On convex metric spaces III

by

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§ 1. Introduction. Let X be a continuum with a metric ϱ. For each two subsets A and B of X there is defined a number

$$(1)\qquad \varrho^1(A, B) = \max[\sup_{a\in A}\varrho(a, B), \sup_{b\in B}\varrho(A, b)],$$

where $\varrho(x, Z) = \varrho(Z, x) = \inf_{z\in Z}\varrho(x, z)$ ([1], p. 291 and [3], p. 106). This number is called a *Hausdorff metric* and has the following property: each family of closed and non-empty subsets of X is metrized by it. In particular, we shall denote by $C(X)$ the so-called hyperspace of all non-empty subcontinua of X metrized by a Hausdorff metric, and by 2^X the so-called hyperspace of all non-empty and closed subsets of X metrized in the same manner ([3], p. 326). It is well known that the hyperspaces $C(X)$ and 2^X are continua (theorem of Mazurkiewicz [6], see also [2], theorem 2.7). Evidently $C(X) \subset 2^X$.

For every subset Z of X and every number $\eta \geqslant 0$ let $Q(Z, \eta)$ be a generalized solid sphere of centre Z and radius η, i.e.

$$Q(Z, \eta) = \{x\colon x \in X, \varrho(x, Z) \leqslant \eta\}.$$

Obviously $Z \subset Q(Z, \eta)$ for every $\eta \geqslant 0$, $Q(Z, \eta) = Q(Z_1, \eta) \cup Q(Z_2, \eta)$ for $Z = Z_1 \cup Z_2$, and $Q(Z, 0) = Z$ for $Z = \overline{Z}$.

The formula

$$(2)\qquad \varrho^1(A, B) = \inf_{\eta\geqslant 0}\{[A \subset Q(B, \eta)]\cdot[B \subset Q(A, \eta)]\}$$

is equivalent to formula (1) ([2], p. 22).

I use in this paper the notion of convexity in the well-known sense of Menger ([7], p. 81, see also [5], p. 184). Also other notions and notations derive from [5].

The purpose of this paper is to determine the connexions between convexity of continuum X and convexity of its hyperspaces $C(X)$ and 2^X.

§ 2. Auxiliary theorems. Let A and B be two subsets of X such that for each two points $a \in A$ and $b \in B$ there is at least one segment $\overline{ab}$ in X. For $a = b$ this segment is reduced to one point. I call a *junction in X between A and B*, and denote by $J(A, B)$, the union $\bigcup\limits_{\substack{a \in A \\ b \in B}} \overline{ab}$ of such segments, i.e. a set containing at least one segment $\overline{ab}$ with each pair of points $a \in A$ and $b \in B$. I call a *bridge in X between A and B*, denoting it by $P(A, B)$, every compact junction in X between A and B.

It follows that $A \cup B \subset J(A, B)$ for every junction $J(A, B)$ in X between A and B and $A \cup B \subset P(A, B)$ for every bridge $P(A, B)$ in X between A and B. Clearly, if X is not a strongly convex continuum (see [5], p. 184), then the junction may not contain with any two points $a \in A$ and $b \in B$ every segment $\overline{ab}$ joining in X these points; consequently, the junction $J(A, B)$ is not unique in this case. It is also obvious that every bridge is a continuum (if it exists of course).

2.1. *If X is a metric continuum, A and B are closed subsets of X, and $J(A, B)$ is a junction in X between A and B, then the closure of $J(A, B)$ is a bridge in X between A and B.*

Proof. Since the space X is compact and $J(A, B) \subset X$ by hypothesis, we have $\overline{J(A, B)} \subset X$. Let $p \in \overline{J(A, B)}$. Then there exists a sequence of points $\{p_n\}_{n=1,2,\ldots}$ of the junction $J(A, B)$ such that

$$p = \lim_{n \to \infty} p_n . \tag{3}$$

Every point p_n of this sequence determines the family of all such segments $\overline{ab} \subset J(A, B)$ that $a \in A$, $b \in B$ and $p_n \in \overline{ab}$. Applying the axiom of choice we have for $n = 1, 2, \ldots$

$$p_n \in \overline{a_n b_n} \subset J(A, B) , \tag{4}$$

where $\overline{a_n b_n}$ is the unique segment of the class determined by p_n.

