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ON THE DEFINITION OF ALGEBRAIC OPERATIONS IN FINITARY ALGEBRAS

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1. Let A be a (empty or non-empty [1]) set. We deal with *finitary* [2] *operations* f in A, i. e. with one-valued mappings from some A^k into A, A^k being the set of all k-tuples $(x_1, \ldots, x_k)$ of elements $x_\varkappa \in A$. The natural number k is called the *type* of the operation [3]. Like many authors [4], we allow k to assume the value 0; as there is one and only one 0-tuple (namely the empty set), the operations of type 0, which may be called *constants* [5], are in a natural one-one correspondence with the elements of A.

There are two methods of compounding such finitary operations in A to new ones. A very well-known method is that one which might be called *direct composition* or *composition on equal arguments*. Let g be an operation of type k and let $h_1, \ldots, h_k$ be operations of type l; then by

$$g^l(h_1, \ldots, h_k)(x_1, \ldots, x_l) = g\big(h_1(x_1, \ldots, x_l), \ldots, h_k(x_1, \ldots, x_l)\big), \tag{1}$$

we get an operation $g^l(h_1, \ldots, h_k)$ [6] of type l in A; thus, g^l may be considered as an operation of type k in the set

$$O^{(l)}A = A^{(A^l)}$$

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(1) Some authors prefer to exclude the empty set as a fundamental set of a general algebra. This will unavoidably lead to difficulties in the theory of subalgebras of algebraic systems: in many cases, one is unavoidably forced to consider the empty subset as a subalgebra.

(2) Birkhoff [2], p. 312; [3], p. vii.

(3) Birkhoff [1], p. 439, index of the operation.

(4) Cf. e. g. Birkhoff [2], p. 311.

(5) These constants occur very often in algebra and its applications.

(6) Most frequent notation: $g(h_1, \ldots, h_k)$. Marczewski [6], p. 47, introduces the symbol $\hat{g}$ to make a distinction between the operation g in A and the operation induced by g in $O^{(l)}(A)$; he allows the sign $\wedge$ to be omitted if no confusion can arise.

of all operations of type l in A. By identifying operations of type 0 and elements of the fundamental set, definition (1) will get the form

$$g^l(x_1, \ldots, x_l) = g$$

in the case $k = 0$. (The reader is asked to discuss the special case $k = 0$ for himself at all forthcoming similar occasions.)

Another method not so frequently used is that one which might be called *tensorial composition* or *composition on different arguments*. Let g be an operation of type k in A and let $h_1, \ldots, h_k$ be operations of the (not necessarily equal) types $l_1, \ldots, l_k$ respectively; then by

$$(2) \quad g^\infty(h_1, \ldots, h_k)(x_{11}, \ldots, x_{1l_1}, \ldots, x_{k1}, \ldots, x_{kl_k}) \\ = g\big(h_1(x_{11}, \ldots, x_{1l_1}), \ldots, h_k(x_{k1}, \ldots, x_{kl_k})\big)$$ [7],

we get an operation $g^\infty(h_1, \ldots, h_k)$ of type $l = l_1 + \ldots + l_k$ in A; thus, g^∞ may be regarded as an operation of type k in the set

$$\boldsymbol{O}(A) = \bigcup_{l=0}^{\infty} \boldsymbol{O}^{(l)}(A)$$

of all finitary operations in A.

There are some especially important operations of type $l \geqslant 1$: the trivial or identity operations [8]

$$(3) \qquad e_\lambda^l(x_1, \ldots, x_l) = x_\lambda$$

($\lambda = 1, \ldots, l$); let $\boldsymbol{E}^{(l)}$ be the set of these operations. Then $\boldsymbol{E}^{(1)}$ consists of the identical mapping

$$(4) \qquad e = e_1^1$$

of A into itself only. From this simplest operation, the set

$$\boldsymbol{E} = \bigcup_{l=1}^{\infty} \boldsymbol{E}^{(l)}$$

of all identity operations can be derived by means of a further method of generating new operations, namely the method of *general transformation of variables*.

(7) Cf. Sierpiński [8], p. 169; Birkhoff [3], p. viii, (3).

(8) Marczewski [6], p. 46; in [5], p. 732, as in McKinsey-Tarski [7], p. 160, Birkhoff [2], p. 321: identity functions.

