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ON OPEN THEORIES

BY

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The first part of this paper contains a topological characterization of open theories. This characterization was announced, without proofs, in [7]. The second part contains a topological proof of the Herbrand [1] theorem for open theories in the most general form. The proof is based on an idea from my earlier paper [5].

§ 1. Let $\mathscr{S}$ be a two-valued first-order predicate calculus containing the following primitive symbols: an infinite set V of free individual variables denoted by the letters x, y (with indices), an infinite set (disjoint from V) of bound individual variables denoted by the letters ξ, η, a set of functors, a non-empty set of predicates, the logical connectives $\cup$ (or), $\cap$ (and), $\rightarrow$ (if... then...), $-$ (not), and the existential and universal quantifiers $\bigcup$ and $\bigcap$. The cardinals of the sets of variables, functors and predicates are arbitrary.

The sets of all terms and formulas in $\mathscr{S}$ will be denoted by T and F respectively. Terms are denoted by the letter τ, and formulas — by α, β, γ, δ.

Let $\mathscr{T}$ be a formalized theory based on $\mathscr{S}$, $\mathscr{A}$ denoting an assumed set of axioms for $\mathscr{T}$.

The symbol $L(\mathscr{T})$ will denote the Lindenbaum algebra of the theory $\mathscr{T}$, i. e. the Boolean algebra obtained from F by identification of formulas α, β if and only if both $\alpha \rightarrow \beta$ and $\beta \rightarrow \alpha$ are theorems in $\mathscr{T}$. For every formula α in F, the symbol $|\alpha|_{\mathscr{T}}$ will denote the corresponding element in $L(\mathscr{T})$. We recall that the join, meet and complement in $L(\mathscr{T})$ are defined by the equalities

(1) $$|\alpha|_{\mathscr{T}} \cup |\beta|_{\mathscr{T}} = |\alpha \cup \beta|_{\mathscr{T}}, \quad |\alpha|_{\mathscr{T}} \cap |\beta|_{\mathscr{T}} = |\alpha \cap \beta|_{\mathscr{T}}, \quad -|\alpha|_{\mathscr{T}} = |-\alpha|_{\mathscr{T}},$$

and

(2) $$|\alpha|_{\mathscr{T}} \leqslant |\beta|_{\mathscr{T}} \text{ if and only if } \alpha \rightarrow \beta \text{ is a theorem in } \mathscr{T},$$

where $\leqslant$ is the Boolean partial ordering in $L(\mathcal{T})$. We recall also that, for every formula $\alpha(x)$,

(3) $$|\bigcup_\xi \alpha(\xi)|_{\mathcal{T}} = \bigcup_{\tau\in T}|\alpha(\tau)|_{\mathcal{T}}, \qquad |\bigcap_\xi \alpha(\xi)|_{\mathcal{T}} = \bigcap_{\tau\in T}|\alpha(\tau)|_{\mathcal{T}},$$

where $\alpha(\xi)$ and $\alpha(\tau)$ denote respectively the result of substitution of a bound individual variable ξ and a term τ for the free individual variable x in $\alpha(x)$, respectively.

By a Q-*filter* in $L(\mathcal{T})$ we shall understand any prime filter $\mathfrak{f}$ in $L(\mathcal{T})$ such that for every formula $\alpha(x)$

$$|\bigcup_\xi \alpha(\xi)|\in\mathfrak{f} \text{ implies that } |\alpha(\tau)|\in\mathfrak{f} \text{ for a term } \tau.$$

The symbol $\mathcal{L}(\mathcal{T})$ will denote the set of all Q-filters in $L(\mathcal{T})$. For every formula α, let $\|\alpha\|_{\mathcal{T}}$ be the set

$$\|\alpha\|_{\mathcal{T}} = \{\mathfrak{f}\in\mathcal{L}(\mathcal{T}) \colon |\alpha|_{\mathcal{T}}\in\mathfrak{f}\}.$$

By definition, for every formula α and every $\mathfrak{f}\in\mathcal{L}_{\mathcal{T}}$

(4) $$\mathfrak{f}\in\|\alpha\|_{\mathcal{T}} \text{ if and only if } |\alpha|_{\mathcal{T}}\in\mathfrak{f}.$$

(i) *The mapping $H_{\mathcal{T}}$ defined by the formula*

$$H_{\mathcal{T}}(|\alpha|_{\mathcal{T}}) = \|\alpha\|_{\mathcal{T}}$$

is a Boolean homomorphism from $L(\mathcal{T})$ into the Boolean algebra of all subsets of $\mathcal{L}(\mathcal{T})$, i. e.,

$$\|\alpha\cup\beta\|_{\mathcal{T}} = \|\alpha\|_{\mathcal{T}}\cup\|\beta\|_{\mathcal{T}}, \qquad \|\alpha\cap\beta\|_{\mathcal{T}} = \|\alpha\|_{\mathcal{T}}\cap\|\beta\|_{\mathcal{T}}, \qquad -\|\alpha\|_{\mathcal{T}} = \|-\alpha\|_{\mathcal{T}}.$$

The homomorphism $H_{\mathcal{T}}$ preserves also all the infinite joins and meets (3) *corresponding to logical quantifiers, i. e. for every formula $\alpha(x)$*

(5) $$\|\bigcup_\xi \alpha(\xi)\|_{\mathcal{T}} = \bigcup_{\tau\in\mathcal{T}}\|\alpha(\tau)\|_{\mathcal{T}}, \qquad \|\bigcap_\xi \alpha(\xi)\|_{\mathcal{T}} = \bigcap_{\tau\in\mathcal{T}}\|\alpha(\tau)\|_{\mathcal{T}},$$

where $\bigcup_{\tau\in\mathcal{T}}$ and $\bigcap_{\tau\in\mathcal{T}}$ denote respectively the set-theoretical union and intersection.

