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HEREDITY OF THE GENERALIZED THEOREM ON THREE CONTINUA

BY

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The theorem proved as a "théorème sur trois continus" by C. Kuratowski (1929) has subsequently been studied by E. Čech (1931), S. Eilenberg (1936), R. H. Bing (1946) and C. Kuratowski (1949).

Let us denote by E^2 the Euclidean plane and by S^2 the 2-dimensional sphere.

We say that the subset A of the space X *separates* the points p and q in X if the set $X-A$ is not connected between p and q, i. e., there exists a decomposition $X-A = M \cup N$ such that $p \in M$, $q \in N$ and $\overline{M} \cap N = 0 = M \cap \overline{N}$ (cf. [6], p. 89).

We say that the subset A of the space X *cuts* X between the points p and q of X if $p, q \in X-A$ and, for every connected closed set $C \subset X$ [1] such that $p, q \in C$, we have $C \cap A \neq 0$ (cf. [6], p. 129).

Evidently, if A separates p and q in X, then A cuts X between p and q, but not inversely.

The theorem originally proved by Kuratowski [4] follows:

THEOREM 1 (Kuratowski). *If C_1, C_2, C_3 are connected closed subsets of the plane E^2 such that $C_1 \cap C_2 \cap C_3 \neq 0$ and no set $C_i \cup C_j$ cuts E^2 between the points $p, q \in E^2$ $(i, j = 1, 2, 3;\ i \neq j)$, then the set $C_1 \cup C_2 \cup C_3$ does not cut E^2 between p and q.*

This theorem has later been generalized by Čech [2] and Eilenberg [3] as follows:

THEOREM 2 (Čech-Eilenberg). *If C_1, C_2, C_3 are connected subsets of the sphere S^2 such that $C_1 \cap C_2 \cap C_3 \neq 0$ and no set $C \cup C_j$ cuts S^2 between*

(1) The connected closed sets are sometimes called *continua*, whence the name of Kuratowski's Theorem is derived. But we do not use here the term "continua", reserving it for connected compact sets, according to the terminology noted in the paper [6].

the points $p, q \in S^2$ $(i, j = 1, 2, 3;\ i \neq j)$, *then the set* $C_1 \cup C_2 \cup C_3$ *does not cut* S^2 *between* p *and* q.

The question of Theorem 2 being in connection with the topological characterization of the sphere S^2 has recently been raised by B. Knaster. In view of our Theorem 6, the validity of Theorem 2 at any rate is not sufficient condition for a space to be topologically S^2. Namely, Theorem 2 constitutes a hereditary property of the space, for some class of spaces (see p. 77).

It may easily be proved by induction that Theorem 2, as well as Theorem 1, also holds for an arbitrary finite number of sets $C_1, \ldots, C_n$ instead of sets C_1, C_2, C_3. Theorem 1 in that form has been generalized, in another direction, by Bing [1] as follows:

THEOREM 3 (Bing). *If* $C_1, \ldots, C_n$ *are connected subsets of the plane* E^2 *such that* $C_1 \cap \ldots \cap C_n \neq 0$, *all but at most two of sets* $C_1, \ldots, C_n$ *are bounded and no set* $C_i \cup C_j$ *cuts* E^2 *between the points* $p, q \in E^2$ $(i, j = 1, \ldots, n;\ i \neq j)$, *then the set* $C_1 \cup \ldots \cup C_n$ *does not separate* p *and* q *in* E^2. *If, moreover, all sets* $C_1, \ldots, C_n$ *are bounded, then the set* $C_1 \cup \ldots \cup C_n$ *does not cut* E^2 *between* p *and* q.

Remark that Bing's proof of Theorem 3 can also be used, with a few inessential changes, as a short proof that Theorem 1 implies Theorem 2.

Theorem 2 is a particular case (for $n = 3$) of the following theorem proved by Kuratowski [5] (all indices being reduced mod n):

THEOREM 4 (Kuratowski). *If* $A_0, \ldots, A_{n-1}$ *are subsets of the sphere* S^2 $(n \geqslant 3)$ *such that each set* $C_i = A_{i+1} \cup \ldots \cup A_{i+n-2}$ *is connected,* $C_0 \cap \ldots \cap C_{n-1} \neq 0$ *and no set* $A_{i+1} \cup \ldots \cup A_{i+n-1}$ *cuts* S^2 *between the points* $p, q \in S^2$ $(i = 0, \ldots, n-1)$, *then the set* $A_0 \cup \ldots \cup A_{n-1}$ *does not cut* S^2 *between* p *and* q.

We shall show that Theorem 4 (thus also Theorem 1 and 2) may be more strongly formulated in such a way that a junction from p to q (i. e. a connected closed set containing p and q), which lies outside the set $A_0 \cup \ldots \cup A_{n-1}$, may already be found in the union of n of junctions from p to q, lying outside the sets $A_{i+1} \cup \ldots \cup A_{i+n-1}$, respectively, and being given by the hypotheses.

More precisely, denote by $T_n(X)$ the theorem obtained from Theorem 4 by putting X instead of S^2 and establishing an integer $n \geqslant 3$, for an arbitrary topological metrizable space X. Hence $T_3(S^2)$ or $T_n(S^2)$ is Theorem 2 or 4, respectively.