The sets A and B are compact by hypothesis, and so is $A \times B$. Then we may assume that the sequence $\{a_n, b_n\}$ tends to $(a, b) \in A \times B$. Since, moreover, the hyperspace $C(X)$ is compact, we may also assume that the sequence of segments $\{\overline{a_n b_n}\}_{n=1,2,\ldots}$ is also convergent.

The topological limit of the sequence of these segments is a segment between the limits of the sequences of theirs ends ([7], p. 92). We then have $\operatorname*{Lim}\limits_{n \to \infty} \overline{a_n b_n} = \overline{ab}$, whence we infer by virtue of (3) and (4) that $p \in \overline{ab} \subset \overline{J(A, B)}$.

2.2. *If X a metric space, A and B are closed subsets of X, $P(A, B)$ is a bridge in X between A and B, and $H \subset P(A, B)$ is closed, then*

(5) *the set $\overline{\bigcup \overline{ac}}$, where $a \in A$, $c \in H$ and $\overline{ac} \subset \overline{ab} \subset P(A, B)$, is a bridge in X between A and H,*

and symmetrically

(6) *the set $\overline{\bigcup \overline{cb}}$, where $c \in H$, $b \in B$ and $\overline{cb} \subset \overline{ab} \subset P(A, B)$, is a bridge in X between H and B.*

Proof. By symmetry of hypothesis it suffices to prove that the set (5) is a bridge in X between A and H. It follows at once from $H \subset P(A, B)$ that for each pair of points $a \in A$ and $c \in H$ there exists a point $b \in B$ such that $c \in \overline{ab}$ and $\overline{ab} \subset P(A, B)$. Then there exists a segment $\overline{ac} \subset \overline{ab}$ for each pair of points $a \in A$ and $c \in H$. The union $\bigcup \overline{ac}$ of these segments is therefore a junction in X between A and H. Hence its closure, i.e. the set (5), is by virtue of 2.1 a bridge in X between A and H.

2.3. *If X is a metric space, A and B are closed subsets of X such that*

$$\text{there exists a bridge } P(A, B) \text{ in } X \text{ between } A \text{ and } B, \tag{7}$$

and if ε is a number such that

$$0 \leqslant \varepsilon \leqslant \varrho^1(A, B) , \tag{8}$$

then the set

$$H = P(A, B) \cap Q(A, \varepsilon) \cap Q[B, \varrho^1(A, B) - \varepsilon] \tag{9}$$

satisfies the conditions

$$H \in 2^X , \tag{10}$$

$$\varrho^1(A, H) = \varepsilon , \tag{11}$$

$$\varrho^1(H, B) = \varrho^1(A, B) - \varepsilon . \tag{12}$$

Proof. It follows from definition (1) that there exists in the bridge $P(A, B)$ a segment of length at most $\varrho^1(A, B)$ and with ends belonging to A and B. By (8) and (9) such a segment must have common points with the set H, whence $H \neq 0$. Since, moreover, H is the common part of three compact sets, each of which is contained in X (the first by virtue of (7), the second and third by the definition of the generalized solid sphere, see p. 23), then $H \subset X$ and $H = \overline{H}$. This means that the non-empty set H satisfies (10).

Put $\varrho^1(A, B) = d$. It is easy to see that $A \subset Q(B, d)$, whence $H = P(A, B) \cap A \cap Q(B, d) = P(A, B) \cap A = A$ for $\varepsilon = 0$ by virtue of (9). Similarly, $H = B$ for $\varepsilon = d$. Thus in both extreme cases equalities (11) and (12) are true.

