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A note on embedding problems

by

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1. Introduction. It is well known (ref. [2]) that not every semi-group with cancellation can be embedded in a group and not every ring without zero divisors can be embedded in a (skew) field. Satisfactory tests for the problem of embedding a semi-group in a group have been given by A. I. Malcev ([4]), J. Lambek ([1]), and D. Tamari ([9]) while the corresponding problem for rings in relation to skew fields has been settled only in particular cases. In the present paper we use an argument of the lower predicate calculus in order to prove the existence of a test of well-defined form for the latter problem (section 2). While our method does not yield explicit expressions for the test it does enable us to establish certain concrete algebraic results (section 3). We observe that our approach is distinct from that of Malcev ([5]), J. Łoś ([2]), and B. H. Neumann ([6], compare ref. [8]) who used the lower predicate calculus, or a fragment of it, essentially in order to establish that an algebraic structure will possesses certain properties if the properties in question are shared by all finitely generated subsystems of the given structure.

I am indebted to Michael Rabin and Dov Tamari for some stimulating discussions on the subject of the present paper.

2. The embedding test. A structure M is given by its set of individual constants, $A = \{a, b, c, \ldots\}$ and by the set of relations which are defined in M, $Q = \{R, S, T, \ldots\}$, such that for every element of Q, $R(x_1, \ldots, x_n)$ say, and for every n-ple of elements of P, $(a_1, \ldots, a_n)$, $R(a_1, \ldots, a_n)$ either holds or does not hold in M. The set of atomic sentences, $R(a_1, \ldots, a_n)$ etc., which hold in M is called the *positive diagram* of M and will be denoted by N^+ (see [7]). The set of negations of atomic sentences, $\sim R(a_1, \ldots, a_n)$ etc. for $R \in Q$ and $a_i \in P$, $i = 1, \ldots, n$, such that $R(a_1, \ldots, a_n)$ does not hold in M is called the *negative diagram* of M, N^-. The union of N^+ and N^-, $N = N^+ \cup N^-$ is the *diagram* of M. Equality is regarded as an ordinary two-place relation $E(x, y)$ such that the characteristic properties of equality (reflexity, transitivity, symmetry, substitutivity) are satisfied or (in the appropriate context) are laid down by axioms. There is no need to introduce symbols (functors) for algebraic operations. Thus we may formulate a set of axioms K_0 for the concept of a general ring without zero divisors in terms of the relations of equality $E(x,y)$ (i.e. $x=y$), addition $S(x,y,z)$ (i.e. $x+y=z$) and multiplication $P(x,y,z)$ i (i.e. $xy=z$), together with a single individual constant, 0, which stands for the neutral element with respect to addition. For instance the axiom for the absence of zero-divisors may be formulated as

$$(x)(y)[P(x,y,0)\supset E(x,0)\vee E(y,0)]$$

Let K be a consistent set of sentences which contain the relations and the individual constant that occur in K_0 such that K entails K_0, $K\vdash K_0$. K may contain additional relations but it is a convenient and inessential limitation to suppose that K does not contain any additional individual constants. Given any model M of K_0 (i.e. any ring without zero divisors) we may ask whether M is embeddable in a model M' of K. If K is a set of axioms for the concept of a skew field this is the problem discussed in the introduction.

A necessary and sufficient conditions for a model M' of K to be an extension of M is that the sentences of the diagram N of M hold in N'. Hence M is embeddable in a model M' of K if and only if $K\cup N$ is consistent. If $K\cup N$ is is contradictory then there exist finite subsets N_0^+, N_0^- of N^+ and N^- respectively such that $K\cup N_0^+\cup N_0^-$ is contradictory. We are going to show by means of a process of reduction that we may suppose that N_0^- consists of a single sentence of the form $\sim E(a,0)$, where a is in M, by the definition of N^-.

At any rate, N^- cannot be empty for $K\cup N^+$ is consistent, being satisfied by any model M' of K if we interpret all the individual constants which occur in N^+ as the zero of M'. If N_0^- contains any element of the form $\sim P(a,b,c)$ then we add $P(a,b,d)$ to N_0^+, where $d=ab$ in M, and replace $\sim P(a,b,c)$ in N_0^- by $\sim E(c,d)$. We adopt a similar procedure to eliminate sentences of the form $\sim S(a,b,c)$. Next we replace any element of (the modified) N_0^- which is of the form $\sim E(a,b)$, where $b\neq 0$ in M by $\sim E(c,0)$ where $c=a-b$ in M, and we add $S(c,b,a)$ to N_0^+. We are now left with a set N_0^- which consists of a number of sentences $\sim E(a_1,0),\ldots,\sim E(a_k,0)$, $k\geqslant 1$. If $k=1$ then we have finished. If $k>1$, then we replace the sentence just mentioned by $\sim E(b,0)$ where $b=a_1a_2\ldots a_k$ (so that $\sim E(a,0)$ belongs to N^-). At the same time we add the sentences $P(a_1,a_2,b_2), P(b_2,a_3,b_3),\ldots,P(b_{n-1},a_n,b_n)$ to N_0^+, where $a_1a_2=b_2$, $b_2a_3=b_3$, …, $b_{n-1}a_n=b_n$ in M.