Let h be any operation of type k in A, l a natural number and σ a one-valued mapping from the set $\{1, 2, \ldots, k\}$ into the set $\{1, 2, \ldots, l\}$ [9], then by

$$(5) \qquad h_{\sigma,l}(x_1, \ldots, x_l) = h(x_{\sigma(1)}, \ldots, x_{\sigma(k)})$$ [10],

we get an operation $h_{\sigma,l}$ of type l in A. One might say that σ induces a mapping from $\boldsymbol{O}^{(k)}(A)$ into $\boldsymbol{O}^{(l)}(A)$. There are three important special cases [11] of this general transformation of variables:

1. σ injective (one-one), but not projective (onto), therefore necessarily $l > k$: then the transition from h to $h_{\sigma,l}$ (eventually for isotone σ only) is called *introduction of new variables*;

2. σ bijective, i. e. injective and projective, therefore necessarily $l = k$: *permutation of variables*;

3. σ projective, but not injective, therefore necessarily $l < k$: then the transition from h to $h_{\sigma,l}$ (eventually for isotone σ only) is called *identification of variables*.

Now we have the easy

THEOREM 1. *The set* $\boldsymbol{E}$ *of all identity operations is the smallest set* $\boldsymbol{H}' \subseteq \boldsymbol{O}(A)$ *such that*

$$\begin{cases} e \in \boldsymbol{H}', \\ \textit{if } h \in \boldsymbol{H}', \textit{ then } h_{\sigma,l} \in \boldsymbol{H}' \end{cases}$$ [12].

The most simple proof is obtained at once by means of the equation

$$(6) \qquad (e_\varkappa^k)_{\sigma,l} = e_{\sigma(\varkappa)}^l$$

and its special case $k = \varkappa = 1$,

$$(7) \qquad e_\lambda^l = e_{\sigma,l}$$

where $\sigma(1) = \lambda$.

2. Let $(f_i)_{i \in I}$ be a family of operations of types k_i in A; then we call $\big(A, (f_i)_{i \in I}\big)$ an *algebra of type* $(k_i)_{i \in I}$ [13], the operations f_i being called the *primitive* or *fundamental operations* [14] of this algebra. Let $\boldsymbol{F}$ be the set

(9) In the case $k = 0$, σ is the empty mapping, l an arbitrary natural number $\geqslant 0$, whereas in the case $k \geqslant 1$ one has necessarily $l \geqslant 1$.

(10) Cf. Marczewski [6], § 1.2, (viii) ((iv)—(vii) as special cases).

(11) Cf. Marczewski [4], § 4; [5], § 1, (ii)—(iv); [6], § 1.2, (iv), (v). Permutation and identification of variables are also used by Sierpiński [8].

(12) Here it will be sufficient to regard strictly isotone transformations σ (introduction of new variables) only.

(13) Birkhoff [1], p. 439: algebra of species $(k_i)_{i \in I}$.

(14) Marczewski [4]: primitive; [5], [6]: fundamental.

of these operations. The subject of this note is to give different equivalent definitions of the so-called algebraic operations derived from $\boldsymbol{F}$ by the methods described above. We start with

THEOREM 2. *The smallest set* $\boldsymbol{H}' \subseteqq \boldsymbol{O}(A)$ *such that*

$$\mathrm{I}\begin{cases} e \in \boldsymbol{H}', \\ \text{if } f \in \boldsymbol{F} \text{ and } h_1, \ldots, h_k \in \boldsymbol{H}', \text{ then } f^{\infty}(h_1, \ldots, h_k) \in \boldsymbol{H}' \end{cases}$$

is equal to the smallest set $\boldsymbol{H}' \subseteqq \boldsymbol{O}(A)$ *such that*

$$\mathrm{II}\begin{cases} \boldsymbol{F} \cup \{e\} \subseteqq \boldsymbol{H}', \\ \text{if } g, h_1, \ldots, h_k \in \boldsymbol{H}', \text{ then } g^{\infty}(h_1, \ldots, h_k) \in \boldsymbol{H}' \text{ (}^{15}\text{)}. \end{cases}$$