For the proof, cf. Sikorski [6], theorem 24.6.

We assume the following notation:

$$\boldsymbol{L}(\mathcal{T}) = \{\|\alpha\|_{\mathcal{T}} \colon \alpha\in F\},$$

$$\boldsymbol{L}_0(\mathcal{T}) = \{\|\alpha\|_{\mathcal{T}} \colon \alpha\in F \text{ is an open formula}\}.$$

By (i), $H_{\mathcal{T}}$ is a homomorphism from $L(\mathcal{T})$ onto $\boldsymbol{L}(\mathcal{T})$.

The set $\mathcal{L}(\mathcal{T})$ of all Q-filters in $L(\mathcal{T})$ will always be considered as a topological space, the class $\boldsymbol{L}_0(\mathcal{T})$ being assumed as the basis determining the topology in $\mathcal{L}(\mathcal{T})$. By definition, every set $S_0\in\boldsymbol{L}_0(\mathcal{T})$ is clopen, i. e. both open and closed. A set $S\subset\mathcal{L}(\mathcal{T})$ is open (closed) if and only if it is the union (intersection) of some sets $S_0\in\boldsymbol{L}_0(\mathcal{T})$.

It the set $\mathcal{A}$ of axioms for $\mathcal{T}$ is empty, then the theory $\mathcal{T}$ is the predicate calculus $\mathcal{S}$. In that case we shall write simply $\mathcal{L}$, L, $\boldsymbol{L}$, H, $|\alpha|$, $\|\alpha\|$ instead of $\mathcal{L}_{\mathcal{S}}$, $L_{\mathcal{S}}$, $\boldsymbol{L}_{\mathcal{S}}$, $H_{\mathcal{S}}$, $|\alpha|_{\mathcal{S}}$, $\|\alpha\|_{\mathcal{S}}$.

The following theorem is another formulation of Gödel's completeness theorem for the predicate calculus $\mathcal{S}$ (see Rasiowa and Sikorski [2]):

(ii) *The mapping H is an isomorphism from L onto $\boldsymbol{L}$.*

Rieger [3, 4] has proved that

(iii) *The topological space $\mathcal{L}$ is compact and totally disconnected.*

Moreover, $\mathcal{L}$ is homeomorphic with a Cantor discontinuum, viz. with the product of $\mathfrak{m}$ replicas of a two-element Hausdorff space where $\mathfrak{m}$ is the cardinal of the set of all open formulas in $\mathcal{S}$. However, this fact will not play any essential part in our investigations.

If $\mathcal{T}$ is a theory based on the predicate calculus $\mathcal{S}$, then $\nabla_{\mathcal{T}}$ will denote the class of all sets $\|\alpha\|\subset\mathcal{L}$ where α is a theorem in $\mathcal{T}$. It is easy to verify that

(iv) *$\nabla_{\mathcal{T}}$ is a filter in $\boldsymbol{L}$ such that, for every formula $\alpha(x)$,*

(6) $$\textit{if } \|\alpha(x)\|\in\nabla_{\mathcal{T}}, \textit{ then } \|\bigcap_\xi \alpha(\xi)\|\in\nabla_{\mathcal{T}}.$$

More precisely, $\nabla_{\mathcal{T}}$ is the smallest filter having property (6) *and containing all $\|\alpha\|$ where α is in the set $\mathcal{A}$ of axioms of $\mathcal{T}$.*

(v) *$L(\mathcal{T})$ is isomorphic to $\boldsymbol{L}/\nabla_{\mathcal{T}}$, the isomorphism being defined by the formula*

$$h_{\mathcal{T}}(|\alpha|_{\mathcal{T}}) = \|\alpha\|/\nabla_{\mathcal{T}}.$$

In the last equality, $\|\alpha\|/\nabla_{\mathcal{T}}$ denotes the element in $\boldsymbol{L}/\nabla_{\mathcal{T}}$, which is determined by the element $\|\alpha\|\in\boldsymbol{L}$.

For every formula α, the symbol $\bar{\alpha}$ will denote the closure of α, i. e. the formula obtained from α by binding all free individual variables in α by universal quantifiers.

Let $\mathcal{K}(\mathcal{T})$ denote the intersection of all sets $\|\alpha\|\in\nabla_{\mathcal{T}}$. By definition and (iv),

(vi) *$\mathcal{K}(\mathcal{T})$ is the intersection of all sets $\|\bar{\alpha}\|$, where α is any formula in the set $\mathcal{A}$ of axioms of $\mathcal{T}$.*

(vii) *For every* Q-*filter $\mathfrak{f}$ in $\mathcal{L}(\mathcal{T})$ there exists exactly one point $\mathfrak{p}\in\mathcal{K}(\mathcal{T})$ such that*

(7) $$\mathfrak{f} = \{|\alpha|_{\mathcal{T}} \colon \mathfrak{p}\in\|\alpha\|\}.$$

Conversely, for every $\mathfrak{p}\in\mathcal{K}(\mathcal{T})$ formula (7) *defines a* Q-*filter in $\mathcal{L}(\mathcal{T})$.*

Viz., for a given filter $\mathfrak{f}\in\mathcal{L}_{\mathcal{T}}$, the Q-filter $\mathfrak{p}\in\mathcal{L}$ composed of all $|\alpha|$ such that $|\alpha|_{\mathcal{T}}\in\mathfrak{f}$ satisfies (7). This proves the first part of (vii). The second part follows immediately from (7).

The set $\mathscr{K}_{\mathscr{T}}$ will always be considered as a topological space with the topology induced by the topology in $\mathscr{L}$.

