R and λ a complex number with $|\lambda| = 1$, or of the form

$$\Phi(f) = \lambda \frac{f(\sigma) - f(\tau)}{|\sigma - \tau|^\alpha}, \qquad f \in \operatorname{lip}\alpha, \tag{3.3}$$

for σ and τ in R, $0 < \tau - \sigma \leqslant \frac12$ and λ a complex number with $|\lambda| = 1$.

Proof. We shall use the notation established in § 2. The functionals Φ described in the statement of Theorem 3.3 are precisely those of the form

$$\Phi(f) = \lambda \tilde{f}(x), \qquad f \in \operatorname{lip}\alpha, \tag{3.4}$$

for x a point of W and λ a complex number with $|\lambda| = 1$. Lemmas 2.3 and 3.1 applied to $X = W$ and $A = \{\tilde{f} : f \in \operatorname{lip}\alpha\}$ show that each extreme point of the unit sphere of $(\operatorname{lip}\alpha)^*$ is indeed a functional of the form (3.4). To establish the converse, because of Lemma 3.2, it suffices to show that for each point x of W it is possible to find some function f in $\operatorname{lip}\alpha$ with $\tilde{f}$ peaking at x relative to A.

Case I. $x = \varrho$, $0 \leqslant \varrho \leqslant 1$. By the invariance of $\operatorname{lip}\alpha$ and $\|\cdot\|_\alpha$ under translation, we may assume that $0 < \varrho < 1$. Let f be any function in $\operatorname{lip}\alpha$ satisfying $f(\varrho) = 1$, $|f(\sigma)| < 1$ if $\sigma - \varrho$ is not an integer, and $|f(\sigma) - f(\tau)| \leqslant \frac12\ |\sigma - \tau|^\alpha$ for $\sigma, \tau \in R$. Then $\tilde{f}(x) = 1$ and $|\tilde{f}(y)| < 1$ if $y \in W$ and $y \neq x$, so $\tilde{f}$ peaks at x relative to A.

Case II. $x = (\sigma, \tau)$, $0 \leqslant \sigma \leqslant 1$, $0 < \tau - \sigma < \frac12$. By the invariance of $\operatorname{lip}\alpha$ and $\|\cdot\|_\alpha$ under translation, we may assume that $\sigma = 0$. Let f be the function in $\operatorname{lip}\alpha$ that satisfies $f(0) = 0$, $f(\tau) = -\tau^\alpha$, $f(1) = 0$, and is linear in the intervals $[0, \tau]$ and $[\tau, 1]$. Let x' be the point $(1, 1+\tau)$ of W. Then $\tilde{f}(x) = \tilde{f}(x') = 1$, $|\tilde{f}(y)| < 1$ if $y \in W$, $y \neq x$, $y \neq x'$, and $\tilde{g}(x) = \tilde{g}(x')$ for all $g \in \operatorname{lip}\alpha$, so $\tilde{f}$ peaks at x relative to A.

Case III. $x = (\sigma, \tau)$, $0 \leqslant \sigma \leqslant 1$, $\tau - \sigma = \frac12$. By the invariance of $\operatorname{lip}\alpha$ and $\|\cdot\|_\alpha$ under translation, we may assume that $(\sigma, \tau) = (\frac14, \frac34)$. Let f be the function in $\operatorname{lip}\alpha$ that satisfies $f(\frac14) = 0$, $f(\frac34) = -(\frac12)^\alpha$, $f(\frac54) = 0$, and is linear in the intervals $[\frac14, \frac34]$ and $[\frac34, \frac54]$. Let x' be the point $(\frac34, \frac54)$ of W. Then $\tilde{f}(x) = 1$, $\tilde{f}(x') = -1$, $|\tilde{f}(y)| < 1$ if $y \in W$, $y \neq x$, $y \neq x'$, and $\tilde{g}(x) = -\tilde{g}(x')$ for all $g \in \operatorname{lip}\alpha$, so $\tilde{f}$ peaks at x relative to A.

This completes the proof of Theorem 3.3.