Proof. Let $\boldsymbol{H}_{\mathrm{I}}$, $\boldsymbol{H}_{\mathrm{II}}$ be the smallest $\boldsymbol{H}'$ having the properties I or II respectively. Due to the implication II $\to$ I, we have $\boldsymbol{H}_{\mathrm{I}} \subseteqq \boldsymbol{H}_{\mathrm{II}}$. We shall show that $\boldsymbol{H}' = \boldsymbol{H}_{\mathrm{I}}$ has properties II. First, because of

$$f = f^{\infty}(\underbrace{e, \ldots, e}_{k \text{ times}}) \tag{8}$$

(k = type of f), any fundamental operation f belongs to $\boldsymbol{H}_{\mathrm{I}}$: $\boldsymbol{H}_{\mathrm{I}}$ has the first of the two properties II. We prove the second by "induction on g", namely by the inductive method connected with the definition of $\boldsymbol{H}_{\mathrm{I}}$ [16]. Because of

$$e^{\infty}(h) = h, \tag{9}$$

the identity operation e is one of these g's. Assume $g_1, \ldots, g_k$ (of types $l_1, \ldots, l_k$ respectively) to be such g's; we show $g = f^{\infty}(h_1, \ldots, k_k)$, where $f \in \boldsymbol{F}$ (of type k), also to be such g. For this purpose, let $h_{11}, \ldots, h_{1l_1}, \ldots, h_{k1}, \ldots, h_{kl_k} \in \boldsymbol{H}' = \boldsymbol{H}_{\mathrm{I}}$. By inductive assumption, $h_{\varkappa} = g_{\varkappa}^{\infty}(h_{\varkappa 1}, \ldots, h_{\varkappa l_{\varkappa}})$ ($\varkappa = 1, \ldots, k$) belongs to $\boldsymbol{H}_{\mathrm{I}}$; therefore

$$g^{\infty}(h_{11}, \ldots, h_{1l_1}, \ldots, h_{k1}, \ldots, h_{kl_k}) = f^{\infty}(h_1, \ldots, h_k) \tag{10}$$

also belongs to $\boldsymbol{H}_{\mathrm{I}}$.

We write $\boldsymbol{H}_{\mathrm{I}} = \boldsymbol{H}_{\mathrm{II}} = \boldsymbol{H}_0$ and call the operations $h \in \boldsymbol{H}_0$ the *algebraic operations* (of algebra $(A, (f_i)_{i \in I})$) *in the narrower sense*. The definition

[15] For the sake of simplicity conditions of this kind will be understood to hold as far as the conclusion makes sense; thus, f is tacitly understood to be of type k here.

[16] There seems to be no logical or practical need of introducing numbers of "rank", "degree", or "order" (as has been done by several authors, e. g. Birkhoff [1], p. 439; [2], p. 312; [3], p. viii; McKinsey-Tarski [7], p. 162; Marczewski [6], p. 47 f.) in order to reduce this "algebraic" induction to ordinary complete induction on the rank numbers (besides, complete induction on natural numbers is a special case of general "algebraic" induction only).

$\boldsymbol{H}_0 = \boldsymbol{H}_{\mathrm{I}}$ might be interpreted: $\boldsymbol{H}_0$ is the subalgebra generated by $\boldsymbol{E}^{(1)}$ of algebra $(\boldsymbol{O}(A), (f_i^{\infty})_{i \in I})$ (of the same type $(k_i)_{i \in I}$ as the original algebra $(A, (f_i)_{i \in I})$). In particular, all fundamental operations f belong to $\boldsymbol{H}_0$. One notices that

(i) if the types of all fundamental operations are > 0, then so are all $h \in \boldsymbol{H}_0$;

(ii) if the types of all fundamental operations are > 1, then so are the $h \in \boldsymbol{H}_0$, $h \neq e$;

(iii) in general, any operation $h \in \boldsymbol{H}_0$, $h \neq e$, may be represented in the form

$$h = f^{\infty}(h_1, \ldots, h_k)$$

such that $f \in \boldsymbol{F}$, $h_1, \ldots, h_k \in \boldsymbol{H}_0$.

