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Existence of Lyapunov functions

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1. Introduction. We consider here the stability of the solutions of a non-autonomous system of differential equations $\dot{x} = X(x, t)$, where (1) X is continuous on $H\times J_0, H \subset E^n$ being a connected open set. There is no loss of generality to assume that the solution whose stability is being considered is $x = 0$, so that $X(0, t) \equiv 0$. We moreover assume throughout this article: for each $(x_0, t_0) \in H\times J_0$ there exists a unique solution $x = x(t; x_0, t_0)$ in H which depends continuously on (x_0, t_0), equals x_0 for $t = t_0$, and is defined for all $t \geqslant 0$; thus, we can take $H = E^n$ with no loss in generality.

The definition of uniform stability of $x = 0$ can be stated in the following (normalized (2)) way [1]: For each cylinder $C_n \equiv S(0, 1/n)\times \times \mathring{J}_{1-1/n}$, $n = 1, 2, \ldots$, there exists a C_m, $m \geqslant n$, such that every trajectory entering C_m remains thereafter in C_n; this type of stability is characterized by special properties of a Lyapunov function on $E^n\times J_0$, that is, a non-negative continuous real-valued function on $E^n\times J_0$ vanishing on $0\times J_1$, bounded positively below outside each C_k, and having a continuous non-positive trajectory derivative (3) on $E^n\times J_0$. Now, if instead of cylinders, one is given a sequence $\{U_n\}$ of connected open sets in $E^n\times J_0$ with $0\times J_1 = \bigcap_n U_n$, then, replacing C_n by U_n in each of the above state-

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(1) Throughout this article, E^n denotes Euclidean n-space, and $J_a \subset E^1$ the subspace $\{t \mid t \geqslant a\}$. Vector notation is used. $S(x, \varepsilon)$ is the spherical neighborhood (nbd) of x with radius ε; $\dot{A}$ = boundary of A; $\mathring{A}$ = interior of A; $\complement A$ = complement of A; $(x \mathrel{\bar{\in}} A) \equiv (x \in \complement A)$.

(2) The normalization consists in having the cylinders pinch down on $0\times J_1$ rather than on some other $0\times J_a$.

(3) For each $(x_0, t_0) \in E^n\times J_0$, the trajectory derivative $V'(x_0, t_0)$ is the derivative $\frac{d}{dt}V[x(t; x_0, t_0), t]$ evaluated at $t = t_0$. We call a Lyapunov function *proper* (or simply "Lyapunov function") if $V'(x_0, t_0)$ is continuous on $E^n\times J_0$; it is called "split" if $V'(x_0, t_0)$ is continuous only on $\complement(0\times J_0)$.

ments, one is led to the more general notions of uniform $\{U_n\}$-stability, and $\{U_n\}$-Lyapunov function. It is evident that $x = 0$ can be uniformly $\{U_n\}$-stable—in fact, there is always at least one such choice—without being uniformly stable, i.e., uniformly $\{C_n\}$-stable. The question arises: For a given sequence $\{U_n\}$ of connected open sets expressing $0 \times J_1$ as a G_δ, does there exist a $\{U_n\}$-Lyapunov function and, if so, to what extent does it characterize the uniform $\{U_n\}$-stability of $x = 0$?

In part I, a condition which is necessary and sufficient for the existence of a $\{U_n\}$-Lyapunov function is obtained (Theorem 6.2), which clarifies the relation between the sets $\{U_n\}$ and the geometry of the trajectories. §§ 2-4 are primarily terminological.

In part II, a necessary and sufficient condition for the uniform $\{U_n\}$-stability of $x = 0$ is obtained; for $U_n \equiv C_n$ this gives a slight generalization of Kurzweil's theorem [4], in that the continuity of the trajectory derivative only on $\mathcal{C}(0 \times J_0)$, and not on the entire $E^n \times J_0$ appears to be relevant. An application to ordinary stability is also given.

The approach is based on the technique of rectifying the flow associated with the given differential equation, which has been presented in a joint paper with H. Antosiewicz.