Let

$$\boldsymbol{K}(T) = \{\|\alpha\| \cap \mathscr{K}(\mathscr{T}) \colon \alpha \in F\},$$

$$\boldsymbol{K}_0(T) = \{\|\alpha\| \cap \mathscr{K}(\mathscr{T}) \colon \alpha \in F \text{ is an open formula}\}.$$

By definition, the class $\boldsymbol{K}_0(\mathscr{T})$ is a basis for the topology in $\mathscr{K}(\mathscr{T})$. The classes $\boldsymbol{K}(\mathscr{T})$ and $\boldsymbol{K}_0(\mathscr{T})$ are Boolean algebras of subsets of $\mathscr{K}(\mathscr{T})$.

Theorem (vii) defines a natural one-to-one mapping

$$(8) \qquad f_{\mathscr{T}}(\mathfrak{p}) = \mathfrak{f}$$

from $\mathscr{K}(\mathscr{T})$ onto $\mathscr{L}(\mathscr{T})$.

(viii) *The mapping* (8) *is a homeomorphism of* $\mathscr{K}_{\mathscr{T}}$ *onto* $\mathscr{L}_{\mathscr{T}}$. *The mapping which, to every* $\|\alpha\|_{\mathscr{T}} \in \boldsymbol{L}(\mathscr{T})$ *assigns the set* $f^{-1}(\|\alpha\|_{\mathscr{T}}) = \|\alpha\| \cap \mathscr{K}(\mathscr{T}) \in \boldsymbol{K}(\mathscr{T})$ *is an isomorphism from* $\boldsymbol{L}(\mathscr{T})$ *onto* $\boldsymbol{K}(\mathscr{T})$. *This isomorphism maps* $\boldsymbol{L}_0(\mathscr{T})$ *onto* $\boldsymbol{K}_0(\mathscr{T})$.

The proof is by an easy verification.

Since the space $\mathscr{L}$ is totally disconnected (see (iii)), so is its subspace $\mathscr{K}(\mathscr{T})$. Consequently, by the first part of (viii),

(ix) *The space* $\mathscr{L}(\mathscr{T})$ *is totally disconnected.*

Theorem (ix) can also be easily proved directly.

By definition of $\mathscr{K}(\mathscr{T})$, for every formula α

$$\text{if } \|\alpha\| \in \nabla_{\mathscr{T}}, \text{ then } \mathscr{K}(\mathscr{T}) \subset \|\alpha\|.$$

(x) *The mapping* $H_{\mathscr{T}}$ (see (i)) *is an isomorphism from* $L(\mathscr{T})$ *onto* $\boldsymbol{L}(\mathscr{T})$ *if and only if, for every formula* α,

$$(9) \qquad \mathscr{K}(\mathscr{T}) \subset \|\alpha\| \textit{ implies } \|\alpha\| \in \nabla_{\mathscr{T}}.$$

Condition (9) can also be formulated in the following equivalent form:

$$(9') \qquad \mathscr{K}(\mathscr{T}) \subset \|\alpha\| \text{ if and only if } \|\alpha\| \in \nabla_{\mathscr{T}}.$$

By (v) and the second part of (viii), the mapping $H_{\mathscr{T}}$ is an isomorphism if and only if the homomorphism $H'_{\mathscr{T}}$ defined by the equality

$$H'_{\mathscr{T}}(\|\alpha\|/\nabla_{\mathscr{T}}) = \|\alpha\| \cap \mathscr{K}(\mathscr{T}) \qquad (\alpha \in F)$$

is an isomorphism from $\boldsymbol{L}/\nabla_{\mathscr{T}}$ onto $\boldsymbol{K}(\mathscr{T})$. $H'_{\mathscr{T}}$ is an isomorphism if and only if condition (9) holds. The proof of the last statement is similar to the proof of theorem 28.1 in Sikorski [6].

A theory $\mathscr{T}$ is said to be *open* provided it has a set $\mathscr{A}$ of axioms which are open formulas.

(xi) *If* $\mathscr{T}$ *is an open theory, then* $\mathscr{K}(\mathscr{T})$ *is a closed subset of* $\mathscr{L}$. *If* $\mathscr{T}$ *is open and consistent, then* $\mathscr{K}(\mathscr{T})$ *is closed and non-empty.*

The first part follows immediately from (vi) since if α is an open formula, then $\|\bar{\alpha}\|$ is closed by (5) as the intersection of clopen sets. The second part of (xi) also follows from (vi) since if $\mathscr{T}$ is consistent, then no conjunction $\bar{\alpha}_1 \cap \ldots \cap \bar{\alpha}_n$, where $\alpha_1, \ldots, \alpha_n$ are open axioms in the set $\mathscr{A}$, is refutable, i. e.

$$\|\bar{\alpha}_1\| \cap \ldots \cap \|\bar{\alpha}_n\| = \|\bar{\alpha}_1 \cap \ldots \cap \bar{\alpha}_n\| \neq 0.$$

Thus the class of closed sets $\|\bar{\alpha}\|$, where $\alpha \in \mathscr{A}$, has the finite intersection property. By (iii), the intersection $\mathscr{K}(\mathscr{T})$ of all those sets is not empty.

(xii) *Let* $\mathscr{T}$ *be an open theory such that the sets of all terms and of all free individual variables have the same power. If* α_0 *is a formula irrefutable in* $\mathscr{T}$, *then there exists a point* $\mathfrak{p} \in \mathscr{K}(\mathscr{T})$ *such that* $\mathfrak{p} \in \|\alpha_0\|$.