It is not difficult to check that at each step the set $K\cup N_0^+\cup N_0^-$ obtained is entailed by the corresponding set before modification. We conclude that if M is not embeddable in a model of K then there exists a finite subset N_0^+ of N^+ and an element $a\neq 0$ in M such that $K\cup N_0^+\cup$ $\cup\{\sim E(a,0)\}$ is contradictory. Clearly, N_0^+ must be non-empty. Let Y be the conjunction of the elements of the set N_0^+ then K entails $Y\supset E(a,0)$. Replace the individual constants other then 0 which appear in this sentence by variables, replacing in particular a by x_0. Let Z be obtained from Y in this way, and quantify the variables in $Z\supset E(x_0,0)$ universally. Since K does not contain any individual constants other than 0, we see that the resulting sentence

2.1. $(x_0)\ldots(x_j)[Z\supset E(x_0,0)]$

is deducible from K. At the same time, the negation of 2.1 holds in M.

Now let $S(K)$ be the set of all sentences of the form 2.1 which are deducible from K where Z is a non-empty conjunction of atomic sentences which involve the atomic relations of M only. Then

2.2. THEOREM. *In order that a ring without zero-divisors, M, be embeddable in a model of K it is necessary and sufficient that M satisfies the sentences of $S(K)$.*

Proof. Suppose that M is not embeddable in K. As shown above, there exists a sentence of the form 2.1, i.e. an element of $S(K)$ whose negation holds in M. This shows that the condition of the theorem is sufficient. Again, suppose that M is embeddable in a model M' of K. Then the sentences of $S(K)$ hold in M'. But these sentences are universal and M is a subset of M'. It follows that the sentences are satisfied also by M. This proves 2.2.

In order to throw some light upon the special nature of this result we shall now consider a situation which is of a somewhat more general character. Let Q be a set of relations and A set of individual constants and let K' be a set of sentences which contain the relation of Q and possibly some other relations, as well as the individual constants of A but (for simplicity, as before) no other individual constant. We consider structures whose set of relations is Q and whose set of individual constants includes A. Let $S'(K')$ be the set of all universal sentences.

2.3. $(x_1)\ldots(x_n)[Y\supset Z],\quad n\geqslant 0$

which contain constants from A and Q only, such that Y is a conjunction of atomic sentences (which may be empty) and Z is a disjunction of atomic sentences (which may be empty and) which are deducible from K'. If Y is empty then the sentence is to be interpreted as $(x_1)\ldots(x_n)Z$ while if Z is empty then the sentence is read as $(x_1)\ldots(x_n)[\sim Y]$. Adopting (part of) the method used in the proof of 2.2, the reader will have no difficulty in establishing the following result which is essentially due to J. Łoś ([2]).

2.4. THEOREM. *In order that a structure M as above be embeddable in a model of K' it is necessary and sufficient that M be a model of $S'(K')$.*

2.4 is weaker than 2.2 for the particular case considered there since the single atomic sentences $E(a,0)$ are now replaced by disjunctions.

There is another important class of sets K' for which the introduction of disjunctions is unnecessary. This includes the case that K' is the set of axioms for a group, or for any other type of algebra in Birkhoff's sense.

2.5. THEOREM. *Let K' be a set of sentences (as before) which is closed under the formation of finite direct sums. In order that a model M (as above) be embeddable in a model of K' it is necessary and sufficient that M satisfy all sentences of $S'(K')$ such that the implicate contains at most one atomic formula.*

Proof. Let $S''(K')$ be the set of sentences of $S'(K')$ whose implicate contains at most one atomic formula. In order to prove 2.5, it is sufficient to show that $S'(K')$ is deducible from $S''(K')$. In 2.3, let

$$Y = Y_1 \wedge \ldots \wedge Y_l \quad \text{and} \quad Z = Z_1 \wedge \ldots \wedge Z_m$$

where the Y_i, Z_i are atomic formulae, $l > 0, m \geqslant 2$. Any one of the sentences

2.6. $(x_1) \ldots (x_n)[Y \supset Z_i], \quad i = 1, \ldots, n$

entails 2.3. Accordingly, we only have to show that one of these sentences must be deducible from K'.