Part I

2. Rectifying homeomorphisms. Because of the standing hypotheses, the solutions of $\dot{x} = X(x, t)$ can be regarded geometrically as the trajectories of a flow F on E^{n+1} associated with the differential equation. In fact, defining f: $E^{n+1} \times J_0 \to E^{n+1}$ by $f[(x_0, t_0), \tau] = [x(\tau + t_0; x_0, t_0), \tau + t_0]$ one verifies that f is continuous and that for each $p = (x_0, t_0) \in E^{n+1}$ one has $f(p, 0) = p$ and $f[f(p, t), \tau] = f(p, t+\tau)$; f determines the associated flow F. F can be transformed to a standard form, wherein each trajectory corresponds to a straight line parallel to the t-axis intersecting $E^n \times 0$ at the same point that the corresponding trajectory does:

2.1. Theorem. *There exists a homeomorphism h of $E^n \times J_0$ onto itself which keeps $E^n \times 0$ pointwise fixed and such that*

$$h[x(t; x_0, t_0), t] = [x(0; x_0, t_0), t] .$$

Proof. $h(x, t) = [x(0; x, t), t]$ is the desired map, since it is continuous (because of the standing assumptions) and has the continuous map $g(\xi, \tau) = [x(\tau; \xi, 0), \tau]$ as inverse.

h is called a *rectifying homeomorphism*, and plays a key role in this discussion (see § 4).

3. Positive definite functions.

3.1. Definition. Let Y be an arbitrary space, and $M \subset Y$ a closed G_δ with a given representation [4] $M = \bigcap_n U_n$, $\overline{U}_{n+1} \subset U_n$. A continuous real-valued function f: $Y \to J_0$ is called *positive definite rel* $\{U_n\}$ if (1) $f^{-1}(0) = M$ and (2) for each U_n there is a real $\mu(U_n) > 0$ such that $f(y) \geqslant \mu(U_n)$ for $y \,\bar{\in}\, U_n$.

In general M has many G_δ representations and, unless M is compact, a given f can be positive definite rel one representation but not rel another.

Note that if f: $Y \to J_0$ satisfies $f^{-1}(0) = M$ and if for each $\lambda > 0$ one defines $Q(\lambda) = \{x \mid f(x) < \lambda\}$, then for any sequence $\lambda_n \to 0$ the open sets $\{Q(\lambda_n)\}$ satisfy (1) $\bigcap_n Q(\lambda_n) = M$, (2) if $\lambda < \lambda'$ then $\overline{Q(\lambda)} \subset Q(\lambda')$ and (3) f is positive definite rel $\{Q(\lambda_n)\}$. This observation leads to a simple criterion for positive definiteness. One first makes

3.2. Definition. Let $M = \bigcap_n U_n = \bigcap_n W_n$ be two G_δ representations of M. $\{U_n\}$ *cushions* $\{W_n\}$ if for each W_i there exists a $U_{n_i} \subset W_i$. If $\{U_n\}$, $\{W_n\}$ cushion each other, the representations are called *interlacing*.

and then one has the simple

3.3. Theorem. *Let f: $Y \to J_0$ satisfy $f^{-1}(0) = M$. Then f is positive definite rel $\{U_n\}$ if and only if $\{Q(\lambda_n)\}$ cushions $\{U_n\}$ for some one (hence all) sequence $\lambda_n \to 0$.*

Proof. Assume f positive definite rel $\{U_n\}$. For each U_n let $\lambda_n = \frac{1}{2}\mu(U_n)$; then $Q(\lambda_n) \subset U_n$ since if $y \,\bar{\in}\, U_n$ then $f(y) \geqslant \mu(U_n) > \lambda_n$ so $y \,\bar{\in}\, Q(\lambda_n)$ either. For the converse, given U_n select $Q(\lambda_{i_n}) \subset U_n$ and define $\mu(U_n) = \lambda_{i_n}$.

4. L-functions. Certain functions on $E^n \times J_0$ play an important role in the approach given here.

4.1. Definition. Let $0 \times J_1 = \bigcap_n U_n$ be a G_δ representation. A continuous real-valued function $v(x, t)$ on $E^n \times J_0$ with the properties (1) v is positive definite rel $\{U_n\}$, (2) $v(x, t) \leqslant 1$ on $E^n \times J_0$, and (3) $v_t(x, t) \leqslant 0$ and is continuous on $\mathcal{C}(0 \times J_0)$, will be called a "split" L-*function rel* $\{U_n\}$; if v_t is continuous on $E^n \times J_1$ it is termed "proper" [2].