For $\boldsymbol{H}'$, consisting of all operations $h \in \boldsymbol{O}(A)$ of type > 0, or of e and all operations $h \in \boldsymbol{O}(A)$ of type > 1, or of e and all operations $f^{\infty}(h_1, \ldots, h_k)$ such that $f \in \boldsymbol{F}$, $h_1, \ldots, h_k \in \boldsymbol{H}_0$ respectively, has properties I of the theorem. Therefore, if all fundamental operations are of type > 1, then

(i) there is no $h \in \boldsymbol{H}_0$ of type 0;

(ii) there is precisely one $h \in \boldsymbol{H}_0$ of type 1, namely e;

(iii) the operations $h \in \boldsymbol{H}_0$ of types $l \geqslant 2$ are just those of the form

$$h = f^{\infty}(h_1, \ldots, h_k)$$

such that $f \in \boldsymbol{F}, h_1, \ldots, h_k \in \boldsymbol{H}_0$, of types $l_1, \ldots, l_k < 1$.

Thus, in this special case we are able to prove theorems on algebraic operations in the narrower sense by ordinary *induction on their type numbers* [17]. For instance, let there be one primitive operation f only, the type of f being 2. Then in general, there are 2 algebraic operations in the narrower sense of type 3, namely

$$f^{\infty}(f, e)\colon (x_1, x_2, x_3) \to f(f(x_1, x_2), x_3),$$
$$f^{\infty}(e, f)\colon (x_1, x_2, x_3) \to f(x_1, f(x_2, x_3)).$$

In a similar manner, the operations $h \in \boldsymbol{H}_0^{(l)}$, i. e. the $h \in \boldsymbol{H}_0$ of type l, correspond (possibly one-one) roughly speaking to the different ways of bracketing the "product" of l different arguments. Per definitionem, (A, f) is a semigroup if $\boldsymbol{H}_0^{(3)}$ consists of one element only. The *general associative law* might be given as an application of the above notions in the following manner: in a semigroup, for any $l \geqslant 1$, $\boldsymbol{H}_0^{(l)}$ consists of precisely one element; the proof will be obtained by induction on the type l.

[17] This induction on the type numbers being possible also in other cases.

3. In general algebra, one makes frequent use of a wider class of operations, as described in

THEOREM 3. *The following sets of operations are equal:*
the smallest set $\boldsymbol{H}' \subseteqq \boldsymbol{O}(A)$ *such that*

$$\mathrm{III}\begin{cases}\boldsymbol{E} \subseteqq \boldsymbol{H}',\\ \text{if } f \in \boldsymbol{F} \text{ and } h_1, \ldots, h_k \in \boldsymbol{H}', \text{ then } f^l(h_1, \ldots, h_k) \in \boldsymbol{H}';\end{cases}$$

the smallest set $\boldsymbol{H}' \subseteqq \boldsymbol{O}(A)$ *such that*

$$\mathrm{IV}\begin{cases}\boldsymbol{F} \cup \boldsymbol{E} \subseteqq \boldsymbol{H}',\\ \text{if } g, h_1, \ldots, h_k \in \boldsymbol{H}', \text{ then } g^l(h_1, \ldots, h_k) \in \boldsymbol{H}';\end{cases}$$

the smallest set $\boldsymbol{H}' \subseteqq \boldsymbol{O}(A)$ *such that*

$$\mathrm{V}\begin{cases}e \in \boldsymbol{H}',\\ \text{if } f \in \boldsymbol{F} \text{ and } h_1, \ldots, h_k \in \boldsymbol{H}', \text{ then } f^l(h_1, \ldots, h_k) \in \boldsymbol{H}',\\ \text{if } h \in \boldsymbol{H}', \text{ then } h_{\sigma,l} \in \boldsymbol{H}';\end{cases}$$

the smallest set $\boldsymbol{H}' \subseteqq \boldsymbol{O}(A)$ *such that*

$$\mathrm{VI}\begin{cases}\boldsymbol{F} \cup \{e\} \subseteqq \boldsymbol{H}',\\ \text{if } g, h_1, \ldots, h_k \in \boldsymbol{H}', \text{ then } g^l(h_1, \ldots, h_k) \in \boldsymbol{H}',\\ \text{if } h \in \boldsymbol{H}', \text{ then } h_{\sigma,l} \in \boldsymbol{H}';\end{cases}$$