§ 4. The isometries of $\operatorname{lip}\alpha$. The ϱ be a real number and λ a complex number with $|\lambda| = 1$. It is clear that the linear mappings U and V of $\operatorname{lip}\alpha$ onto itself defined by

$$Uf(\sigma) = \lambda f(\varrho + \sigma), \qquad \sigma \in R,$$

and

$$Vf(\sigma) = \lambda f(\varrho - \sigma), \qquad \sigma \in R,$$

satisfy

$$\|Uf\|_\alpha = \|f\|_\alpha, \qquad f \in \operatorname{lip}\alpha,$$

and

$$\|Vf\|_\alpha = \|f\|_\alpha, \qquad f \in \operatorname{lip}\alpha.$$

In this section [3] we establish the following result, which shows that $\operatorname{lip}\alpha$ has no further isometries:

THEOREM 4.1. *Let T be a linear isometry of $\operatorname{lip}\alpha$ onto itself. Then there is a real number ϱ and a complex number λ with $|\lambda| = 1$ so that either*

$$Tf(\sigma) = \lambda f(\varrho + \sigma), \qquad \sigma \in R,$$

for all f in $\operatorname{lip}\alpha$, or

$$Tf(\sigma) = \lambda f(\varrho - \sigma), \qquad \sigma \in R,$$

for all f in $\operatorname{lip}\alpha$.

The remainder of the section is devoted to the proof of this theorem.

(3) This work was supported in part by the Society for the Preservation of the Norm.

We shall denote by ext S^* the set of extreme points of the unit sphere of $(\operatorname{lip}\alpha)^*$. Since T is a linear isometry of $\operatorname{lip}\alpha$ onto $\operatorname{lip}\alpha$, its adjoint T^* is a linear isometry of $(\operatorname{lip}\alpha)^*$ onto $(\operatorname{lip}\alpha)^*$ and satisfies

$$T^*(\operatorname{ext} S^*) = \operatorname{ext} S^*. \tag{4.1}$$

LEMMA 4.2. *Let f be a function in $\operatorname{lip}\alpha$. Then f is a constant function if and only if*

$$\{|\Phi(f)|: \Phi \in \operatorname{ext} S^*\} \tag{4.2}$$

consists of at most two numbers.

Proof. If f is constant, that (4.2) has at most two elements is clear from Theorem 3.3. For the converse, suppose that (4.2) consists of at most two numbers. Since f is in $\operatorname{lip}\alpha$, 0 is in the closure of

$$\left\{\frac{f(\sigma)-f(\tau)}{|\sigma-\tau|^\alpha}: \sigma, \tau \in R, \sigma \neq \tau\right\},$$

and thus by Theorem 3.3, 0 must be in (4.2). If there is no other element in (4.2), by Theorem 3.3 f must be the zero function and we are finished. So we may assume that (4.2) is $\{0, \varrho\}$ where $\varrho > 0$. Since $f \in \operatorname{lip}\alpha$, there exists an $\varepsilon > 0$ so that

$$\frac{|f(\sigma)-f(\tau)|}{|\sigma-\tau|^\alpha} < \varrho$$

if $|\sigma-\tau| < \varepsilon$. But since (4.2) is $\{0, \varrho\}$, because of Theorem 3.3, each number

$$\frac{|f(\sigma)-f(\tau)|}{|\sigma-\tau|^\alpha}$$

is equal to either 0 or ϱ. Thus $f(\sigma) = f(\tau)$ if $|\sigma-\tau| < \varepsilon$ and f is constant.