The relation between L-functions and Lyapunov functions for flows is the following: Recall [1] that a Lyapunov function for $0 \times J_1$ on a flow F in $E^n \times J_0$ is a continuous real-valued function V on $E^n \times J_0$ which is positive definite rel $\{S(0, 1/n) \times \mathring{J}_{1-1/n}\}$ and has a continuous non-positive

[4] A sequence of open sets $\{U_n\}$, with $\overline{U}_{i+1} \subset U_i$ for each index i, will be called *descending*. We take all G_δ representations to be descending; since the applications are all to metric spaces Y, this involves no loss in generality.

trajectory derivative (3). Under the rectifying homeomorphism h of § 2, the cylinders $\{S(0, 1/n)\times \mathring{J}_{1-1/n}\}$ transform to open sets $\{U_n\}$ giving a G_δ representation of $0\times J_1$.

Assume now that v is an L-function rel $\{U_n\}$, and define $V(x, t) \equiv vh(x, t) \equiv v[x(0; x, t), t]$. Due to the standing hypotheses stated in the introduction, V will be a Lyapunov function for F: only the existence (3) and continuity of $V'(x_0, t_0)$ needs to be verified, and this follows from the identity $V[x(t; x_0, t_0), t] \equiv v[x(0; x_0, t_0), t]$ together with the assumed existence and continuity of $v_t(x_0, t_0)$. This argument is clearly reversible: If V is a Lyapunov function for F, then defining $v(x, t) \equiv Vh^{-1}(x, t) \equiv V[x(t; x, 0), t]$ gives v as an L-function rel $\{U_n\}$, the existence and continuity of $v_t(x_0, t_0)$ coming easily from the identity $v(x_0, t) \equiv V[x\{t; x(t_0; x_0, 0), t_0\}, t]$ and the assumed existence and continuity of $V'[x(t_0; x_0, 0), t_0]$.

In view of this relationship, the question of the existence of a Lyapunov function for a flow reduces to the following question about L-functions: Under what conditions does an L-function rel a given representation of $0\times J_1$ as a G_δ exist? This is answered in § 6; it should be noted that, with this formulation of the problem, the fact that the $h^{-1}(U_n)$ are cylinders is completely irrelevant to the issue at hand.

5. Sectional cylinders. In this section it will be shown that, under certain conditions, an open set U containing $0\times J_1$ can be "approximated" by an open set having the graph of a continuous real-valued function for boundary.

5.1. Definition. (a) A *cylindroid* is a connected open set in $E^n\times J_0$ containing $0\times J_1$.

(b) A cylindroid C_1 "captures" for C_2 if for each $(p, t_0)\in\bar{C}_1$, $(p, t_0+\tau)\in\bar{C}_2$ for all $\tau\geqslant 0$. If $C_1 = C_2$, C_1 is called "self-capturing".

(c) A line (p_0, t) is "ultimately in" a given cylindroid C if there exists a $T > 0$ with $(p_0, \tau)\in C$ for all $\tau > T$.

(d) A cylindroid whose boundary is the graph $\{(x, \varphi(x))\}$ of a continuous real-valued function φ defined on an open $U\subset E^n$ is called a "sectional cylinder"; the cylindroid determined by φ is $\{(x, t)\,|\,x\in U,\ t>\varphi(x)\}$ and is evidently self-capturing.

The significance of sectional cylinders is a consequence of

5.2 Lemma. *Let $\bar{C}_1\subset C_2$ be two cylindroids, where C_1 captures for C_2. Then there exists a self-capturing cylindroid C such that $\dot{C}$ separates E^{n+1} and with $C_1\subset C\subset\bar{C}_2$.*

Proof. Let

$$D = \bigcup_{(p,t)\in\bar{C}_1} (p\times J_t).$$

Clearly, $C_1\subset \operatorname{Int} D$ since C_1 is an open set contained in D; further, $D\subset\bar{C}_2$ since for each $(p, t)\in D$ there is a $(p, t')\in\bar{C}_1$ with $t'\leqslant t$ and C_1 captures for C_2. We also have that D is closed. For, let $(p_0, \tau)\,\bar{\in}\, D$. Since (p_0, τ) lies on an open half-ray not meeting $\bar{C}_1$, there is an $\eta > 0$ with $[p_0\times \overline{\complement J}_{\tau+\eta}]\cap [\bar{C}_1\cap (E^n\times\overline{\complement J}_{\tau+\eta})] = \emptyset$; the first set being compact and the second closed, there is a positive distance $\delta > 0$ between them, so that $S(p_0, \delta)\times\{t\,|\,0 < t < \tau+\eta\}$ provides a neighborhood of (p_0, τ) not meeting D. $C = \operatorname{Int} D$ is therefore the required cylindroid.