There remains the case $0 < \varepsilon < d$. By symmetry of definition (1) we may restrict this case to the following one:

$$\varrho^1(A, B) = \sup_{b \in B} \varrho(A, b) . \tag{13}$$

Hence by virtue of $\varepsilon < d$ we have

$$B - Q(A, \varepsilon) \neq 0 . \tag{14}$$

Let F be the boundary of a generalized solid sphere $Q(A, \varepsilon)$:

(15) $$F = \{x\colon \varrho(x, A) = \varepsilon\}\,.$$

Hence

(16) $$\varrho(a, p) \geqslant \varepsilon \quad \text{for each} \quad a \in A \text{ and } p \in F\,.$$

We infer by obvious inclusions $B \subset P(A, B)$ and $B \subset Q(B, d-\varepsilon)$ that $B \subset P(A, B) \cap Q(B, d-\varepsilon)$. Hence $B \cap Q(A, \varepsilon) \subset P(A, B) \cap Q(A, \varepsilon) \cap \cap Q(B, d-\varepsilon)$, i.e., by virtue of (9), $B \cap Q(A, \varepsilon) \subset H$, whence

(17) $$Q(A, \varepsilon) \cap (B-H) = 0\,.$$

Every connected set which has common points with the generalized solid sphere $Q(A, \varepsilon)$ and with the set $B-H$, lying by (17) in the complementary of $Q(A, \varepsilon)$, must have common points with the boundary $\mathrm{Fr}\big(Q(A, \varepsilon)\big)$ ([4], p. 80). Hence, by virtue of the connectivity of segments $\overline{ab}$ and of the obvious inclusions $A \subset Q(A, \varepsilon)$ and $\mathrm{Fr}\big(Q(A, \varepsilon)\big) \subset F$ (see (15)), we conclude that

(18) $$F \cap \overline{ab} \neq 0 \quad \text{for each pair of points } a \in A \text{ and } b \in B-H.$$

It follows that $B-H \neq 0$ by the hypothesis that $A \neq 0$ and by (9). Then there exist in the bridge $P(A, B)$ segments whose ends belong to A and $B-H$. Since by virtue of (8) and of definition (1)

(19) for each point $b \in B$ there exists a point $a \in A$ such that $\varrho(a, b) \leqslant d$,

there is among these segments a segment $\overline{ab} \subset P(A, B)$ such that $\varrho(a, b) \leqslant d$. By (18) we have $\overline{ab} \cap F \neq 0$. Let $p \in \overline{ab} \cap F$, whence

(20) $$\varrho(a, b) = \varrho(a, p) + \varrho(p, b)\,.$$

It follows by $\varrho(a, b) \leqslant d$, (16) and (20) that $\varrho(p, b) \leqslant d-\varepsilon$, whence $\overline{ab} \cap F \subset Q(B, d-\varepsilon)$ by virtue of the free choice of the point $p \in \overline{ab} \cap F$. We have by (15) $\overline{ab} \cap F \subset Q(A, \varepsilon)$. We then obtain $\overline{ab} \cap F \subset P(A, B) \cap \cap Q(A, \varepsilon) \cap Q(B, d-\varepsilon)$, i.e. $\overline{ab} \cap F \subset H$ by virtue of (9). It is thus proved that

(21) $\varrho(a, b) \leqslant d$ implies $\overline{ab} \cap F \subset H$ for every segment $\overline{ab} \subset P(A, B)$, where $a \in A$ and $b \in B-H$.

Applying (18) we have $\overline{ab} \cap F \neq 0$, whence by virtue of (21)

(22) $$F \cap H \neq 0\,.$$

We shall now prove that

(23) $$\sup_{b \in B} \varrho(b, H) = d-\varepsilon\,.$$

In fact, supposing first that

(24) $$\sup_{b \in B} \varrho(b, H) > d-\varepsilon\,,$$

there would exist such a point $b_0 \in B-H$ that

(25) $$\varrho(b_0, H) > d-\varepsilon\,.$$

As a continuous function, ϱ attains its suprema on the compact set $A \times (b_0)$. Let $a_0 \in A$ be the nearest point to the point $b_0 \in B-H$, i.e.