Our result is the following (all indices, except the index n of $T_n(X)$, being reduced mod n):

THEOREM 5. *Let* X *be a connected, locally connected and unicoherent space such that* $T_n(X)$ *holds for an integer* $n \geqslant 3$, *and let* $A_0, \ldots, A_{n-1}$ *and* $K_0, \ldots, K_{n-1}$ *be subsets of* X, *satisfying the following five conditions:*

(i) $C_i = A_{i+1} \cup \ldots \cup A_{i+n-2}$ *is a connected set for* $i = 0, \ldots, n-1$,

(ii) $C_0 \cap \ldots \cap C_{n-1} \neq 0$,

(iii) K_i *is a connected closed set for* $i = 0, \ldots, n-1$,

(iv) $K_i \subset X - (A_{i+1} \cup \ldots \cup A_{i+n-1})$ *for* $i = 0, \ldots, n-1$,

(v) $p, q \in K_0 \cap \ldots \cap K_{n-1}$.

Then there exists a connected closed set

$$K \subset (K_0 \cup \ldots \cup K_{n-1}) - (A_0 \cup \ldots \cup A_{n-1})$$

such that $p, q \in K$.

Proof. We assume that all indices in the sequel, except the index n of $T_n(X)$ and the indices being Greek letters, are reduced mod n. According to (iv), we have $K_i \cap A_j = 0$ for $i \neq j$, whence

$$A_i \subset X - (K_{i+1} \cup \ldots \cup K_{i+n-1})$$

for $i = 0, \ldots, n-1$. Denote by $\{R_\lambda^i\}$, where $\lambda \in \Lambda^i$, the collection of components of the set

$$X - (K_{i+1} \cup \ldots \cup K_{i+n-1}),$$

which intersect the set A_i, and put

$$G_i = \bigcup_{\lambda \in \Lambda^i} R_\lambda^i$$

for $i = 0, \ldots, n-1$. Therefore

$$A_i \subset G_i \subset X - (K_{i+1} \cup \ldots \cup K_{i+n-1}) \tag{1}$$

for $i = 0, \ldots, n-1$. Since the space X is locally connected, it follows from (iii) that

$$\begin{aligned} \mathrm{Fr}(G_i) &\subset \overline{\bigcup_{\lambda \in \Lambda^i} \mathrm{Fr}(R_\lambda^i)} \subset \mathrm{Fr}[X - (K_{i+1} \cup \ldots \cup K_{i+n-1})] \\ &= \mathrm{Fr}(K_{i+1} \cup \ldots \cup K_{i+n-1}) \subset K_{i+1} \cup \ldots \cup K_{i+n-1} \end{aligned} \tag{2}$$

(see [6], p. 168-169), and that each R_λ^i is an open set (see [6], p. 163).

Thus, each G_i is an open set, whence

$$X - (G_0 \cup \ldots \cup G_{n-1}) \text{ is a closed set.} \tag{3}$$

Moreover, the set

$$D_i = G_{i+1} \cup \ldots \cup G_{i+n-2}$$

is a union of connected sets R_λ^i, each of whose intersects the set

C_i, and $C_i \subset D_i$, by (i) and (1). Therefore D_i is, by (i), a connected set for $i = 0, \ldots, n-1$ (see [6], p. 82). Furthermore, it follows from (i), (ii) and (1) that

$$0 \neq C_0 \cap \ldots \cap C_{n-1} \subset D_0 \cap \ldots \cap D_{n-1}$$

and $G_i \cap K_j = 0$ for $i \neq j$. Hence

$$K_i \subset X - (G_{i+1} \cup \ldots \cup G_{i+n-1})$$

for $i = 0, \ldots, n-1$. Therefore, by (iii) and (v), no sum $G_{i+1} \cup \ldots \cup G_{i+n-1}$ cuts X between p and q $(i = 0, \ldots, n-1)$.

Now applying $T_n(X)$ for $A_i = G_i$ and $C_i = D_i$, $i = 0, \ldots, n-1$, we conclude that there exists a connected closed set H such that

(4) $$p, q \in H \subset X - (G_0 \cup \ldots \cup G_{n-1}).$$

Write shortly

$$K_* = K_0 \cup \ldots \cup K_{n-1}$$

and denote by $\{R_\lambda\}$, where $\lambda \in \Lambda$, the collection of components of the set $X - K_*$. Let

$$\Lambda_1 = \{\lambda\colon \lambda \in \Lambda,\ H \cap R_\lambda \neq 0\}, \quad \Lambda_2 = \Lambda - \Lambda_1$$

and consider the sets

$$U = H \cup \bigcup_{\lambda \in \Lambda_1} R_\lambda,$$

$$V = K_* \cup \bigcup_{\lambda \in \Lambda_2} R_\lambda.$$

We obviously have

(5) $$U \cup V = X \quad \text{and} \quad U \cap V = H \cap K_*.$$

The set U is evidently connected (see [6], p. 82) and the sets R_λ are open as components of the set $X - K_*$ which, according to (iii), is open (see [6], p. 163). Therefore none of the sets

$$\bigcup_{\lambda \in \Lambda_1} R_\lambda, \quad \bigcup_{\lambda \in \Lambda_2} R_\lambda$$

contains a limit point of the other. It follows (see [6], p. 83) that the set V is connected and closed, because the space X is connected and the set K_* is connected and closed, according to (iii) and (v).