the smallest set $\boldsymbol{H}' \subseteqq \boldsymbol{O}(A)$ *such that*

$$\mathrm{VII}\begin{cases}e \in \boldsymbol{H}',\\ \text{if } f \in \boldsymbol{F} \text{ and } h_1, \ldots, h_k \in \boldsymbol{H}', \text{ then } f^\infty(h_1, \ldots, h_k) \in \boldsymbol{H}',\\ \text{if } h \in \boldsymbol{H}', \text{ then } h_{\sigma,l} \in \boldsymbol{H}';\end{cases}$$

the smallest set $\boldsymbol{H}' \subseteqq \boldsymbol{O}(A)$ *such that*

$$\mathrm{VIII}\begin{cases}\boldsymbol{F} \cup \{e\} \subseteqq \boldsymbol{H}',\\ \text{if } g, h_1, \ldots, h_k \in \boldsymbol{H}', \text{ then } g^\infty(h_1, \ldots, h_k) \in \boldsymbol{H}',\\ \text{if } h \in \boldsymbol{H}', \text{ then } h_{\sigma,l} \in \boldsymbol{H}'.\end{cases}$$

Proof. First we show, for any $\boldsymbol{H}' \subseteqq \boldsymbol{O}(A)$,

$$(*) \qquad \mathrm{IV} \leftrightarrow \mathrm{VI} \leftrightarrow \mathrm{VIII} \rightarrow \mathrm{V} \leftrightarrow \mathrm{VII} \rightarrow \mathrm{III}.$$

The implication VIII → VII (as well as VI → V, IV → III) is trivial. V → III and VI → IV are immediate consequences of theorem 1. IV → VI is a consequence of the equation

$$(11) \qquad h_{\sigma,l} = h^l(e^l_{\sigma(1)}, \ldots, e^l_{\sigma(k)})$$

(k = type of h). The equivalences V ↔ VII and VI ↔ VIII will be immediate consequences of the following

LEMMA. *Let* $\boldsymbol{G}$, $\boldsymbol{H}'$ *be sets of operations,* $\boldsymbol{H}'$ *being closed with respect to the general transformation of variables. Then the following closure properties are equivalent:*

(a) *if* $g \in \boldsymbol{G}$ *and* $h_1, \ldots, h_k \in \boldsymbol{H}'$, *then* $g^l(h_1, \ldots, h_k) \in \boldsymbol{H}'$;

(b) *if* $g \in \boldsymbol{G}$ *and* $h_1, \ldots, h_k \in \boldsymbol{H}'$, *then* $g^\infty(h_1, \ldots, h_k) \in \boldsymbol{H}'$.

Proof of the lemma. (a) → (b): Let $g \in \boldsymbol{G}$ (of type k), and let $h_1, \ldots, h_k \in \boldsymbol{H}'$ (of types $l_1, \ldots, l_k$). By hypothesis, $(h_\varkappa)_{\sigma_\varkappa, l} \in \boldsymbol{H}'$, where $l = l_1 + \ldots + l_k$ and $\sigma_\varkappa(\lambda) = l_1 + \ldots + l_{\varkappa-1} + \lambda$ ($\varkappa = 1, \ldots, k$; $\lambda = 1, \ldots, l_\varkappa$); therefore,

$$(12) \qquad g^\infty(h_1, \ldots, h_k) = g^l\big((h_1)_{\sigma_1,l}, \ldots, (h_k)_{\sigma_k,l}\big)$$

belongs to $\boldsymbol{H}'$ too.

(b) → (a): Let $g \in \boldsymbol{G}$ (of type k) and let $h_1, \ldots, h_k \in \boldsymbol{H}'$ (of equal type l). By hypothesis, $g^\infty(h_1, \ldots, h_k)$ belongs to $\boldsymbol{H}'$; therefore,

$$(13) \qquad g^l(h_1, \ldots, h_k) = \big(g^\infty(h_1, \ldots, h_k)\big)_{\sigma,l},$$

where $\sigma\big((\varkappa-1)\cdot l + \lambda\big) = \lambda$ ($\varkappa = 1, \ldots, k$; $\lambda = 1, \ldots, l$) belongs to $\boldsymbol{H}'$ too.