The fundamental result is

5.3. Theorem. *Let $C_1\subset C_2\subset C_3\subset C_4$ be four cylindroids such that $\bar{C}_i\subset C_{i+1}$ and C_i captures for C_{i+1}, $i = 1, 2, 3$. Then there exists a sectional cylinder S such that*

1. $C_1\subset S\subset\bar{C}_4$,
2. $\dot{S}$ *separates* E^{n+1},
3. $\dot{S}$ *passes through every ray ultimately in* C_3,
4. $\dot{S}$ *passes only through rays ultimately in* $\bar{C}_4$.

Proof. By 5.2, there are self-capturing cylindroids K_0, K_i with $C_3\subset K_0\subset\bar{C}_4$ and $C_1\subset K_i\subset\bar{C}_2$. We will construct a sectional cylinder with boundary in the open set $\Sigma = K_0 - \bar{K}_i$. Letting π denote the projection $\pi(x, t) = (x, 0)$ the domain of definition of the required function φ will be the open set $D = \pi K_0$. Note that if $p\in D$ then $(p, t_0)\in K_0\subset\bar{C}_4$ for suitable t_0 and, because K_0 is self-capturing, (p, t) is ultimately in $\bar{C}_4$. Further, if (p, t) is ultimately in C_3 then $(p, t)\in K_0$ for large enough t so that $p\in D$. Now define on D

$$u(x) = \inf\{t\,|\,(x, t)\in\Sigma\},\qquad l(x) = \sup\{t\,|\,(x, t)\in\Sigma\}.$$

Then $u(x) < l(x)$ on D, otherwise the line $x_0\times E^1$ intersects Σ at a single point, contradicting that Σ is open. Further, $\{x\,|\,u(x) < k\}$ is open for each k, i.e., u is upper semi-continuous. In fact, let $\xi\in\{x\,|\,u(x) < k\}$; then there is a point in Σ with coordinates $(\xi, u(\xi)+\delta)$, where $0 < \delta < k-u(\xi)$ and, since Σ is open, there is a cubical nbd N of $(\xi, u(\xi)+\delta)$ lying entirely in Σ; then πN is a nbd of ξ and $\pi N\subset\{x\,|\,u(x) < k\}$ since for any $x'\in\pi N$, $u(x')\leqslant u(\xi)+\delta < k$; this shows $\{x\,|\,u(x) < k\}$ open. Similarly $\{x\,|\,l(x) > k\}$ is open, so that l is lower semi-continuous. Under these conditions, a theorem of Dowker (5) [2] applies to give a continuous φ on D with $u(x) < \varphi(x) < l(x)$ for every $x\in D$; the sectional cylinder determined by φ is clearly the desired one.

6. Existence of L-functions. The requirement 3 of 4.1 is the one of basic importance, since the other two can easily be satisfied for any G_δ representation. Since 3 says that values do not increase along

(5) Theorem 4, p. 223. Any metric space is paracompact.

any line (x, τ), the $\{Q(\lambda_n)\}$ of § 3 for an L-function have the additional property: if $(x, t) \in Q(\lambda_n)$ then $(x, t+\tau) \in Q(\lambda_n)$ for all $\tau \geqslant 0$. Thus, because of Theorem 3.3, it is to be expected that the given $\{U_n\}$ should possess some analogous property.

6.1. Lemma. *Let* $0 \times J_1 = \bigcap_n U_n$. *Then a split* L-*function rel* $\{U_n\}$ *exists if and only if there is a descending sequence of sectional cylinders cushioning* $\{U_n\}$.

