Wir erhalten nun endlich

$$\max_{\omega-\omega_0\leqslant\omega_1\leqslant\omega}\left|e^{\omega_1/2}H(e^{\omega_1/2})\right| \geqslant \sqrt{T}\, e^{-3\frac{\log T\log\log\log\log T}{\log\log T}}$$

und daraus wegen (4.2), (3.2), (3.3) und der Definition von $H(\beta)$

$$\max_{Te^{-\varphi(T)}\leqslant\beta\leqslant T}\left|\beta\sum_{1}^{\infty}\frac{\mu(n)}{n}e^{-(\beta/2)^2}\right| \geqslant \sqrt{T}\, e^{-3\frac{\log T\log\log\log\log T}{\log\log T}}$$

für

$$T > \max(c_{25}, e^{\sqrt{a}}), \qquad \varphi(T) = \frac{\log T\log\log\log T}{\log\log T}.$$

Daraus bekommt man einfach (1.6) und damit ist unser Satz bewiesen.

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Arithmetical notes, IX. On the set of integers representable as a product of a prime and a square

by

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1. Introduction. In this note we obtain an asymptotic determination of the number $A(x)$ of positive integers n not exceeding x which are expressible as the product of a prime by a square. In particular, we show that

$$A(x) \sim \frac{\pi^2}{6}\cdot\frac{x}{\log x} \quad \text{as} \quad x\to\infty. \tag{1.1}$$

Actually, a more precise approximation to $A(x)$ is deduced (Theorem 2.1.) The proof is based upon the prime number theorem and a simple factorization principle (Remark, § 2). Using the theorem of § 2 on $A(x)$, a similar result is derived for a related problem in § 3.

2. The main result. Let A denote the set of all n which can be represented as a prime multiplied by a square. Since every n is uniquely expressible as a product of a square-free number and a square, we have immediately the

Remark. If n is in A, then n has a factorization $n = pQ$, p prime, Q square, and this representation is unique.

In addition to this remark we shall use the prime number theorem in the form

$$\pi(x) = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right), \tag{2.1}$$

where $\pi(x)$ denotes, as usual, the number of primes $\leqslant x$ (Landau [2], § 54).

Theorem 2.1. *If $x \geqslant 2$, then*

$$A(x) = \frac{\pi^2}{6}\cdot\frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right). \tag{2.2}$$

Proof. In the proof, variables of summation are all assumed positive, and the variable p is restricted to prime values. By the remark above, we may write

$$A(x) = \sum_{\substack{n\leqslant x\\ n\in A}} 1 = \sum_{pe^2\leqslant x} 1, \tag{2.3}$$

so that

$$A(x) = \sum_{\substack{pe^2\leqslant x\\ e\leqslant \sqrt[4]{x}}} 1 + \sum_{\substack{pe^2\leqslant x\\ e>\sqrt[4]{x}}} 1 = A_1 + A_2, \tag{2.4}$$

let us say. Evidently

$$A_1 = \sum_{e\leqslant\sqrt[4]{x}} \sum_{p\leqslant xe^{-2}} 1 = \sum_{n\leqslant\sqrt[4]{x}} \pi(x/n^2).$$

Application of (2.1) yields

$$A_1 = A_{11} + A_{12}, \qquad A_{11} = x\sum_{n\leqslant\sqrt[4]{x}} \frac{1}{n^2\log(x/n^2)}, \tag{2.5}$$

$$A_{12} = O\left(x\sum_{n\leqslant\sqrt[4]{x}} \frac{1}{n^2\big(\log(x/n^2)\big)}\right). \tag{2.6}$$

By (2.5) one may write, after a simple calculation,

$$A_{11} = \frac{x}{\log x}\sum_{n\leqslant\sqrt[4]{x}} \frac{1}{n^2}\left(1-\left(\frac{\log n^2}{\log x}\right)\right)^{-1}.$$