Without any restriction we can assume that α_0 is in the normal prenex form

$$\bigcap_{\xi_1} \bigcup_{\eta_1} \ldots \bigcap_{\xi_k} \bigcup_{\eta_k} \beta(\boldsymbol{x}, \boldsymbol{\xi}_1, \boldsymbol{\eta}_1, \ldots, \boldsymbol{\xi}_k, \boldsymbol{\eta}_k),$$

where $\boldsymbol{x}$, $\boldsymbol{\xi}_i$, $\boldsymbol{\eta}_i$ are abbreviations:

$$\boldsymbol{x} = (x_1, \ldots, x_m),$$

$$\boldsymbol{\xi}_i = (\xi_{i1}, \ldots, \xi_{im_i}), \qquad \boldsymbol{\eta}_i = (\eta_{i1}, \ldots, \eta_{in_i}) \qquad \text{for} \quad i = 1, \ldots, k;$$

$\bigcap_{\xi_i}, \bigcap_{\eta_i}$ are abbreviations for

$$\bigcap_{\xi_{i1}} \ldots \bigcap_{\xi_{im_i}}, \qquad \bigcup_{\eta_{i1}} \ldots \bigcup_{\eta_{in_i}}$$

respectively, $\beta(\boldsymbol{x}, \ldots)$ does not contain any quantifier, and $x_1, \ldots, x_m$ are all free individual variables in α_0.

Let $\mathscr{S}'$ be the predicate calculus obtained from $\mathscr{S}$ by adding some new individual constants

$$c_1, \ldots, c_m$$

and some $(m_1 + \ldots + m_i)$-argument functors

$$\varphi_{ij} \qquad (j = 1, \ldots, n_i,\ i = 1, \ldots, k).$$

Let γ_0 be the formula in $\mathscr{S}'$

$$\beta(\boldsymbol{c}, \boldsymbol{x}_1, \boldsymbol{\varphi}_1(\boldsymbol{x}_1), \ldots, \boldsymbol{x}_k, \boldsymbol{\varphi}_k(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k)),$$

where the following abbreviations are used:

$$\boldsymbol{c} = (c_1, \ldots, c_m), \qquad \boldsymbol{x}_i = (x_{i1}, \ldots, x_{im_i}) \qquad (i = 1, \ldots, k),$$

where all x_{ij} are distinct from one another, and distinct from $x_1, \ldots, x_m$;

$$\varphi_i(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_i) = \big(\varphi_{i1}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_i), \ldots, \varphi_{in_i}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_i)\big),$$

where

$$\varphi_{ij}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_i) = \varphi_{ij}(x_{11}, \ldots, x_{1m_1}, \ldots, x_{i1}, \ldots, x_{im_i})$$

for $j = 1, \ldots, n_i$ and $i = 1, \ldots, k$.

Let $\mathscr{T}'$ be the open theory based on $\mathscr{S}'$ whose set of axioms is composed of the formula γ_0 and all axioms of $\mathscr{T}$. By a known theorem, the hypothesis that α_0 is irrefutable in $\mathscr{T}$ implies that the open theory $\mathscr{T}'$ is consistent. By (xi), there exists a Q-filter $\mathfrak{p}' \epsilon \mathscr{K}(\mathscr{T})'$.

It is easy to see that the set T' of all terms in $\mathscr{S}'$ has the same power as the set V of all individual variables. Thus there exists a one-to-one mapping g from V onto T'. Moreover, we may assume that

$$g(x_j) = c_j \quad \text{for} \quad j = 1, \ldots, m. \tag{10}$$

If α is a formula in F and $y_1, \ldots, y_n$ are all free individual variables appearing in α, let α' denote the formula

$$\alpha\begin{pmatrix} g(y_1), \ldots, g(y_n) \\ y_1, \ldots, y_n \end{pmatrix},$$

i. e. the result of the indicated substitution in α. Clearly α' is a formula in $\mathscr{S}'$.

It is easy to see that the mapping which assigns $|\alpha'|_{\mathscr{S}'}$ to $|\alpha|$ is a homomorphism from L into $L(\mathscr{S}')$. It follows from the hypothesis that g maps V onto T' and that this mapping preserves also infinite joins and meets corresponding to logical quantifiers. Hence it follows that the set

$$\mathfrak{p} = \{|\alpha| : \alpha \epsilon F \text{ and } |\alpha'|_{\mathscr{S}'} \epsilon \mathfrak{p}'\}$$

is a Q-filter in $\mathscr{L} = \mathscr{L}(\mathscr{S})$, i. e. $\mathfrak{p} \epsilon \mathscr{L}$. By definition,

$$\mathfrak{p} \epsilon \|\alpha\| \text{ if and only if } \mathfrak{p}' \epsilon \|\alpha'\|_{\mathscr{S}'},$$

where $\|\alpha'\|_{\mathscr{S}'} = \{\mathfrak{f}' \epsilon \mathscr{L}(\mathscr{S}') : |\alpha'|_{\mathscr{S}'} \epsilon \mathfrak{f}'\}$, according to the definition on p. 172.

For every closed formula α in $\mathscr{S}$, the formula α' coincides with α. Taking as α the closure of any axiom in $\mathscr{T}$, we infer that $\mathfrak{p} \epsilon \mathscr{K}(\mathscr{T})$ since $\mathfrak{p}' \epsilon \mathscr{K}(\mathscr{T}')$.

Since $\mathfrak{p}' \epsilon \|\gamma\|_{\mathscr{S}'}$, where γ is any substitution of γ_0, we have (see (5))

$$\mathfrak{p}' \epsilon \|\alpha_0'\|_{\mathscr{S}'},$$

where α_0' is the formula

$$\bigcap_{\xi_1}\bigcup_{\eta_1} \ldots \bigcap_{\xi_k}\bigcup_{\eta_k} \beta(\boldsymbol{c}, \xi_1, \eta_1, \ldots, \xi_k, \eta_k).$$

Hence, by (10),

$$\mathfrak{p} \epsilon \|\alpha_0\|.$$

This completes the proof of (xii).