Proof. Only if: Let $\mu = \mu(U_1)$ and define $Q_n = Q(\mu/n)$; by 3.2, $\{Q_n\}$ cushions $\{U_n\}$. Since each Q_n is self-capturing and $\bar{Q}_{n+1} \subset Q_n$, using four Q-cylindroids interposed (6) in $Q_n - \bar{Q}_{n+1}$, an application of 5.3 produces a sectional cylinder S_n with $Q_{n+1} \subset S_n \subset Q_n$; the $\{S_n\}$ interlaces $\{Q_n\}$ and are the required sets.

If: This part of the proof is related to work of Krasovskii, [3]. Let $\{S_n\}$ be a descending (4) sequence of sectional cylinders cushioning $\{U_n\}$; the function determining S_n is φ_n, is defined on πS_n and $\pi S_{n+1} \subset \pi S_n$. For each pair S_n, S_{n-1}, interpolate an auxiliary sectional cylinder L_n by the function

$$l_n = \varphi_n(x) - \tfrac{1}{2}[\varphi_n(x) - \varphi_{n-1}(x)]\,, \qquad x \in \pi S_n\,.$$

Now let $\Phi(\xi)$ be any C^∞ function of one real variable such that

$$\Phi(\xi) = \begin{cases} 1\,, & \xi \leqslant 0\,,\\ \text{monotone decreasing}\,, & 0 < \xi < 1\,,\\ 0\,, & \xi \geqslant 1\,.\end{cases}$$

We can also assume (7) that $|\Phi'(\xi)| \leqslant B < \infty$ for all ξ. For each k define on E^{n+1} a function

$$v_k(x, t) = \begin{cases} 1\,, & (x, t) \,\bar{\in}\, L_k\,,\\ 0\,, & (x, t) \in \bar{S}_k\,,\\ \Phi\left[\dfrac{t - l_k(x)}{\varphi_k(x) - l_k(x)}\right]\,, & \text{otherwise}\end{cases}$$

(6) That is $Q\left[\mu \cdot \left\{\dfrac{1}{n+1} + \dfrac{i}{5}\left(\dfrac{1}{n(n+1)}\right)\right\}\right]$, $\quad i = 1, \ldots, 4$.

(7) For example, noting first that the function

$$u(x) = \begin{cases} \exp[-\csc^2 \pi x]\,, & 0 < x < 1\,,\\ 0\,, & \text{otherwise}\end{cases}$$

is of class C^∞ on E^1, we can use

$$\Phi(x) = \frac{\int_x^\infty u(\xi)d\xi}{\int_{-\infty}^\infty u(\xi)d\xi}\,, \qquad x \in E^1\,.$$

which is legitimate, since $\varphi_k(x) - l_k(x) \neq 0$ on πS_k. This function is easily seen to be continuous on E^{n+1} with $(v_k)_t \leqslant 0$ and also continuous on E^{n+1}.

Finally, choose any convergence factors $\{a_n\}$, say $0 < a_n \leqslant 1/2^n$ and set

$$v(x, t) = \sum_{n=1}^\infty a_n v_n(x, t)\,.$$

This is the desired split L-function. In fact, because $0 \leqslant v_n(x, t) \leqslant 1$, the series converges uniformly, hence represents a continuous function on E^{n+1} with $0 \leqslant v(x, t) \leqslant 1$. Further,

a. $v(0, t) \equiv 0$ for $t \in J_1$ since $(0, t) \in S_n$ for all n, hence all $v_n(0, t) = 0$.

b. If $(x, t) \in \bar{S}_n$, then $(x, t) \in \bar{S}_i$ for all $i \leqslant n$, hence $v_i(x, t) = 0$, $i \leqslant n$, and therefore $v(x, t) \leqslant \sum_{n+1}^\infty a_j$.

c. If $(x, t) \,\bar{\in}\, \bar{S}_n$, then $(x, t) \,\bar{\in}\, \bar{S}_i$ for all $i \geqslant n$, so that $v_i(x, t) = 1$ certainly for all $i \geqslant n+1$, showing $v(x, t) \geqslant \sum_{n+1}^\infty a_j$.

