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REMARKS ON DEPENDENCE RELATIONS AND CLOSURE OPERATORS

BY

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1. We adopt here for abstract dependence relations the terminology and notation of general closure operators, i. e. we say "x belongs to the closure of E" instead of "x depends on E" (see Bourbaki [4], p. 107, and Schmidt [12], p. 234-235).

Let C be a *closure operator* in a fixed set X, i. e. an extensive, monotone, and idempotent function which associates a subset $C(E)$ of X with each subset E of X (cf. e. g. Birkhoff [1], p. 49, Schmidt [11] and [12]).

We consider the following two properties of a closure operator (cf. e. g. Schmidt [12], p. 236-239):

Finite character: for each $E \subset X$

$$C(E) = \bigcup C(F),$$

where F runs over all finite subsets of E.

Exchange property: for each $x, y \in X$ and $E \subset X$, if $y \notin C(E)$ and $y \in C(E \cup \{x\})$ then $x \in C(E \cup \{y\})$.

A very important example of C is the case when X is an abstract algebra and $C(E)$ the subalgebra generated by $E \subset X$. If all operations of the algebra in question are finitary, then obviously C is of finite character. Moreover, in some algebras, e. g. in vector spaces, or more generally in so-called v-algebras and v^*-algebras (see Marczewski [7] and Narkiewicz [9]), the operator C has the exchange property.

The definition of a closure operator of finite character with the exchange property forms a basis for the theory of abstract linear dependence. All other approaches to this notion by Haupt, van der Waerden, Whitney, Haupt-Nöbeling-Pauc, Rado, etc., in papers quoted in [2], [5], and [12] are logically equivalent to it (only the "mehrstufige Austauschoperator" of Schmidt [12] is more general).

If $G \subset A \subset X$ and $C(G) \supset A$, we say that G is a set of *C-generators* of A (or else, that G is *C-dense* in A). We say that I is *C-independ-*

ent (1) (or *C-isolated*) if I is a minimal set of generators of $C(I)$ (Schmidt [11], p. 38). Two sets A an B are C-equivalent if $C(A) = C(B)$.

2. It is known (see Kertesz [5], Bleicher-Preston [3]; cf. also Bourbaki [2], p. 107) that some theorems on vector spaces, fields, etc. are special cases of the following proposition:

(o) If C is of finite character and has the exchange property, then any two minimal sets of C-generators of X and, more generally, any two C-independent and C-equivalent sets have the same cardinal number.

The main purpose of this paper is to remark that (o) may be decomposed into the following two propositions:

(i) *If C has the exchange property and one of two C-independent and C-equivalent sets is finite, then the other has the same cardinal number.*

(ii) *If C is of finite character and one of two C-independent and C-equivalent sets is infinite, then the other has the same cardinal number.*

Let us denote by $|S|$ the cardinal number of S.

Proposition (i) results from the following one:

(i′) *If C has the exchange property, I is a finite C-independent set, and $C(G) \supset I$, then $|G| \geqslant |I|$.*

The proof of (i′) is the same as in the case of ordinary linear independence; see e. g. Pickert [10], p. 66, propositions 1 and 3.

The proof of (ii) is simple; e. g. it suffices to apply the argument used in [8], p. 50.

3. The following examples show that *the hypotheses in* (i) *and* (ii) *cannot be exchanged.*

The first example shows that proposition (i) is false for the case of infinite sets: Let I denote the set of all irrational numbers, J — the set of all rational numbers and $X = I \cup J$. For any subset E of X we put $C(E) = X$ whenever

$$(*) \qquad \begin{cases} \text{either } I \setminus E \text{ is finite and } |E \setminus I| \geqslant |I \setminus E| \\ \text{or } \quad J \setminus E \text{ is finite and } |E \setminus J| \geqslant |J \setminus E|. \end{cases}$$

Otherwise we put $C(E) = E$.