(26) $$\inf_{a \in A} \varrho(a, b_0) = \varrho(a_0, b_0)\,,$$

whence by virtue of (19)

(27) $$\varrho(a_0, b_0) \leqslant d\,.$$

Therefore $\overline{a_0 b_0} \cap F \subset H$ by (21). By virtue of (18) there exists a point $p \in \overline{a_0 b_0} \cap F$, and thus a point of H, satisfying (20) for $a = a_0$ and $b = b_0$. It follows from (16) and $a_0 \in A$ that

(28) $$\varrho(a_0, p) \geqslant \varepsilon\,,$$

and from (25) and $p \in H$ that

(29) $$\varrho(p, b_0) > d-\varepsilon\,.$$

Let us now add inequalities (28) and (29). By virtue of (20) we would have $\varrho(a_0, b_0) > d$, contrary to (27). Supposition (24) thus leads to a contradiction.

Let us suppose next that

(30) $$\sup_{b \in A} \varrho(b, H) < d-\varepsilon\,.$$

For each $b \in B$ there would exist such a point $x_b \in H$ that

(31) $$\varrho(b, x_b) < d-\varepsilon\,.$$

It follows by (9) that $H \subset Q(A, \varepsilon)$. Thus for each $x_b \in H$ there would exist a point $a \in A$ such that

(32) $$\varrho(x_b, a) \leqslant \varepsilon\,.$$

Let us now add inequalities (31) and (32). We obtain $\varrho(a, b) < d$ by virtue of the inequality of the triangle. It means that for each point $b \in B$ there would exist a point $a \in A$ whose distance from b is less than d. Therefore, as previously, we infer by the continuity of the function ϱ and by the compactness of the sets A and B that $\sup_{b \in B} \varrho(A, b) < d$, contrary to (13). Supposition (30) thus also leads to a contradiction.

Therefore condition (23) is proved.

Using it we now prove condition (12). For this purpose let us observe that $H \subset Q(B, d-\varepsilon)$ by definition (9), whence

$$\sup_{x \in H} \varrho(x, B) \leqslant d - \varepsilon . \tag{33}$$

It suffices now to substitute H instead of A in definition (1), and to apply (23) and (33).

It remains to prove condition (11). We infer by (9) that $H \subset Q(A, \varepsilon)$, whence $\sup_{x \in H} \varrho(x, A) \leqslant \varepsilon$. We infer also by formulae (22) and (16) that $\sup_{x \in H} \varrho(x, A) \geqslant \varepsilon$. Hence

$$\sup_{x \in H} \varrho(x, A) = \varepsilon . \tag{34}$$

Given some point $a \in A$, let the point b be nearest to a point of the set B. By definition (1), $\overline{ab} \subset P(A, B)$ implies $\varrho(a, b) \leqslant d$, whence $Q(A, \varepsilon) \cap Q(b, d-\varepsilon) \cap \overline{ab} \neq 0$. Thus $Q(A, \varepsilon) \cap Q(b, d-\varepsilon) \cap \overline{ab} \subset H$ by (9). Choosing some point c of this subset of H we have $c \in H$ and $\varrho(a, c) \leqslant \varepsilon$, whence a fortiori $\varrho(a, H) \leqslant \varepsilon$. Hence we have by virtue of the free choice of the point $a \in A$

$$\sup_{a \in A} \varrho(a, H) \leqslant \varepsilon . \tag{35}$$

Condition (11) follows from (1), similarly to (12), by applying (34) and (35).

Thus the proof of 2.3 is complete.