(xiii) *If $\mathscr{T}$ is an open theory such that $\overline{\overline{V}} = \overline{\overline{T}}$, then condition* (9) *holds.*

Suppose that α is a formula such that $\|\alpha\| \notin \nabla_{\mathscr{T}}$. Then the negation α_0 of α is irrefutable. By (xii) there exists a point $\mathfrak{p} \epsilon \mathscr{K}(\mathscr{T})$ such that $\mathfrak{p} \epsilon \|\alpha_0\|$, i. e. $\mathfrak{p} \notin \|\alpha\|$. Thus $\mathscr{K}(\mathscr{T}) \not\subset \|\alpha\|$. This proves that (9) holds.

(xiv) *Suppose $\overline{\overline{V}} = \overline{\overline{T}}$. In order that the theory $\mathscr{T}$ be open it is necessary and sufficient that $\mathscr{K}(\mathscr{T})$ be a closed subset of $\mathscr{L}$ and that condition* (9) *hold.*

The necessity (under the additional hypothesis that $\overline{\overline{V}} = \overline{\overline{T}}$) follows from (xi) and (xiii).

Suppose that $\mathscr{K}(\mathscr{T})$ is closed and that (9) holds. Let $\mathscr{A}'$ be the set of all open formulas such that $\mathscr{K}(\mathscr{T}) \subset \|\alpha\|$, and let $\mathscr{T}'$ be the open theory with $\mathscr{A}'$ as the set of axioms. By definition, $\mathscr{K}(\mathscr{T}) \subset \mathscr{K}(\mathscr{T}')$. On the other hand, if $\mathfrak{p} \notin \mathscr{K}(\mathscr{T})$, there exists an open formula β such that $\mathfrak{p} \epsilon \|\beta\|$ and $\mathscr{K}(\mathscr{T})$ is disjoint from $\|\beta\|$ (this follows from the fact that $\mathscr{K}(\mathscr{T})$ is closed and from the fact that sets $\|\beta\|$, where β is an open formula, form a basis for $\mathscr{L}$). The negation α of β has the properties: $\mathscr{K}(\mathscr{T}) \subset \|\alpha\|$, and $\mathfrak{p} \notin \|\alpha\|$. This implies that $\mathfrak{p} \notin \mathscr{K}(\mathscr{T}')$. Consequently

$$\mathscr{K}(\mathscr{T}') = \mathscr{K}(\mathscr{T}). \tag{11}$$

Since the theory is open, it satisfies also condition (9) (i. e. (9')) by the part of (xiii) which has just been proved. In other words, by (11),

a formula α is a theorem in $\mathscr{T}'$ if and only if $\mathscr{K}(\mathscr{T}) \subset \|\alpha\|$.

Since $\mathscr{T}$ satisfies condition (9),

a formula α is a theorem in $\mathscr{T}$ if and only if $\mathscr{K}(\mathscr{T}) \subset \|\alpha\|$.

This proves that $\mathscr{T}$ and $\mathscr{T}'$ have the same sets of theorems, i. e. $\mathscr{T}$ is identical with the open theory $\mathscr{T}'$. Thus $\mathscr{T}$ is open.

(xv) *Suppose that $\overline{\overline{V}} = \overline{\overline{T}}$. In order that the theory $\mathscr{T}$ be open it is necessary and sufficient that the space $\mathscr{L}(\mathscr{T})$ be compact and the homomorphism $H_{\mathscr{T}}$ be an isomorphism.*

This immediately follows from (xiii) on account of (iii), (viii) and (x).

§ 2. Let α be a formula in $\mathscr{S}$. Without any restriction of generality we can suppose that α is in the prenex form

$$\bigcup_{\xi_1}\bigcap_{\eta_1} \ldots \bigcup_{\xi_k}\bigcap_{\eta_k} \beta(\boldsymbol{x}_0, \xi_1, \eta_1, \ldots, \xi_k, \eta_k), \tag{12}$$

where $\beta(\boldsymbol{x}_0, \ldots)$ is an open formula. We use here the same abbreviations as in the proof of (xiii). In particular,

$$\boldsymbol{x}_0 = (x_{01}, \ldots, x_{0n}),$$

where $x_{01}, \ldots x_{0n}$ are all free individual variables appearing in α, and

$$\boldsymbol{\xi}_i = (\xi_{i1}, \ldots, \xi_{im_i}), \quad \boldsymbol{\eta}_i = (\xi_{i1}, \ldots, \xi_{in_i}) \quad \text{for} \quad i = 1, \ldots, k.$$

In the sequel we shall use the notation

$$\boldsymbol{x}_i = (x_{i1}, \ldots, x_{in_i}),$$

where x_{ij} are any free individual variables, and

$$\boldsymbol{\tau}_i = (\tau_{i1}, \ldots, \tau_{im_i}),$$

where τ_{ij} are any terms ($i = 1, \ldots, k$).

Denote by Z_0 the set composed of the formula α only. For $r = 1, \ldots, k$, let Z_r be the set of all formulas

(13) $$\bigcup_{\boldsymbol{\xi}_{r+1}} \bigcap_{\boldsymbol{\eta}_{r+1}} \ldots \bigcup_{\boldsymbol{\xi}_k} \bigcap_{\boldsymbol{\eta}_k} \beta(\boldsymbol{x}, \boldsymbol{\tau}_1, \boldsymbol{x}_1, \ldots, \boldsymbol{\tau}_r, \boldsymbol{x}_r, \boldsymbol{\xi}_{r+1}, \boldsymbol{\eta}_{r+1}, \ldots, \boldsymbol{\xi}_k, \boldsymbol{\eta}_k).$$

In particular, Z_r is the set of all open formulas of the form

(14) $$\beta(\boldsymbol{x}_0, \boldsymbol{\tau}_1, \boldsymbol{x}_1, \ldots, \boldsymbol{\tau}_k, \boldsymbol{x}_k).$$

Let Z be the union of the sets $Z_0, Z_1, \ldots, Z_k$.

By a *Herbrand disjunction for* α we shall understand any disjunction

(15) $$\alpha_1 \cup \ldots \cup \alpha_l,$$

where $\alpha_1, \ldots, \alpha_l$ are distinct formulas in Z.