By (5), $X = \bar{U} \cup V$ is thus a decomposition of the space into two connected closed subsets. Since the space X is unicoherent (see [6], p. 104), their common part

$$K = \bar{U} \cap V$$

is a connected closed set and $p, q \in K$, according to (v) and (4).

Now, let $\lambda \in \Lambda_1$. Since we have

$$0 \neq H \cap R_\lambda \subset R_\lambda - (G_0 \cup \ldots \cup G_{n-1}),$$

by (4), and

$$R_\lambda \cap \operatorname{Fr}(G_0 \cup \ldots \cup G_{n-1}) \subset R_\lambda \cap [\operatorname{Fr}(G_0) \cup \ldots \cup \operatorname{Fr}(G_{n-1})]$$
$$\subset R_\lambda \cap (K_0 \cup \ldots \cup K_{n-1}) = R_\lambda \cap K_* = 0,$$

by (2), we get (see [6], p. 80) the inclusion

$$R_\lambda \subset X - (G_0 \cup \ldots \cup G_{n-1}).$$

It follows, according to (4), that

$$U \subset X - (G_0 \cup \ldots \cup G_{n-1}),$$

whence we infer, by (1) and (3), that

$$K \subset \bar{U} \subset X - (G_0 \cup \ldots \cup G_{n-1}) \subset X - (A_0 \cup \ldots \cup A_{n-1}).$$

Finally, since H is a closed set and the space X is locally connected, we have

$$\bar{U} = H \cup \bigcup_{\lambda \in \Lambda_1} R_\lambda \cup \operatorname{Fr}\Big(\bigcup_{\lambda \in \Lambda_1} R_\lambda\Big) \subset U \cup \overline{\bigcup_{\lambda \in \Lambda_1} \operatorname{Fr}(R_\lambda)} \subset U \cup \overline{\operatorname{Fr}(X - K_*)}$$
$$= U \cup \operatorname{Fr}(K_*) \subset U \cup K_*,$$

according to (iii) (see [6], pp. 168-169). Hence we conclude, by (5), that

$$K = \bar{U} \cap V \subset (U \cup K_*) \cap V = (U \cap V) \cup (K_* \cap V)$$
$$= (H \cap K_*) \cup (K_* \cap V) \subset K_* = K_0 \cup \ldots \cup K_{n-1},$$

and Theorem 5 is thus proved.

It immediately implies

THEOREM 6. *If $n \geqslant 3$ is an integer, X is a connected, locally connected and unicoherent space such that $T_n(X)$ holds, and $A \subset X$, then $T_n(A)$ holds.*

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ON COMPACTIFICATIONS OF SOME SUBSETS OF EUCLIDEAN SPACES

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Let S_n be the unit sphere, i. e. the sphere with centre 0 and radius 1 in the $(n+1)$-dimensional Euclidean space E^{n+1}. I say that a set $X \subset S_n$ is *densely connected* in S_n if the set $R \cap X$ is connected for every connected open subset R of S_n. Obviously, each set in S_n (where $n = 0, 1, \ldots$) that is non-degenerate (i. e. containing at least two distinct points) and densely connected in S_n is dense in S_n, but not inversely.

THEOREM. *If a non-degenerate set* $X \subset S_n$ *is densely connected in* S_n $(n = 0, 1, \ldots)$, Y *is a compact metric space and* $h\colon X \to Y$ *is a homeomorphism such that* $\dim[Y - h(X)] \leqslant 0$, *then* $n \leqslant \dim Y$.

Proof. Let $p, q \in X$ and $p \neq q$. Since the sphere S_n is topologically homogeneous, we can assume that p, q are the poles p_N (north) and p_S (south) of S_n, respectively. The set $Y - h(X)$ being empty or 0-dimensional, there exists (see [3], p. 164) an open neighbourhood G of $h(p)$ in Y such that

$$\mathrm{Fr}(G) \subset h(X) \tag{1}$$

and $h(q) \in Y - \overline{G}$ (1). Then neither $h(p)$ nor $h(q)$ belongs to $\mathrm{Fr}(G)$ and so there are such sufficiently small open neighbourhoods P and Q of p and q in S_n, respectively, that

$$\mathrm{Fr}(G) \subset Y - [\overline{h(P \cap X)} \cup \overline{h(Q \cap X)}]. \tag{2}$$

The theorem being evidently true for $n = 0$, let us assume that $n > 0$ and denote by r the projection of $S_n - \{p_N, p_S\}$ onto the equator S_{n-1} of S_n along the meridians of S_n. Since r is a continuous mapping

(1) $\overline{G}$ and $\mathrm{Fr}(G)$ denote the closure and the boundary of G in Y, respectively. The notation from [3] and [4] is used throughout in this proof.