2.4. *There is no metrization of a simple closed curve such that every subcontinuum of it be convex.*

Proof. If the metrization of a simple closed curve T is not convex, then T itself is a non-convex subcontinuum. And if the metrization of T is convex, then, given any point $p \in T$, let the point $q \in T$ be the farthest from p. There exists a segment $\overline{pq} \subset T$ by the convexity of T. Let $r \in T - (\overline{pq})$ and let $\overline{qr} \subset T$ be a segment between q and r. The arc $(\overline{pq}) \cup (\overline{qr})$ is not convex, because its length $\varrho(p, q) + \varrho(q, r)$ is greater than the distance $\varrho(p, r)$ between its ends.

2.5. *If X is a metric convex continuum and every subcontinuum of X is convex, then the following sets are strongly convex:*

(a) *the continuum X,*

(b) *the generalized solid sphere $Q(A, \varepsilon)$ for every continuum $A \subset X$ and every $\varepsilon \geqslant 0$,*

(c) *the bridge $P(A, B)$ for every two subcontinua A and B of X.*

Proof. (a) If a continuum X is not strongly convex, then (see [7], p. 104) it contains two points p and q and two joining segments $(\overline{pq})_1$ and $(\overline{pq})_2$ such that $(\overline{pq})_1 \cap (\overline{pq})_2 = (p) \cup (q)$. The union $(\overline{pq})_1 \cup (\overline{pq})_2$ is then a simple closed curve and contains by 2.4 subcontinua which are not convex. Then the continuum X has not the assumed property.

(b) By the definition of the generalized solid sphere we have $Q(A, \varepsilon) = \bigcup \overline{ax}$, where the pair (a, x) runs through all such pairs of points $a \in A$ and $x \in X$ that $\varrho(a, x) \leqslant \varepsilon$. It follows that $Q(A, \varepsilon)$ is connected ([4], p. 82). As a compact set, $Q(A, \varepsilon)$ is thus a continuum, and as a subcontinuum of X it is strongly convex by virtue of the assertion (a) just proved.

(c) The bridge $P(A, B)$ is by definition a subcontinuum of X. Hence it is strongly convex also by virtue of the assertion (a).

§ 3. Sufficient conditions for the convexity of hyperspace 2^X and necessary conditions for the convexity of hyperspaces $C(X)$ and 2^X.

THEOREM 3.1. *If X is a metric continuum, A and B are closed subsets of X and there is a bridge $P(A, B)$, then there exists in 2^X at least one segment between A and B.*

If, moreover, A and B are subcontinua of X and every subcontinuum of X is convex, then there exists in $C(X)$ at least one segment between A and B.

Proof. Let $A = H_0 \in 2^X$ and $B = H_1 \in 2^X$. By virtue of 2.3 there exists for $\varepsilon = 2^{-1} \cdot \varrho^1(H_0, H_1)$ a set $H_{1/2}$ defined by formula (9), i.e. such that

$$H_{1/2} \subset P(H_0, H_1)$$

and

$$\varrho^1(H_0, H_{1/2}) = \varrho^1(H_{1/2}, H_1) = 2^{-1} \cdot \varrho^1(H_0, H_1) .$$

We shall use induction. Suppose that for some natural n the sets $H_{k/2^n}$, where $k = 0, 1, \ldots, 2^n$, and $H_{2k/2^n} = H_{k/2^{n-1}}$, are compact and satisfy the conditions

$$H_{(2m+1)/2^n} \subset P(H_{2m/2^n}, H_{(2m+2)/2^n}) \quad \text{for} \quad m = 0, 1, \ldots, 2^{n-1}-1, \tag{36}$$

$$\varrho^1(H_{k/2^n}, H_{m/2^n}) = \frac{|k-m|}{2^n} \varrho^1(H_0, H_1) \quad \text{for} \quad k, m = 0, 1, \ldots, 2^n. \tag{37}$$