Let (15) be a Herbrand disjunction for α. Suppose that α_j is of the form (13). Let α_j' be the formula

(16) $$\bigcup_{\boldsymbol{\xi}_r} \bigcap_{\boldsymbol{\eta}_r} \ldots \bigcup_{\boldsymbol{\xi}_k} \bigcap_{\boldsymbol{\eta}_k} \beta(\boldsymbol{x}, \boldsymbol{\tau}_1, \boldsymbol{x}_1 \ldots, \boldsymbol{\tau}_{r-1}, \boldsymbol{x}_{r-1}, \boldsymbol{\xi}_r, \boldsymbol{\eta}_r, \ldots, \boldsymbol{\xi}_k, \boldsymbol{\eta}_k).$$

If α_j' is not identical with one of the formulas $\alpha_1, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_l$, then the disjunction

(17) $$\alpha_1 \cup \ldots \cup \alpha_{j-1} \cup \alpha_j' \cup \alpha_{j+1} \cup \ldots \cup \alpha_l$$

is said to be a *direct derivative* of (15). If α_j' coincides with one of the formulas $\alpha_1, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_l$, then the disjunction

(18) $$\alpha_1 \cup \ldots \cup \alpha_{j-1} \cup \alpha_{j+1} \cup \ldots \cup \alpha_l$$

is said to be a *direct derivative* of (15). Note that each direct derivative of (15) is a Herbrand disjunction for α.

A formula δ is said to be a *derivative* of a Herbrand disjunction (15) for α if there exists a sequence $\delta_1, \ldots, \delta_q$ of formulas such that δ_1 is identical with (15), δ_q is identical with δ, and δ_{j+1} is a direct derivative of δ_j ($j = 1, \ldots, q-1$). Then δ is also a Herbrand disjunction for α.

A Herbrand disjunction (15) for α is said to be *reducible* if there exists an integer j such that α_j is of form (13) with $r \geqslant 1$, and

a) all the individual variable $x_{r1}, \ldots, x_{r,n_r}$ are distinct from one another,

b) all the individual variables are distinct from all the individual variables x_{ij} where $j = 1, \ldots, n_i$ and $i = 0, \ldots, r-1$, and are distinct from all the individual variables appearing in all the formulas $\alpha_1, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_l$ and terms $\tau_1, \ldots, \tau_r$.

(xvi) *Suppose that γ is a closed formula, δ is a reducible Herbrand disjunction for α, and $\gamma \to \delta$ is a tautology. Then there exists a direct derivative δ' of δ such that $\gamma \to \delta'$ is also a tautology.*

Suppose that δ is of form (15) and that α_j satisfies conditions a) and b). It follows from a) and b) and from the rule of introduction of the universal quantifiers that the following formula is a tautology:

$$\gamma \to (\alpha_1 \cup \ldots \cup \alpha_{j-1} \cup \alpha_j'' \cup \alpha_{j+1} \cup \ldots \cup \alpha_l),$$

where α_j'' is the formula

$$\bigcap_{\boldsymbol{\eta}_{r+1}} \bigcup_{\boldsymbol{\xi}_{r+1}} \bigcap_{\boldsymbol{\eta}_{r+1}} \ldots \bigcup_{\boldsymbol{\xi}_k} \bigcap_{\boldsymbol{\eta}_k} \beta(\boldsymbol{x}, \boldsymbol{\tau}_1, \boldsymbol{x}_1, \ldots, \boldsymbol{\tau}_{r-1}, \boldsymbol{x}_{r-1}, \boldsymbol{\tau}_r, \boldsymbol{\eta}_r, \boldsymbol{\xi}_{r+1}, \boldsymbol{\eta}_{r+1}, \ldots, \boldsymbol{\xi}_k, \boldsymbol{\eta}_k).$$

Since the implication $\alpha_j'' \to \alpha_j'$, where α_j' is defined by (16), is a tautology, the implication

$$\gamma \to \delta',$$

where δ' is the direct derivative (17) or (18) of δ, is also a tautology.

A Herbrand disjunction δ for α is said to be *hereditarily reducible* if δ and each of its derivatives are reducible or coincide with α.

(xvii) *Suppose that γ is a closed formula, δ is a hereditarily reducible Herbrand disjunction for α, and $\gamma \to \delta$ is a tatuology. Then the implication $\gamma \to \alpha$ is also a tautology.*

By (xvi) we can define, by induction, a sequence (δ_q) of Herbrand disjunctions for α such that δ_1 is identical with δ, δ_{q+1} is a direct derivative of δ_q and $\gamma \to \delta_q$ is a tautology ($q = 1, 2, \ldots$). We can always define δ_{q+1} provided δ_q is not identical with α. On the other hand, every sequence (δ_q) such that δ_{q+1} is a direct derivative of δ_q must be finite. Therefore our inductive definition has to stop at an integer, say q_0. Hence it follows that δ_{q_0} is identical with α. Thus $\gamma \to \alpha$ is a tautology.

A Herbrand disjunction (15) for a is said to be *proper* provided (15) is hereditarily reducible and all the formulas $a_1, \ldots, a_l$ are open (i. e. they are in Z_r).

(xviii) *If δ is a proper Herbrand disjunction for a, then $\bar{\delta} \to a$ is a tautology, $\bar{\delta}$ denoting the closure of δ.*

(xviii) follows immediately from (xvii) where γ is the closed formula $\bar{\delta}$.

For any set A and an integer m, A^m will denote the Cartesian product $A \times \ldots \times A$ m-times. By definition (see p. 178)

$$\boldsymbol{x}_i \in V^{n_i} \quad (i = 0, 1, \ldots, k), \qquad \boldsymbol{\tau}_i \in T^{m_i} \quad (i = 1, \ldots, k).$$

The letter Φ_i (Φ_i^*) will denote the set of all mappings

$$\boldsymbol{f}_i \text{ from } T^{m_1} \times \ldots \times T^{m_i} \text{ into } V^{n_i} \text{ (into } T^{n_i}) \qquad (i = 1, \ldots, k).$$

By definition, $\Phi_i \subset \Phi_i^*$.