If we place $r = \log n^2/\log x$ for the moment, it is plain, for the values of n in the latter summation that $0\leqslant r\leqslant 1/2$ and hence that $1\leqslant 1/(1-r)\leqslant 2$. By the geometric series expansion, one obtains therefore

$$\begin{aligned}
A_{11} &= \frac{x}{\log x}\sum_{n\leqslant\sqrt[4]{x}} \frac{1}{n^2}\left(1+O\left(\frac{\log n^2}{\log x}\right)\right)\\
&= \frac{x}{\log x}\sum_{n\leqslant\sqrt[4]{x}} \frac{1}{n^2} + O\left(\frac{x}{\log^2 x}\sum_{n\leqslant\sqrt[4]{x}} \frac{\log n}{n^2}\right)\\
&= \frac{x}{\log x}\left\{\sum_{n=1}^{\infty} \frac{1}{n^2} + O\left(\sum_{n>\sqrt[4]{x}} \frac{1}{n^2}\right)\right\} + O\left(\frac{x}{\log^2 x}\right)\\
&= \frac{x}{\log x}\left\{\zeta(2) + O\left(\int_{\sqrt[4]{x}}^{\infty} \frac{ds}{s^2}\right)\right\} + O\left(\frac{x}{\log^2 x}\right),
\end{aligned}$$

where $\zeta(s)$ denotes the Riemann zeta-function. Since the integral is $O(x^{-1/4})$, one obtains

$$A_{11} = \frac{\pi^2}{6}\cdot\frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right). \tag{2.7}$$

As for A_{12}, it follows by (2.6) that

$$A_{12} = O\left(\frac{x}{(\log\sqrt{x})^2}\sum_{n\leqslant\sqrt[4]{x}} \frac{1}{n^2}\right) = O\left(\frac{x}{\log^2 x}\right). \tag{2.8}$$

Returning to A_2, it follows from (2.4) that p cannot exceed $\sqrt{x}$ in the A_2-summation. Therefore

$$A_2 = \sum_{\substack{pe^2\leqslant x\\ e>\sqrt[4]{x},\, p\leqslant\sqrt{x}}} 1 \leqslant \sum_{\substack{pe^2\leqslant x\\ p\leqslant\sqrt{x}}} 1 \leqslant \sum_{\substack{ne^2\leqslant x\\ n\leqslant\sqrt{x}}} 1 = \sum_{n\leqslant\sqrt{x}} [\sqrt{x/n}] \leqslant \sqrt{x}\sum_{n\leqslant\sqrt{x}} n^{-1/2}.$$

Since $A_2\geqslant 0$, it results that

$$A_2 = O\left(\sqrt{x}\int_1^{\sqrt{x}} s^{-1/2}\,ds\right) = O(x^{3/4}) = O\left(\frac{x}{\log^2 x}\right). \tag{2.9}$$

The theorem is now a consequence of (2.4), (2.5), (2.7), (2.8) and (2.9).

3. A related result. Let A^* denote the set of integers n which admit of a representation as a product of a prime p and a square, such that p is a unitary divisor of n; in other words, the integers n of the form $n = pe^2$, p prime $p\nmid e$. Also let $A^*(x)$ denote the enumerative function of A^*, that is, the number of $n\leqslant x$ contained in A^*. We prove the following "unitary" analogue of Theorem 2.1.

THEOREM 3.1. *If $x\geqslant 2$, then*

$$A^*(x) = \frac{\pi^2}{6}\cdot\frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right). \tag{3.1}$$

Proof. Let n have distinct prime divisors $p_1, \ldots, p_t$, and place

$$n = p_1^{e_1}\ldots p_t^{e_t}. \tag{3.2}$$

The set A consists of the n in (3.2) with exactly one odd exponent. The set A^* consists of those n with all exponents e_i even except for a single exponent equal to 1. We note that $A^*\subset A$ and that the complement, $A-A^*$, of A^* in A, is contained in the set E of the n in (3.2) with all exponents $e_i\geqslant 2$. Denoting by $E(x)$ the enumerative function of E, it therefore follows that

$$0\leqslant A(x) - A^*(x)\leqslant E(x).$$

But it is a well-known elementary fact that $E(x) = O(\sqrt{x})$, (see [1] and the bibliography listed there). Hence $A^*(x) = A(x) + O(E(x)) = A(x) + O(\sqrt{x})$, and (3.1) follows on the basis of Theorem 2.1.

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