Using these sets we define the sets $K_{k/2^{n+1}}$, where $k = 0, 1, \ldots, 2^{n+1}$, in such a way that conditions (36) and (37) hold also for $n+1$. Namely we prove that for each pair of sets $H_{k/2^n}, H_{(k+1)/2^n}$, where $k = 0, 1, \ldots, 2^n-1$, there exists a set $H_{(2k+1)/2^{n+1}}$ such that

$$H_{(2k+1)/2^{n+1}} \subset P(H_{k/2^n}, H_{(k+1)/2^n}) , \tag{38}$$

$$\begin{aligned} \varrho^1(H_{k/2^n}, H_{(2k+1)/2^{n+1}}) &= \varrho^1(H_{(2k+1)/2^{n+1}}, H_{(k+1)/2^n}) \\ &= 2^{-1} \cdot \varrho^1(H_{k/2^n}, H_{(k+1)/2^n}) . \end{aligned} \tag{39}$$

First, put

$$H_{2k/2^{n+1}} = H_{k/2^n} \quad \text{for} \quad k = 0, 1, \ldots, 2^n. \tag{40}$$

We then have by virtue of (36) $H_{(k+1)/2^n} \subset P(H_{k/2^n}, H_{(k+2)/2^n})$ for $k = 2m$ and $H_{k/2^n} \subset P(H_{(k-1)/2^n}, H_{(k+1)/2^n})$ for $k = 2m+1$. At all events then there exists by virtue of 2.2 a bridge $P(H_{k/2^n}, H_{(k+1)/2^n})$.

Secondly let $H_{(2k+1)/2^{n+1}}$ be a set having properties (38) and (39), the existence of which follows from 2.3. By (38) we then have (36) for $n+1$ and by (40), (37) and (39) we have also (37) for $n+1$ ([7], p. 88).

The family of closed sets $\{H_{k/2^n}\}$, where $n = 0, 1, \ldots$ and $k = 0, 1, \ldots, 2^n$, satisfying (37) is then defined. The closure in 2^X of this family is a segment between A and B ([7], p. 87-89).

If, moreover, every subcontinuum of X is convex, $A = H_0 \in C(X)$ and $B = H_1 \in C(X)$, then the set $H_{1/2}$ is a continuum (even a strongly convex one) as a common part by (9) of three continua which are strongly convex by virtue of the assertions (b) and (c) of 2.5 ([7], p. 104). Hence $H_{1/2} \in C(X)$. For the same reason every of the sets $H_{k/2^n}$, where $n = 0, 1, \ldots$ and $k = 0, 1, \ldots, 2^n$, is a continuum. Therefore the closure in $C(X)$ of the family $\{H_{k/2^n}\}$ is a segment in $C(X)$ between A and B ([7], p. 87-89, see also [4], p. 110).

From Theorem 3.1 just proved we obtain at once the following

THEOREM 3.2. *If X is a metric convex continuum and every subcontinuum of X is convex, then the hyperspace $C(X)$ is convex.*

The contrary implication is an open problem (see P2, p. 33)

3.3. *If X is a metric space (not necessarily a continuum), $p \in X$, $q \in X$, and $0 \leqslant \varepsilon \leqslant \varrho(p, q)$, then*

(41) $$z \in Q(p, \varepsilon) \frown Q[q, \varrho(p, q) - \varepsilon]$$

implies

(42) $$\varrho(p, z) = \varepsilon \quad \text{and} \quad \varrho(q, z) = (p, q) - \varepsilon .$$

Proof. If (42) does not hold, then we have by (41) the inequalities $\varrho(p, z) < \varepsilon$ or $\varrho(q, z) < \varrho(p, q) - \varepsilon$, and we receive $\varrho(p, q) < \varepsilon + \varrho(p, q) - \varepsilon = \varrho(p, q)$ by the law of the triangle.