Suppose on the contrary that none of the sentences 2.6 are deducible from K'. Then there exist models M_j of K', $j = 1, \ldots, m$ such that 2.6 does not hold in M_j for $i = j$. If follows that we can find individual constants $a_1^j, \ldots, a_n^j$ in M_j such that the sentence $Y^j \wedge \sim Z_j^j$ holds in M_j where Y^j, Z_i^j are the results of substituting $a_1^j, \ldots, a_n^j$ for $x_1, \ldots, x_n$ in Y and the Z_i, respectively.

Now consider the direct sum $M = M_1 \oplus M_2 \oplus \ldots \oplus M_m$. By the assumption of the theorem, M is a model of K'. The elements of M are given by the vectors $(b^1, \ldots, b^m)$ where $b^i \in M_i$. More precisely, to every such vector $(b^1, \ldots, b^m)$ there corresponds an individual constant b of M such that if $R(x_1, \ldots, x_p)$ is a relation in M (or a conjunction of relations) and $b_1, \ldots, b_p$ are individual constants of M, corresponding to vectors $(b_1^1, \ldots, b_1^m), \ldots, (b_p^1, \ldots, b_p^m)$, then $R(b_1, \ldots, b_n)$ holds in M if and only if the sentences $R(b_1^i, \ldots, b_p^i)$ hold in M_i, $i = 1, \ldots, n$.

Let a_i be the individual constants of M which correspond to the vectors $(a_1^i, \ldots, a_m^i)$, $i = 1, \ldots, n$ and let Y', Z', Z_j' be the results of substituting $a_1, \ldots, a_n$ for $x_1, \ldots, x_n$ in Y, Z, Z_j respectively, $j = 1, \ldots, m$. Then Y' holds in M_1 but none of the Z_j' do. It follows that Z' does not hold in M_2, and the same then applies to $Y' \supset Z'$ and to 2.6. This is contrary to assumption and proves the theorem.

We observe that if K' is a set of axioms for the concept of a group then $S'(K')$ cannot contain any sentence with empty implicate. For such a sentence is equivalent to a sentence of the form $(x_1) \ldots (x_n)[\sim Y]$ where Y is a conjunction of atomic formulae and any conjunction of atomic formulae is satisfied in a group if we substitute the unit element for all variables.

3. Ultraprime ideals. We return to the case considered at the beginning of section 2, so that K is a set of axioms which entails K_0 where K_0 defines the concept of a ring without zero divisors. Thus, K may be a set of axioms for the concept of a skew field.

Let R be an arbitrary ring, and let J be a (left-and-right) ideal in R. If J is prime then the quotient ring R/J has no zero divisors, and conversely. We shall say that J is ultraprime (with respect to the given K) if R/J is embeddable in a model of K. An ultraprime ideal is prime.

The elements of a quotient ring R/J, J an ideal in R, are usually regarded as equivalence classes of elements of R. However, it is equally possible, and is actually advantageous for our present purposes, to suppose that $R' = R/J$ contains the same individual constants as R but that the relations E, S, P hold in R' according to the following definitions.

$$\begin{array}{lll} E(a, b) & \text{iff} & a \equiv b(J)\,,\ (^1) \\ S(a, b, c) & \text{iff} & a+b \equiv c(J)\,, \\ P(a, b, c) & \text{iff} & ab \equiv c(J)\,. \end{array}$$

Thus, the atomic sentences which occur in the diagrams $N = N^+ \cup N^-$ and $N' = N'^+ \cup N'^-$ of R and R' respectively coincide, while $M'^+ \supseteq N^+$, $N'^- \subseteq N^-$.

Let R be any ring without zero divisors, then the K-radical, J_0, of R is defined as follows. An element a of R belongs to J_0 if there exists a sentence of $S(K)$, as given by 2.1., such that if we replace x_0 by a, and the remaining variables of 2.1 by certain individual constants of R, then the sentence which results from the implicans Z after substitution, holds in R.

It is not difficult to verify that J_0 is an ideal. For instance, if the fact that the elements a, b belong to J_0 follows by specialization from the sentences

$$X = (x_0) \ldots (x_n)[Z \supset E(x_0, 0)]$$

and

$$X' = (x_0') \ldots (x_m')[Z' \supset E(x_0', 0)]$$

as described, where X and X' belong to $S(K)$ then the fact that $c = a+b$ belongs to J_0 follows by specialization from the sentence

$$(x_0) \ldots (x_n)(x_0') \ldots (x_m')(x_0'')[Z \wedge Z' \wedge S(x_0, x_0', x_0'') \supset E(x_0'', 0)]$$

which also belongs to $S(K)$.