Recall that for $\sigma \in R$, Φ_σ is the functional in $(\operatorname{lip}\alpha)^*$ defined by

$$\Phi_\sigma(f) = f(\sigma), \quad f \in \operatorname{lip}\alpha.$$

COROLLARY 4.3. *There is a complex number λ with $|\lambda| = 1$ so that*

$$T^*\{\Phi_\sigma: \sigma \in R\} = \{\lambda\Phi_\sigma: \sigma \in R\}. \tag{4.3}$$

Proof. Let g be the function in $\operatorname{lip}\alpha$ satisfying

$$g(\sigma) = 1, \quad \sigma \in R.$$

By (4.1) and Lemma 4.2, Tg is also a constant function. Suppose that

$$Tg(\sigma) = \lambda, \quad \sigma \in R.$$

Then, because of Theorem 3.3 and (4.1),

$$\begin{aligned} T^*\{\Phi_\sigma: \sigma \in R\} &= T^*\{\Phi: \Phi \in \operatorname{ext} S^*, \Phi(Tg) = \lambda\} \\ &= \{T^*\Phi: \Phi \in \operatorname{ext} S^*, \Phi(Tg) = \lambda\} = \{T^*\Phi: T^*\Phi \in \operatorname{ext} S^*, T^*\Phi(g) = \lambda\} \\ &= \{\psi: \psi \in \operatorname{ext} S^*, \psi(g) = \lambda\} = \{\lambda\Phi_\sigma: \sigma \in R\}, \end{aligned}$$

so (4.3) holds. Finally, $|\lambda| = 1$ since T is an isometry.

LEMMA 4.4. *If $\sigma, \tau \in R$ and $|\sigma-\tau| \leqslant \frac{1}{2}$, then $\|\Phi_\sigma - \Phi_\tau\|_\alpha^* = |\sigma-\tau|^\alpha$.*

Proof. By (2.1), $\|\Phi_\sigma-\Phi_\tau\|_\alpha^* \leqslant |\sigma-\tau|^\alpha$, so it suffices to establish the reverse inequality. Assume first that $|\sigma-\tau| < \frac{1}{2}$. By the invariance of $\operatorname{lip}\alpha$ and $\|\cdot\|_\alpha$ under translation, we may assume that $\sigma = 0$ and $0 < \tau < \frac{1}{2}$. If f is the function constructed in Case II of Theorem 3.3, $\|f\|_\alpha = 1$ and $|\Phi_\sigma(f)-\Phi_\tau(f)| = |\sigma-\tau|^\alpha$. As a consequence,

$$\|\Phi_\sigma-\Phi_\tau\|_\alpha^* \geqslant |\sigma-\tau|^\alpha \tag{4.4}$$

when $|\sigma-\tau| < \frac{1}{2}$. A similar argument using the function constructed in Case III of Theorem 3.3 establishes the inequality (4.4) for $|\sigma-\tau| = \frac{1}{2}$.

One further lemma is required before we are able to complete the proof of Theorem 4.1. Let λ be the complex number with $|\lambda| = 1$ satisfying (4.3). Then one can find a real number so that $T^*\Phi_0 = \lambda\Phi_\varrho$. Let $\sigma \in R$ satisfy $|\sigma| < \frac{1}{2}$. By the choice of λ, there is some $\tau \in R$ with

$$T^*\Phi_\sigma = \lambda\Phi_\tau, \tag{4.5}$$

and thus a unique $\tau \in R$ satisfying (4.5) and in addition $\varrho - \frac{1}{2} < \tau \leqslant \varrho + \frac{1}{2}$. This unique τ will be denoted by $t(\sigma)$. We have thus defined a mapping

$$t: \{\sigma: -\tfrac{1}{2} < \sigma < +\tfrac{1}{2}\} \to R.$$

LEMMA 4.5. *The mapping t satisfies either*

$$t(\sigma) = \varrho+\sigma, \quad -\tfrac{1}{2} < \sigma < +\tfrac{1}{2}, \tag{4.6}$$

or

$$t(\sigma) = \varrho-\sigma, \quad -\tfrac{1}{2} < \sigma < +\tfrac{1}{2}. \tag{4.7}$$

Proof. Let σ satisfy $|\sigma| < \frac{1}{2}$. Then $|t(\sigma)-\varrho| \leqslant \frac{1}{2}$, so by Lemma 4.4,

$$\begin{aligned} |t(\sigma)-\varrho|^\alpha = \|\Phi_{t(\sigma)}-\Phi_\varrho\|_\alpha^* &= \|\lambda\Phi_{t(\sigma)} - \lambda\Phi_\varrho\|_\alpha^* \\ &= \|T^*(\Phi_\sigma-\Phi_0)\|_\alpha^* = \|\Phi_\sigma-\Phi_0\|_\alpha^* = |\sigma|^\alpha. \end{aligned}$$