Thus the lemma is proved, and we go on with the

Proof of theorem 3. We obtain V ↔ VII and VI ↔ VIII by setting $\boldsymbol{G} = \boldsymbol{F}$ and $\boldsymbol{G} = \boldsymbol{H}'$ respectively, completing the proof of (*). Now, let $\boldsymbol{H}_\varrho$ be the smallest set $\boldsymbol{H}' \subseteqq \boldsymbol{O}(A)$ such that the properties ϱ hold (ϱ = III, IV, V, VI, VII, VIII). From (*) we get

$$\boldsymbol{H}_{\mathrm{III}} \subseteqq \boldsymbol{H}_{\mathrm{V}} = \boldsymbol{H}_{\mathrm{VII}} \subseteqq \boldsymbol{H}_{\mathrm{IV}} = \boldsymbol{H}_{\mathrm{VI}} = \boldsymbol{H}_{\mathrm{VIII}}.$$

We shall have finished when we show that $\boldsymbol{H}' = \boldsymbol{H}_{\mathrm{III}}$ has properties IV. Because of

$$(14) \qquad f = f^k(e^k_1, \ldots, e^k_k)$$

(k = type of f), any fundamental operation f belongs to $\boldsymbol{H}_{\mathrm{III}}$: $\boldsymbol{H}_{\mathrm{III}}$ has the first of the two properties IV. We prove the second by "induction on g", namely by the inductive method connected with the definition of $\boldsymbol{H}_{\mathrm{III}}$. Because of

$$(15) \qquad (e^k_\varkappa)^l(h_1, \ldots, h_k) = h_\varkappa$$

($l =$ common type of the $h_\varkappa$'s), any identity operation $e_\varkappa^k$ is one of those g's. Assume $g_1, \ldots, g_k$ (of type l) to be such g's; we show $g = f^l(h_1, \ldots, h_k)$, where $f \in \boldsymbol{F}$ (of type k), to be such g too. For this purpose, let $h_1, \ldots, h_l$ (of type m) belong to $\boldsymbol{H}' = \boldsymbol{H}_{\mathrm{III}}$. By inductive assumption, $h_\varkappa^* = g_\varkappa^m(h_1, \ldots, h_l)$ ($\varkappa = 1, \ldots, k$) belongs to $\boldsymbol{H}_{\mathrm{III}}$; therefore,

$$g^m(h_1, \ldots, h_l) = f^m(h_1^*, \ldots, h_k^*) \tag{16}$$

belongs to $\boldsymbol{H}_{\mathrm{III}}$ too, completing the proof of theorem 3.

We write $\boldsymbol{H}_{\mathrm{III}} = \boldsymbol{H}_{\mathrm{IV}} = \boldsymbol{H}_{\mathrm{V}} = \boldsymbol{H}_{\mathrm{VI}} = \boldsymbol{H}_{\mathrm{VII}} = \boldsymbol{H}_{\mathrm{VIII}} = \boldsymbol{H}$ and call the operations $h \in \boldsymbol{H}$ the *algebraic operations* (18) (of algebra $(A\,(f_i)_{i \in I})$). The definition $\boldsymbol{H} = \boldsymbol{H}_{\mathrm{III}}$ might be interpreted: $\boldsymbol{H}^{(l)}$, the set of all algebraic operations of type l, is the subalgebra generated by $\boldsymbol{E}^{(l)}$ of algebra $\big(\boldsymbol{O}^{(l)}(A), (f_i^l)_{i \in I}\big)$ (19) (of the same type $(k_i)_{i \in I}$ as the original algebra $(A, (f_i)_{i \in I})$). Because of VII $\to$ I (VIII $\to$ II), all algebraic operations in the narrower sense, especially all fundamental operations, belong to $\boldsymbol{H}$. More precisely: one gets $\boldsymbol{H}$ from $\boldsymbol{H}_0$ by closing $\boldsymbol{H}_0$ with respect to the general transformation of variables. That this can be done in the most simple fashion, will be shown in