Let

$$\boldsymbol{f}_i \in \Phi_i \quad (i = 1, \ldots, k) \tag{19}$$

be given functions. By a *Herbrand $(\boldsymbol{f}_1, \ldots, \boldsymbol{f}_k)$-disjunction* for a we shall understand any disjunction of a finite number of formulas of the form

$$\beta\big(\boldsymbol{x}_0, \boldsymbol{\tau}_1, \boldsymbol{f}_1(\boldsymbol{\tau}_1), \boldsymbol{\tau}_2, \boldsymbol{f}_2(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2), \ldots, \boldsymbol{\tau}_k, \boldsymbol{f}(\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_k)\big). \tag{20}$$

(xix) *Let $\mathscr{T}$ be an open theory such that $\overline{\overline{V}} = \overline{\overline{T}}$. If a is a theorem in $\mathscr{T}$, then for given functions* (19) *there exists a Herbrand $(\boldsymbol{f}_1, \ldots, \boldsymbol{f}_k)$-disjunction which is a theorem in $\mathscr{T}$.*

Since a is a theorem in $\mathscr{T}$, we have

$$\mathscr{K}(\mathscr{T}) \subset \|a\|.$$

By (5) and the distributive laws for sets,

$$\|a\| = \bigcup_{\boldsymbol{\tau}_1 \in T^{m_1}} \bigcap_{\boldsymbol{\sigma}_2 \in T^{m_2}} \ldots \bigcup_{\boldsymbol{\tau}_k \in T^{m_k}} \bigcap_{\boldsymbol{\sigma}_k \in T^{m_k}} \|\beta(\boldsymbol{x}_0, \boldsymbol{\tau}_1, \boldsymbol{\sigma}_1, \ldots, \boldsymbol{\tau}_k, \boldsymbol{\sigma}_k)\|$$

$$= \bigcap_{\boldsymbol{f}_1 \in \Phi_1^*} \ldots \bigcap_{\boldsymbol{f}_k \in \Phi_k^*} \bigcup_{\boldsymbol{\tau}_1 \in T^{m_1}} \ldots \bigcup_{\boldsymbol{\tau}_k \in T^{m_k}} \|\beta(\boldsymbol{x}_0, \boldsymbol{\tau}_1, \boldsymbol{f}_1(\boldsymbol{\tau}_1), \boldsymbol{\tau}_2, \boldsymbol{f}_2(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2), \ldots$$
$$\ldots, \boldsymbol{\tau}_k, \boldsymbol{f}_k(\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_k))\|.$$

Hence it follows that, for the functions $\boldsymbol{f}_1, \ldots, \boldsymbol{f}_k$ mentioned in (xix),

$$\mathscr{K}(\mathscr{T}) \subset \bigcup_{\boldsymbol{\tau}_1 \in T^{m_1}} \ldots \bigcup_{\boldsymbol{\tau}_k \in T^{m_k}} \|\beta(\boldsymbol{x}_0, \boldsymbol{\tau}_1, \boldsymbol{f}_1(\boldsymbol{\tau}_1), \boldsymbol{\tau}_2, \boldsymbol{f}_2(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2), \ldots$$
$$\ldots, \boldsymbol{\tau}_k, \boldsymbol{f}_k(\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_k))\|. \tag{21}$$

Since $\mathscr{K}(\mathscr{T})$ is a closed subset of the compact space $\mathscr{L}$ (see (xi)) and all the sets $\|\beta(\boldsymbol{x}_0, \ldots)\|$ are clopen, $\mathscr{K}(\mathscr{T})$ is contained in a finite union of sets on the right side of inclusion (21). In other words, there exists a Herbrand $(\boldsymbol{f}_1, \ldots, \boldsymbol{f}_k)$-disjunction δ such that $\mathscr{K}(\mathscr{T}) \subset \|\delta\|$. This implies by (xiii) that δ is a theorem in $\mathscr{T}$.

(xx) *If the sets V of all free individual variables and T of all terms have the same cardinal, then there exists functions* (19) *such that every Herbrand $(\boldsymbol{f}_1, \ldots, \boldsymbol{f}_k)$-disjunction for a is proper.*

Let $\mathfrak{m}$ be the common cardinal of T and V. Let V_{ip} ($p = 1, \ldots, n_i$; $i = 1, \ldots, k$) be disjoint subsets of the set $V - \{x_{01}, \ldots, x_{0n_0}\}$ each of which has the cardinal $\mathfrak{m}$. Let $<$ be a well-ordering relation in V such that the ordinal of $(V, <)$ is the smallest ordinal of the power $\mathfrak{m}$. By transfinite induction we define a one-to-one mapping f_{ip} from $T^{m_1} \times \ldots \times T^{m_i}$ into V_{ip} such that, for all $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_i$, the free individual variable $f_{ij}(\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_i)$ is greater (in the ordering $<$) than all free individual variables appearing in $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_i$.