Thus, v is positive definite rel $\{S_n\}$; since $\{S_n\}$ cushions $\{U_n\}$ it is positive definite rel $\{U_n\}$ also. It remains to verify condition 3 of 4.1. Because each $(v_n)_t \leqslant 0$ and is continuous, we need only show that each point $\xi \equiv (x, t) \in \mathcal{C}(0 \times J_0)$ has a nbd on which the derived series converges uniformly. In fact, letting $\varrho = \frac{1}{2}\|x\| > 0$ then, because $\bigcap_n L_n = 0 \times J_1$ and are descending, we must have $S(\xi, \varrho) \cap L_i = \emptyset$ for all $i >$ some $k = k(\xi)$; in $S(\xi, \varrho)$ the derived series reduces to a finite sum of form

$$\sum_0^k \frac{a_i}{\varphi_i(x) - l_i(x)} \Phi'\left[\frac{t - l_i(x)}{\varphi_i(x) - l_i(x)}\right]$$

hence is continuous, and represents v_t on this nbd. The lemma is proved.

Note that the split L-function defined above "tends uniformly to zero on $\{S_n\}$", that is, "diam $v(S_n) \to 0$ as $n \to \infty$"; this is clear from (b) above.

6.2. Theorem. *Let* $0 \times J_1 = \bigcap_n U_n$. *Then a split* L-*function rel* $\{U_n\}$ *exists if and only if there is a descending sequence of cylindroids, each capturing for its predecessor, which cushions* $\{U_n\}$. *Further, the split* L-*function can always be chosen so that it tends to zero uniformly on the cushioning sequence.*

Proof. Necessity is clear, using Theorem 3.3. As for sufficiency, an application of 5.3 gives, for each four successive cushioning cylindroids $C_{4n}, \ldots, C_{4n+3}$ a sectional cylinder S_n with $C_{4n} \subset S_n \subset \bar{C}_{4n+3}$. $\{S_n\}$ thus interlaces $\{C_n\}$ and 6.1, with the final remark, gives the result.

A $\{U_n\}$-Lyapunov function for a flow F is defined formally as in Definition 4.1, except that the "v_t" in condition 3 is replaced by "trajectory derivative". Since the concept of "capturing" is invariant under continuous maps, and under the rectifying homeomorphism a $\{U_n\}$-Lyapunov function for F corresponds to an L-function rel $\{h(U_n)\}$, Theorem 6.2 can be stated directly in terms of $\{U_n\}$ and F.

Part II

7. Uniform $\{U_n\}$-stability.

The application of 6.2 to uniform stability has been indicated in the introduction.

7.1. Definition. Let $\{U_n\}$ be a sequence of cylindroids expressing $0\times J_1$ as [4] a G_δ. The solution $x=0$ is uniformly $\{U_n\}$*-stable* if for each U_n there is a U_m, $m\geqslant n$, such that each trajectory entering $\overline{U}_m$ remains thereafter in $\overline{U}_n$.

This is evidently the classical definition in case $U_n\equiv C_n = S(0,1/n)\times\times\mathring{J}_{1-1/n}$. Note that, if h is a rectifying homeomorphism, then $x=0$ is always uniformly $\{h^{-1}(C_n)\}$-stable; this concept therefore has significance only if the sequence $\{U_n\}$ is specified in advance.

7.2 Theorem. *$x=0$ is uniformly $\{U_n\}$-stable if and only if there exists a split $\{U_n\}$-Lyapunov function V with* $\operatorname{diam} V(U_n)\to 0$ *as* $n\to\infty$.

Proof. Necessity. Under a rectifying homeomorphism for the flow F the open sets $h(U_n)$ give a G_δ representation for $0\times J_1$ and, by 7.1, one can extract a descending subsequence, each capturing for its predecessor. A direct application of 6.2 yields a split L-function v rel $\{h(U_i)\}$ with $\operatorname{diam} v\big(h(U_i)\big)\to 0$ as $i\to\infty$. $V(x,t)=v\big(h(x,t)\big)$ is the desired split $\{U_n\}$-Lyapunov function.

Sufficiency. Since $\operatorname{diam} V(U_n)=\varepsilon_n\to 0$, the sequence $\lambda_n=\varepsilon_n(1++1/n)\to 0$ also, and clearly $U_n\subset Q(\lambda_n)$. Because V is positive definite, 3.3 says that $\{Q(\lambda_n)\}$ cushions $\{U_n\}$. Thus, given U_n select $Q(\lambda_m)\subset U_n$ and then U_m satisfies the requirement of 7.1.