THEOREM 3.4. *If X is a metric continuum and at least one of the hyperspaces $C(X)$ and 2^X is convex, then X is convex.*

Proof. Let $p \in X$ and $q \in X$. Obviously $(p) \subset C(X) \subset 2^X$ and $(q) \subset C(X) \subset 2^X$. At least one of the hyperspaces $C(X)$ and 2^X being convex by hypothesis, there exists in this hyperspace a segment between (p) and (q) composed of subsets of X. It means that the inequality $0 \leqslant \varepsilon \leqslant \varrho^1(p, q)$ implies the existence of a set $Z \subset X$ belonging to this segment and such that

(43) $$\varrho^1(p, Z) = \varepsilon ,$$

(44) $$\varrho^1(Z, q) = \varrho^1(p, q) - \varepsilon .$$

By definition (1) of the metric ϱ^1 and by (43) we have $\varrho(p, x) \leqslant \varepsilon$ for every $x \in Z$, whence $Z \subset Q(p, \varepsilon)$. Similarly, by (44), $Z \subset Q[q, \varrho^1(p, q) - \varepsilon]$. It is obvious that $\varrho(x, y) = \varrho^1(x, y)$ for every $x, y \in X$ (compare (1)). Consequently $Q[q, \varrho^1(p, q) - \varepsilon] = Q[q, \varrho(p, q) - \varepsilon]$. Therefore $Z \subset Q(p, \varepsilon) \frown Q[q, \varrho(p, q) - \varepsilon]$, which implies (41) for every point $z \in Z$. We then obtain (42) by 3.3. Hence the continuum X is convex by virtue of the free choice of the number ε and points p and q.

THEOREM 3.5. *If X is a metric continuum containing isometrically a contour of a square, then the hyperspace $C(X)$ is not convex.*

Proof. Let $K \subset X$ be a continuum isometric with the contour of a unitary square in the plane Oxy, having the opposite vertices $(0, 0)$ and $(1, 1)$. Continuum K is then a union of 4 segments: I having the ends $(0, 0)$ and $(0, 1)$, II having the ends $(0, 1)$ and $(1, 1)$, III having the ends $(1, 1)$ and $(1, 0)$, and IV having the ends $(1, 0)$ and $(0, 0)$. Consider the continua

(45) $$A = I \cup II \cup IV \quad \text{and} \quad B = III \cup II \cup IV .$$

Thus $\varrho^1(A, B) = 2^{-1}$. It suffices to prove that a continuum $H \subset X$ such that

(46) $$\varrho^1(A, H) = \varrho^1(H, B) = 4^{-1}$$

does not exist.

For this purpose it is convenient to begin with $Q(A, 4^{-1})$ and $Q(B, 4^{-1})$. We have by (45)

$$\begin{aligned} Q(A, 4^{-1}) \frown Q(B, 4^{-1}) &= Q(I \cup II \cup IV, 4^{-1}) \frown Q(III \cup I \cup IV, 4^{-1}) \\ &= Q(I, 4^{-1}) \frown Q(III, 4^{-1}) \cup Q(I, 4^{-1}) \frown Q(II, 4^{-1}) \cup Q(I, 4^{-1}) \frown Q(IV, 4^{-1}) \cup \\ &\cup Q(II, 4^{-1}) \frown Q(III, 4^{-1}) \cup Q(II, 4^{-1}) \frown Q(II, 4^{-1}) \cup Q(II, 4^{-1}) \frown Q(IV, 4^{-1}) \cup \\ &\cup Q(IV, 4^{-1}) \frown Q(III, 4^{-1}) \cup Q(IV, 4^{-1}) \frown Q(II, 4^{-1}) \cup Q(IV, 4^{-1}) \frown Q(IV, 4^{-1}) . \end{aligned}$$

The second and the fourth elements of this union are contained in the fifth, the third and the seventh elements—in the ninth, and, by virtue of the obvious equalities $\varrho^1(I, III) = 1$ and $\varrho^1(II, IV) = 1$, the first, the sixth and the eighth elements are vacuous. Hence