The mappings

$$\boldsymbol{f}_i(\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_i) = \big(f_{i1}(\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_i), \ldots, f_{in_i}(\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_i)\big)$$

(where $i = 1, \ldots, k$) have the required property. In fact, $\boldsymbol{f}_i$ maps $T^{m_1} \times \ldots \times T^{m_i}$ into V^{n_i}. Since every derivative of a Herbrand $(\boldsymbol{f}_1, \ldots, \boldsymbol{f}_k)$-disjunction is a disjunction

$$a_1 \cup \ldots \cup a_l, \tag{22}$$

where each of the formulas $a_1, \ldots, a_l$ is of the form

$$\bigcup_{\xi_{r+1}} \bigcap_{\eta_{r+1}} \ldots \bigcup_{\xi_k} \bigcap_{\eta_k} \beta\big(x_0, \boldsymbol{\tau}_1, \boldsymbol{f}_1(\boldsymbol{\tau}_1), \ldots, \boldsymbol{\tau}_r, \boldsymbol{f}_r(\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_r),$$
$$\boldsymbol{\xi}_{r+1}, \boldsymbol{\eta}_{r+1}, \ldots, \boldsymbol{\xi}_k, \boldsymbol{\eta}_k\big), \tag{23}$$

in order to complete the proof it suffices to show that every disjunction (22) of formulas of form (23) is reducible (except the case where (22) is composed only of the formula a). Consider the set of all free individual variables which appear in τ_i in formulas (23) in disjunction (22). Take the greatest element in this set, say x. From all the formulas of (23) (in disjunction (22)) which contain x in some terms $\boldsymbol{\tau}_i$, take one, with a possibly great r. Let a_j be this formula and let (23) be the representation of a_j. Let $\boldsymbol{f}_r(\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_r) = (x_{r1}, \ldots, x_{rn_r})$, i. e. $x_{rp} = f_{rp}(\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_r)$. It follows directly from the definition of f_{ip} that the formula a_j and the variables $x_{r1}, \ldots, x_{rn_r}$ satisfy conditions a) and b) on p. 179.

(xxi) *Let $\mathscr{T}$ be an open theory. If a is a theorem in $\mathscr{T}$, then a proper Herbrand disjunction for a is also a theorem in $\mathscr{T}$.*

Consider first the case where the set of all terms and the set of all individual variables are countable. Let (19) be some functions such that every Herbrand $(\boldsymbol{f}_1, \ldots, \boldsymbol{f}_k)$-disjunction is proper (see (xx)). Apply theorem (xix) to these functions $\boldsymbol{f}_1, \ldots, \boldsymbol{f}_k$. By (xix) there exists a Herbrand $(\boldsymbol{f}_1, \ldots, \boldsymbol{f}_k)$-disjunction which is a theorem in $\mathscr{T}$. This disjunction has all the required properties.

Suppose now that the cardinals of V and T are arbitrary. If α is a theorem in $\mathcal{T}$, there exists an open subtheory $\mathcal{T}_0$ of $\mathcal{T}$ such that α is a theorem in $\mathcal{T}_0$, and the sets of all terms and individual variables in $\mathcal{T}_0$ are countable. By the part of (xxi) which has just been proved, there exists a proper Herbrand disjunction δ for α such that δ is a theorem in $\mathcal{T}_0$. Since $\mathcal{T}_0$ is a subtheory of $\mathcal{T}$, δ is also a theorem in $\mathcal{T}$.

(xxii). *Let $\mathcal{T}$ be an open theory. A formula α is a theorem in $\mathcal{T}$ if and only if a proper Herbrand disjunction for α is a theorem in $\mathcal{T}$.*

This follows immediately from (xviii) and (xxi).

(xxiii). *In order that a theory $\mathcal{T}$ be open it is necessary and sufficient that, for every formula α (in the prenex form (12)), α be a theorem in $\mathcal{T}$ if and only if a proper Herbrand disjunction for α is a theorem in $\mathcal{T}$.*

The necessity follows from (xxii). To prove the sufficiency let us associate with every theorem α in $\mathcal{T}$ a proper Herbrand disjunction δ_α which is also a theorem in $\mathcal{T}$. By (xviii) the implication $\bar{\delta}_\alpha \to \alpha$ is a tautology. This proves that the set of all open formulas δ_α is a set of axioms for $\mathcal{T}$. Thus $\mathcal{T}$ is open.

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A KIND OF CATEGORICITY

BY

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The notion of categoricity has been introduced in order to characterize theories which intentionally have only one model. However, the most elaborated formalization of this notion (categoricity in power introduced by Łoś [2] and Vaught [4]) does not correspond to these intuitions. The arithmetic of natural numbers intentionally related to one model is not categorical in any power. The same can be said about the complete theory of real numbers. The aim of this paper is to define a notion of categoricity according to which the classical elementary theories of arithmetics and geometry (and not too many others) would be categorical.

1. *DEFINITIONS*

Let $\mathrm{Cn}(X)$ be the notion of consequence based on the first order functional calculus. Let $\{A_\Phi\}$ and $\{G_\Phi\}$ be two sequences of constants indexed by the formulas. We define the Skolem forms of a set X ($\mathrm{skl}(X)$) of formulas in the normal prenex form.

If Φ is a formula in the normal prenex form, then

$$\mathrm{skl}'(\Phi) = \begin{cases} \Psi(A_\Phi) \text{ if } \Phi \text{ has the shape } \bigvee x_v \Psi(x_v), \\ \bigwedge x_{k_1}, \ldots, x_{k_n} \Psi(G_\Phi(x_{k_1}, \ldots, x_{k_n})) \text{ if } \Phi \text{ has the shape } \\ \qquad\qquad \bigwedge x_{k_1}, \ldots, x_{k_n} \bigvee x_v \Psi(x_v), \\ \Phi \text{ in other cases.} \end{cases}$$

$\mathrm{skl}(\Phi) = \mathrm{skl}'^n(\Phi)$ for such n that $\mathrm{skl}'^n(\Phi) = \mathrm{skl}'^{n+1}(\Phi)$, $\mathrm{skl}(X)$ $=$ the set of $\mathrm{skl}(\Phi)$ for $\Phi \epsilon X$.

Let X be a set of sentences (formulas without free variables) with extralogical constants: $O_1, \ldots, O_k$ (individual constants), $P_1, \ldots, P_n$ (predicates) and $F_1, \ldots, F_m$ (function-constants).