It is easy to see that C is a closure operator. That C has the exchange property follows from the fact that, for any two elements x and y not belonging to E, if $E \cup \{x\}$ satisfies $(*)$, then $E \cup \{y\}$ also satisfies $(*)$.

(1) In the quoted case, when $C(E)$ is the subalgebra generated by a subset E of an algebra X, the C-independence (denoted in [8] by (G)) is weaker than the independence in the sense treated in [6], [7] and [8].

Nevertheless I and J are minimal sets of C-generators and have different cardinal numbers.

The second example shows that (ii) is false in the case of finite sets. Let $X = \{a, b, c\}$. Put $C(E) = X$ if $\{a, b\} \subset E$ or $\{c\} \subset E$ and $C(E) = E$ otherwise.

C is a closure operator and obviously is of finite character. $\{a, b\}$ and $\{c\}$ are minimal sets of C-generators and have different numbers of elements.

4. By the repeated application of (i′), proposition (i) can be strengthened as follows:

(i″) *If C has the exchange property and one (and, consequently, the second also) of two given C-independent C-equivalent sets is contained in the closure of a finite set, then the two given sets have the same finite number of elements.*

One naturally asks if the modifications of (ii), corresponding to (i′) and (i″) also hold, and in particular, whether — under the hypothesis of the finite character of C — all the sets of C-independent C-generators of a set having a C-independent infinite subset I have the same cardinal numbers, not smaller than $|I|$.

The following example shows the answer to be negative.

Let X be any infinite set. Let a, b, c be three distinct points of X. Define C as in the second example of section 3.

C is clearly a closure operator of finite character. Furthermore, X contains an infinite C-independent set $X \setminus \{a, b, c\}$ and has two finite sets $\{a\}$ and $\{b, c\}$ which are sets of C-independent C-generators and hence C-equivalent, but do not have the same cardinal.

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MAXIMALE UNTERSEMIGRUPPEN UND KONVEXITÄT IN GRUPPEN

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Die gewöhnliche Definition der Konvexität, hier *lineare Konvexität* genannt, läßt sich nicht direkt aus linearen Räumen auf Gruppen übertragen. Um eine solche Verallgemeinerung tatsächlich zu erhalten, kann man aber folgende Bemerkung benutzen:

In lokalkonvexen reellen linearen Räumen sind abgeschlossene konvexe Mengen Durchschnitte von abgeschlossenen Stützhalbräumen, können daher gruppentheoretisch und zwar als Durchschnitte von Translationen abgeschlossener maximaler Untersemigruppen gekennzeichnet werden.

Diese Bemerkung legt schon eine Definition der Konvexität in Gruppen nahe (s. 4). Die beiden Konvexitätsbegriffe, der übliche (lineare Konvexität) und der gruppentheoretische, stimmen für borelsche Mengen in lokalkonvexen linearen Räumen überein (s. 4). Es stellt sich weiter erwartungsgemäß heraus, daß in kompakten zusammenhängenden Gruppen keine konvexe borelsche Menge außer der Gruppe selbst vorhanden ist (s. 5).

Der gruppentheoretische Konvexitätsbegriff erlaubt die meisten Eigenschaften der konvexen Mengen auf diejenigen von maximalen Untersemigruppen zurückzuführen, es ist daher zweckmäßig diese Semigruppen näher zu betrachten.

Es ergibt sich, daß in topologischen Gruppen jede in bezug auf die inneren Automorphismen invariante (d. h. normale) maximale Untersemigruppe, die eine borelsche Menge ist, abgeschlossen sein muß (s. 3).

In Abschnitten 1 und 2 wird die Definition der maximalen Untersemigruppe M nebst ihren einfachsten Eingeschaften und einer vollen Charakterisierung der Faktorgruppe G/M^* angegeben, wo $M^* = M \cap M^{-1}$ gesetzt wird (Satz 1).

1. Maximale Untersemigruppen. Eine Untersemigruppe (USG) M einer abstrakten Gruppe G soll *maximal* heißen (max USG), wenn