(47) $$Q(A, 4^{-1}) \frown Q(B, 4^{-1}) = Q(II, 4^{-1}) \cup Q(IV, 4^{-1}) .$$

At the same time the equality $\varrho^1(II, IV) = 1$ implies that

(48) $$Q(II, 4^{-1}) \frown Q(IV, 4^{-1}) = 0 .$$

We now show that each of the inclusions

(49) $$H \subset Q(II, 4^{-1}) \quad \text{or} \quad H \subset Q(IV, 4^{-1})$$

implies simultaneously

$$\varrho^1(A, H) > 4^{-1} \quad \text{and} \quad \varrho^1(H, B) > 4^{-1}. \tag{50}$$

By virtue of symmetry it suffices to show this for the first inclusion. We have $\varrho(IV, II) \leqslant \varrho(IV, p) + \varrho(p, II)$ for every point $p \in H$ ([3], p. 103, formula (4)), whence $\varrho(IV, p) \geqslant \varrho(IV, II) - \varrho(p, II) \geqslant 1 - 4^{-1} = 3/4$ by the inclusion assumed. It follows by (45) that

$$\sup_{a \in A} \varrho(a, H) \geqslant 3/4 \quad \text{and} \quad \sup_{b \in B} \varrho(H, b) \geqslant 3/4,$$

and a fortiori both inequalities (50) by virtue of definition (1).

Suppose now that there exists a continuum $H \subset X$ satisfying (46). Then by (2) we have $H \subset Q(A, 4^{-1})$ and $H \subset Q(B, 4^{-1})$, whence $H \subset Q(A, 4^{-1}) \cap Q(B, 4^{-1})$. There follows by (47) and (48) one of inclusions (49), whence consequently, as we proved, inequalities (50), contrary to (46).

§ 4. Conditions equivalent to the convexity of hyperspace 2^X and, in euclidean spaces, to the convexity of hyperspace $C(X)$.

THEOREM 4.1. *If X is a metric continuum, then the hyperspace 2^X is convex if and only if X is convex.*

Proof. If the hyperspace 2^X is convex, then by Theorem 3.4 the continuum X is also convex. Inversely, if the continuum X is convex, then there evidently exists, by the definition of the junction (see p. 24), for each two closed subsets A and B of X a junction $J(A, B)$ in X between A and B. Then there exists by 2.1 a bridge $P(A, B)$ in X between A and B, and it follows by Theorem 3.1 that there exists a segment in 2^X joining A and B. Hence 2^X is convex.

THEOREM 4.2. *If a metric continuum X is plungeable isometrically in the n-dimensional euclidean space $\mathcal{E}^n$, where $n \geqslant 1$, and the hyperspace $C(X)$ is convex, then X is a segment—and reciprocally.*

Proof. If X is a segment, then every subcontinuum of it is a segment or a point, and therefore is convex. Hence by Theorem 3.2 the hyperspace $C(X)$ is also convex.

Reciprocally, if $C(X)$ is convex, X is convex by Theorem 3.4, and contains no contour of a square by Theorem 3.5. Therefore $\dim X \leqslant 1$, because every convex and at least 2-dimensional continuum $X \subset \mathcal{E}^n$ contains some square. The unique 1-dimensional convex continuum lying isometrically in a euclidean space is a segment, of course.

§ 5. Problems.

The following two problems remain open:

P1. *Characterize the family of continua whose hyperspace of subcontinua are convex.*

We have given the solution of this problem only for the continua which are isometrically plungeable in euclidean spaces: the characteristic property is to be a segment.

Among continua which are not isometrically plungeable in euclidean spaces, the *dendrites* (i.e. acyclic and locally connected continua), metrized by the length of arcs, have only convex subcontinua and therefore, by Theorem 3.2, their hyperspaces of subcontinua are also convex.

The solution of problem P1 may be obtained from the positive answer to the following problem (see Theorem 3.2):

P2. *Does the convexity of the hyperspace $C(X)$ of a continuum X imply that every subcontinuum of X is convex?*

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