Thus

$$|t(\sigma)-\varrho| = |\sigma|, \quad -\tfrac{1}{2} < \sigma < +\tfrac{1}{2}. \tag{4.8}$$

Furthermore, the mapping t is continuous. For if g is the function in $\operatorname{lip}\alpha$ defined by

$$g(\tau) = e^{2\pi i\tau}, \quad \tau \in R,$$

then Tg is continuous and

$$e^{2\pi i t(\sigma)} = g(t(\sigma)) = \Phi_{t(\sigma)}(g)$$
$$= \lambda^{-1}(T^*\Phi_\sigma)(g) = \lambda^{-1}\Phi_\sigma(Tg) = \lambda^{-1}Tg(\sigma), \quad -\tfrac{1}{2} < \sigma < +\tfrac{1}{2}.$$

It is now clear that t must satisfy either (4.6) or (4.7) since it is one-one continuous and satisfies (4.8).

We are now able to complete the proof of Theorem 4.1. Suppose that the mapping t satisfies (4.6). Then if f is any function in $\operatorname{lip}\alpha$,

$$Tf(\sigma) = \Phi_\sigma(Tf) = (T^*\Phi_\sigma)(f)$$
$$= \lambda\Phi_{t(\sigma)}(f) = \lambda f(t(\sigma)) = \lambda f(\varrho+\sigma), \quad -\tfrac{1}{2} < \sigma < +\tfrac{1}{2},$$

and as a consequence,

$$Tf(\sigma) = \lambda f(\varrho+\sigma), \quad \sigma \in R,$$

for all f in $\operatorname{lip}\alpha$.

Similarly, if the mapping t satisfies (4.7), then

$$Tf(\sigma) = \lambda f(\varrho-\sigma), \quad \sigma \in R,$$

for all f in $\operatorname{lip}\alpha$.

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A remark on an imbedding theorem of Kondrashev type

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1. The present note may be considered as the second part of Paper [1]. An approach developed there in order to obtain an elementary proof of complete continuity of the imbedding of the space $W_m^p(\Omega)$ in $C(\Omega)$ for m large enough (see the definition below) is applied here to study the similar property of the imbedding of $W_m^p(\Omega)$ into the space of functions integrable to the power p over a sufficiently smooth variety contained in Ω, and of a dimension smaller than that of Ω. An elementary proof of the Kondrashev theorem is obtained under conditions imposed on the variety under consideration, which differ from the original ones as presented in [4]. To prove the continuity of the imbedding mentioned, it is natural to impose the geometric conditions I invented by Ehrling; for its complete continuity, the more stringent conditions II seem to be necessary.

Several papers have been published recently in connection with simplifications of imbedding theorems (cf. for references [2]).

In what follows Ω will denote a fixed bounded domain in N-dimensional Euclidean space with points $x, y, \ldots$ and corresponding volume elements $dx, dy, \ldots$; $C(\Omega)$ will denote the space of functions continuous on Ω, $C^\infty(\Omega)$ the space of functions with continuous derivatives of all orders on Ω. In $C^\infty(\Omega)$ we introduce the norm

$$\|f\|_m = \Big(\sum_a \int_\Omega |D_a f|^p\,dx\Big)^{1/p}, \quad p > 1,$$

where the summation is extended over all derivatives of f of order not larger than

$$m\left(D_a f = \frac{\partial^a f}{\partial x_1^{a_1}\ldots\partial x_N^{a_N}}, \quad |a| = a_1+\ldots+a_N\right).$$

By completion of $C^\infty(\Omega)$ in the norm $\|\ \|_m$ we obtain a Banach space $W_m^p(\Omega)$ of all functions of $L^p(\Omega)$ whose generalised derivatives up to order m all belong to $L^p(\Omega)$. In the occurrence of other norms, we shall indicate