In the special case $U_n\equiv C_n$, 7.2 represents a slight generalization of Kurzeweil's theorem [4]. Specifically, it is known [1] that under the standing hypotheses of the introduction, the existence of a proper $\{C_n\}$-Lyapunov function implies uniform stability; for the necessity, however, Kurzweil requires the explicit condition that $X(x,t)$ be locally lipschitzian, nad gives an example to show that, without this explicit requirement, a proper $\{C_n\}$-Lyapunov function need not exist. In view of 7.2, by allowing split Lyapunov functions, we obtain more symmetric necessary and sufficient conditions.

A similar symmetric situation occurs for ordinary stability:

7.3. Theorem. *A necessary and sufficient condition that $x=0$ be stable is that there exist a split $\{C_n\}$-Lyapunov function.*

Proof. Recall that $x=0$ is stable if for each $\varepsilon>0$ and $t_0\in J_0$ there is a $\delta(\varepsilon,t_0)>0$ such that for all $\|x_0\|<\delta$ one has $\|x(t;x_0,t_0)\|<\varepsilon$ for all $t\geqslant t_0$. Let h be a rectifying homeomorphism; the definition immediately shows that $\{h(C_n)\}$ is in fact cushioned by the actual cylinders $\big\{S\big(0,\delta(1/n,0)\big)\times\mathring{J}_{1-1/n}\big\}$; by 6.2, a $\{C_n\}$-Lyapunov function exists. Conversely, if there is a split $\{C_n\}$-Lyapunov function, $\{Q(\lambda_n)\}$ cushions $\{C_n\}$; choosing the sequence $\{\lambda_n\}$ so that $Q(\lambda_n)\subset C_n$ for each n, define $\delta(1/n,t_0)=\varrho$ where $S\big((0,t_0),\varrho\big)\subset Q(\lambda_n)$.

8. Continuity of the trajectory derivative.

8.1. Theorem. *If a split $\{U_n\}$-Lyapunov function exists, then for any preassigned bounded open $K\subset J_0$ there exists a split $\{U_n\}$-Lyapunov function V_k with* [3] *V_k' continuous also on $E^n\times K$.*

Proof. One need establish this only for L-functions. We use the notations in the proof of "If" in Lemma 6.1. Let $N=S(0,1)\times K$ and define $\beta_n=\inf\{|t-t'|\,|\,(x,t)\in\overline{S}_n\cap\overline{N},\ (x,t')\in \complement S_{n-1}\}$; $\beta_n>0$ since $\overline{S}_n\cap\overline{N}$ is compact, does not intersect the closed $\complement S_{n-1}$ hence $0<d(\overline{S}_n\cap\overline{N},\complement S_{n-1})\leqslant\beta_n$. Choose $a_n=\min[1/2^n,\beta_n/2^n]$. Since for $\big(x,\varphi_n(x)\big)\in N$ one has $|\varphi_n(x)-\varphi_{n-1}(x)|\geqslant\beta_n$ the coefficients of the derived series estimate as

$$0<\frac{a_n}{\varphi_n(x)-l_n(x)}=\frac{a_n}{\frac{1}{2}[\varphi_n(x)-\varphi_{n-1}(x)]}\leqslant\frac{\beta_n/2^n}{\frac{1}{2}\cdot\beta_n}=\frac{1}{2^{n-1}},\qquad \big(x,\varphi_n(x)\big)\in N\,.$$

For any nbd W, $(0,t_0)\in W\subset N$, let $k(W)=\sup\{n\,|\,W\cap\complement S_n=\emptyset\}$; then $k(W)\to\infty$ as $\operatorname{diam} W\to 0$, since the $\{S_n\}$ are descending. Thus, if $(x,t)\in W$, using the only possible non-vanishing terms of the derived series gives

$$|v_t(x,t)|\leqslant B\cdot 1/2^{k(W)-1},\qquad (x,t)\in W$$

and, since as $(x_n,t_n)\to(0,t_0)$ one can assume $\operatorname{diam} W\to 0$, it follows that $v_t(x_n,t_n)\to 0$, establishing continuity at $(0,t_0)\in W$.

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