THEOREM 4. *The algebraic operations (in the general sense) are precisely those operations which can be obtained from suitable algebraic operations in the narrower sense h by suitable transformations of variables:*

$$\boldsymbol{H} = \{h_{\sigma,l} \mid h \in \boldsymbol{H}\ ,\ \sigma, l\}.$$

Proof. Owing to $\boldsymbol{H}_0 \subseteq \boldsymbol{H}$ and to the definition of $\boldsymbol{H}$, we have $h_{\sigma,l} \in \boldsymbol{H}$ for any $h \in \boldsymbol{H}_0$. We show that the set of operations $h_{\sigma,l}$, where $h \in \boldsymbol{H}_0$, has properties III. It has the first property: for, because of $e \in \boldsymbol{H}$ and according to (7), any identity operation e_λ^l is such an $h_{\sigma,l}$. It has the second property: for, if $f \in \boldsymbol{F}$ (of type k) and if $h_1, \ldots, h_k \in \boldsymbol{H}_0$ (of types $l_1, \ldots, l_k$),

(18) Marczewski [5], [6]; cf. also McKinsey-Tarski [7], p. 161, where in the special case of closure algebras these operations are called closure-algebraic functions. In these papers, the definition $\boldsymbol{H} = \boldsymbol{H}_{\mathrm{III}}$ has been used. Marczewski [4] essentially uses either $\boldsymbol{H} = \boldsymbol{H}_{\mathrm{VI}}$ or $\boldsymbol{H} = \boldsymbol{H}_{\mathrm{VIII}}$, leaving open the question in which sense "superposition" should be understood there. This coincides with the fact that, in general, the term "superposition" has been used in different meanings in literature; e. g. in Sierpiński [8], "superposition" seems to include transformation of variables, whereas in Marczewski [4] it does not. The result of Sierpiński [8] might be stated as follows: if $\boldsymbol{F} = \boldsymbol{O}^{(2)}(A)$, then, for any $l \geqslant 2$ (even $\geqslant 1$!), $\boldsymbol{O}^{(l)}(A) \subseteq \boldsymbol{H}$ (in the case A is infinite, Sierpiński even shows $\boldsymbol{O}^{(l)}(A) \subseteq \boldsymbol{H}_0$). Whether the "compound" operations of Birkhoff [2], p. 312 (cf. also p. 321, (11)), are our algebraic operations in the wider or in the narrower sense (or even something between the two) seems not to be perfectly clear.

(19) McKinsey-Tarski, loc. cit.

then for given transformations $\sigma_\varkappa$ ($\varkappa = 1, \ldots, k$) and for a given natural number m, every $(h_\varkappa)_{\sigma_\varkappa, m}$, ($\varkappa = 1, \ldots, k$) is such an $h_{\sigma,l}$; but because of $f^\infty(h_1, \ldots, h_k) \in \boldsymbol{H}_0$,

$$f^m\big((h_1)_{\sigma_1,m}, \ldots, (h_k)_{\sigma_k,m}\big) = \big(f^\infty(h_1, \ldots, h_k)\big)_{\sigma,m}, \tag{17}$$

where $\sigma(l_1 + \ldots + l_{\varkappa-1} + \lambda) = \sigma_\varkappa(\lambda)$ ($\varkappa = 1, \ldots, k$; $\lambda = 1, \ldots, l_\varkappa$), is also such an $h_{\sigma,l}$.

REFERENCES

[1] G. Birkhoff, *On the structure of abstract algebras*, Proceedings of the Cambridge Philosophical Society 31 (1935), p. 433-454.

[2] — *Universal algebra*, Proceedings of the First Canadian Mathematical Congress, Montreal 1945, p. 310-326.

[3] — *Lattice theory*, 2nd ed., New York 1948.

[4] E. Marczewski, *Sur les congruences et les propriétés positives d'algèbres abstraites*, Colloquium Mathematicum 2 (1951), p. 220-228.

[5] — *A general scheme of the notions of independence in mathematics*, Bulletin de l'Académie Polonaise des Sciences, Série Math., Astr. et Phys., 6 (1958), p. 731-736.

[6] — *Independence and homomorphisms in abstract algebras*, Fundamenta Mathematicae 50 (1961), p. 45-61.

[7] J. C. C. McKinsey and A. Tarski, *The algebra of topology*, Annals of Mathematics 45 (1944), p. 141-191.

[8] W. Sierpiński, *Sur les fonctions des plusieurs variables*, Fundamenta Mathematicae 33 (1945), p. 169-173.

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