3.1. THEOREM. *The K-radical of a ring R without zero divisors is the intersection of all ultraprime ideals of R with respect to K.*

(1) 'iff' means 'if and only if'.

Proof. Let J be any ultraprime ideal in R (with respect to the given K) and let $a \in J_0$. We have to show that a belongs to J. Now if $a \in J_0$ then there exists a conjunction of atomic sentences which hold in R, Y say, such that $Y \supset E(a, 0)$ is deducible from K. Let R'' be a model of K in which $R' = R/J$ is embedded. Then Y holds in R' and hence in R'', and so $E(a, 0)$ holds in R'' and hence in R'. It follows that $a \equiv 0(J)$, $a \in J$.

Conversely, suppose that $a \in R - J_0$. Then we have to show that R includes an ultraprime ideal which does not contain a. Let $K' = K \cup N^+ \cup \{\sim E(a, 0)\}$. If this set is consistent then there exists a model of it, R'. Let R'' be the subring of R' whose individual constants coincide with the individual constants of R. Then R'' has the same positive diagram as R and is therefore a homomorphic image of R, $R \to R''$, where the homomorphism is onto. Let J be its kernel, then J is ultraprime since $R'' \subseteq R'$. Also $a \notin J$ since $a \neq 0$ in R' and hence in R''. This completes the argument for consistent K'. If K' is contradictory then there exists a conjunction of elements of N^+, Y say, such that $Y \supset E(a, 0)$ is deducible from K. But this implies $a \in J_0$, contrary to assumption.

3.1 yields a "purely mathematical" definition of the K-radical as the intersection of the ultraprime ideals of R (with respect to the given K). We observe that according to our definition $J = R$ is ultraprime but J may then be said to be an improper ultraprime ideal. If $J_0 = R$ then this is obviously the only ultraprime ideal, for the given K.

3.2. THEOREM. *If J_0 is prime then it is ultraprime.*

Proof. Since $R' = R/J_0$ is a ring without zero divisors we only have to show that the sentences of $S(K)$ hold in R'. If not then there exists a conjunction of atomic formulae which hold in R', Y say, and an element a of R such that $a \neq 0$ in R', i.e. $a \notin J$. By 3.1, there exists an ultraprime ideal J in R such that $a \notin J$. But $J \supseteq J_0$ by the definition of J_0 and so the positive diagram of R/J includes the positive diagram of R'. Hence Y holds in R/J but $E(a, 0)$ does not. This contradicts the fact that R/J is embeddable in a model of K and proves the theorem.

3.3. COROLLARY. *If the intersection of all ultraprime ideals with respect to K in the ring R, supposed without zero divisors, is the zero-ideal,* (0), *then R is embeddable in a model of K.*

For in that case, $J_0 = (0)$ and (0) is a prime ideal.

D. Tamari has shown that a Birkhoff-Witt ring with a finite number of indeterminates over a commutative field can be embedded in a skew field ([10]). Moreover, he proved (in the same paper) that if the ring is *locally finite* (that is to say that the number of indeterminates which appear successively in the commutation relations, beginning with a finite number of indeterminates, is finite), then the same conclusion still holds.

It can be shown that the gap between these two results can be bridged by means of 3.3. Using the notation of [10], let J_k^* be the ideal $(x_k, x_{k+1}, \dots)$ for $k = 0, 1, 2, \dots$ Then Tamari's condition of local finiteness implies that for any element a of the ring $F[X]$, $a \in J_k^*$ for sufficiently large k. Also, the rings $F[X]/J_k^*$ are embeddable in skew fields, by Tamari's first result. If follows that the radical J_0 with respect to K—a set of axioms for a skew field—coincides with (0). The second result now follows from 3.3.

The same argument still applies if F is replaced, in both cases, by a ring R which is regular on the right.

The following theorem has been suggested to me by Michael Rabin.

3.4. THEOREM. *Let $\{M_\nu\}$ be a set of models of K where ν ranges over the non-empty index set $I = \{\nu\}$ and let M be the strong direct sum of $\{M_\nu\}$. Let M' be a subring of M which does not contain any zero divisors. Then M' can be embedded in a model of K.*

Proof. The elements of $S(K)$ are universal Horn sentences which hold in the M_ν. It is well known (and can be verified without difficulty) that the validity of such sentences persists under direct sum formation. Hence M and M' satisfy $S(K)$. It now follows from 2.2 that M' is embeddable in a model of K.

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