[8] М. Г. Крейн, *Об одной экстраполяционной проблеме А. М. Колмогорова*, Доклады Акад. Наук. СССР 46 (1945), p. 339-342.

[9] P. Lévy, *A special problem of Brownian motion, and a general theory of Gaussian random functions*, Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, Vol. II. Probability Theory, Berkeley—Los Angeles 1956.

[10] Б. Я. Левин, *Распределение корней целых функций*, Москва 1956.

[11] M. Loève, *Probability theory*, New York 1955.

[12] Ю. А. Розанов, *К экстраполяции обобщенных случайных стационарных процессов*, Теория вероятностей и ее применения IV (4) (1959), p. 465-471.

[13] L. Schwartz, *Théorie des distributions, I, II*, Paris 1950-1951.

[14] E. C. Titchmarsh, *The theory of functions*, Oxford 1932.

[15] N. Wiener, *Extrapolation, interpolation and smoothing of stationary time series*, Cambridge—New York 1949.

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STUDIA MATHEMATICA T. XXI. (1962)

On modified Landau polynomials

by

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This paper contains some theorems on the approximation of continuous functions $f(t)$ in an infinite interval by means of polynomials

$$P_n[f(t);x] = \frac{\int\limits_0^1 f(h_n t)\left[1-\left(t-\frac{x}{h_n}\right)^2\right]^n dt}{2\int\limits_0^1 (1-t^2)^n dt},$$

where $h_n > 0$ and $\lim\limits_{n\to\infty} h_n = \infty$. This kind of polynomials were first introduced by Hsu [1, 2], who also showed their convergence in the case of $h_n = n^{\Theta}$ and $f(t)$ of certain classes of continuous functions. The results given in the present paper are more general.

THEOREM 1. *If $x > 0$, then $\lim\limits_{n\to\infty} P_n(1;x) = 1$ if and only if*

$$\lim_{n\to\infty} \frac{h_n}{\sqrt{n}} = 0.$$

Proof. First we prove the sufficiency. Easy transformations give

$$(1)\qquad P_n(1;x)-1 = \frac{\int\limits_{x\sqrt{n}/h_n}^{\sqrt{n}} \left(1-\frac{u^2}{n}\right)^n du + \int\limits_{(1-x/h_n)\sqrt{n}}^{\sqrt{n}} \left(1-\frac{u^2}{n}\right)^n du}{2\int\limits_0^{\sqrt{n}} \left(1-\frac{u^2}{n}\right)^n du}.$$

Since $(1-u^2/n)^n \leqslant e^{-u^2}$ for $|u| \leqslant \sqrt{n}$, we have

$$0 \leqslant \int\limits_{x\sqrt{n}/h_n}^{\sqrt{n}} \left(1-\frac{u^2}{n}\right)^n du \leqslant \int\limits_{x\sqrt{n}/h_n}^{\sqrt{n}} e^{-u^2} du,$$

$$0 \leqslant \int_{(1-x/h_n)\sqrt{n}}^{\sqrt{n}} \left(1-\frac{u^2}{n}\right)^n du \leqslant \int_{(1-x/h_n)\sqrt{n}}^{\sqrt{n}} e^{-u^2} du$$

for sufficiently large n. Hence $P_n(1;x)-1 \to 0$ for

$$\lim_{n\to\infty} \frac{\sqrt{n}}{h_n} = \infty \quad \text{and} \quad \lim_{n\to\infty} 2\int_0^{\sqrt{n}} \left(1-\frac{u^2}{n}\right)^n du = 2\int_0^{\infty} e^{-u^2} du = \sqrt{\pi}.$$

Now we prove the necessity. By (1), assuming $P_n(1;x) \to 1$, we have in particular $\int_{x\sqrt{n}/h_n}^{\sqrt{n}} (1-u^2/n)^n du \to 0$. Let us suppose the sequence $a_n = \sqrt{n}/h_n$ to have a finite point of accumulation $g \geqslant 0$. Then $a_{n_k} \to g$ for an increasing sequence of positive integers n_k. Obviously $\int_{xa_{n_k}}^{\sqrt{n_k}} (1-u^2/n_k)^{n_k} du \to 0$. Hence $\int_{xa_{n_k}}^{x(g+1)} (1-t^2/n_k)^{n_k} dt \to 0$. Since the sequence $(1-t^2/n)^n$ is convergent to e^{-t^2} uniformly in every finite interval, the inequality $(1-t^2/n_k)^{n_k} \geqslant \frac{1}{2} e^{-t^2}$ is satisfied for every $t \in [0, x(g+1)]$ and sufficiently large k. Hence $\int_{xa_{n_k}}^{x(g+1)} e^{-t^2} dt \to 0$, for $0 \leqslant \frac{1}{2}\int_{xa_{n_k}}^{x(g+1)} e^{-t^2} dt \leqslant \int_{xa_{n_k}}^{x(g+1)} (1-t^2/n_k)^{n_k} dt$. On the other hand, $\int_{xa_{n_k}}^{x(g+1)} e^{-t^2} dt \to \int_{xg}^{x(g+1)} e^{-t^2} dt$ as $k \to \infty$. But this is impossible. Thus the sequence a_n possesses no finite points of accumulation, i. e. $a_n \to \infty$.

Remark. It is easily seen that if $h_n/\sqrt{n} \to 0$, then the sequence $P_n(1;x)$ tends to 1 almost uniformly in $(0,\infty)$, i. e. uniformly in every subinterval of $(0,\infty)$.

THEOREM 2. *If*

(α) $h_n/\sqrt{n} \to 0$ *as* $n \to \infty$,

(β) *the sequence* δ_n *satisfies the conditions* $\delta_n > 0$, $\delta_n \to 0$, $\delta_n\sqrt{n}/h_n \to \infty$,

(γ) *the function* $f(t)$ *is measurable and bounded on each interval* $[0,b]$,

(δ) $f(t)$ *is continuous at a fixed point* $x>0$, *then* $P_n(f;x) \to f(x)$ *if and only if*

$$(2) \qquad I_n(f;x) = \frac{\sqrt{n}}{h_n} \int_{x+\delta_n}^{h_n} f(t)\left[1-\left(\frac{t-x}{h_n}\right)^2\right]^n dt \to 0 \quad \text{as} \quad n\to\infty.$$

Proof. The identity

$$P_n(f;x)-f(x) = P_n(f;x)-f(x)P_n(1;x)+f(x)[P_n(1;x)-1]$$

implies that $P_n(f;x) \to f(x)$ as $n \to 0$ if and only if $P_n(f;x)-f(x)P_n(1;x) \to 0$. Since

$$(3) \qquad P_n(f;x)-f(x)P_n(1;x) = \frac{\int_0^{h_n} [f(t)-f(x)]\left[1-\left(\frac{t-x}{h_n}\right)^2\right]^n dt}{2h_n\int_0^1 (1-t^2)^n dt}$$

$$= I_n(f;x)\frac{1}{2\sqrt{n}\int_0^1 (1-t^2)^n dt} + a_n(x),$$

where

$$(4) \qquad a_n(x) =$$

$$= \frac{\left(\int_{x-\delta_n}^{x+\delta_n} + \int_0^{x-\delta_n}\right)[f(u)-f(x)]\left[1-\left(\frac{u-x}{h_n}\right)^2\right]^n du - f(x)\int_{x+\delta_n}^{h_n}\left[1-\left(\frac{u-x}{h_n}\right)^2\right]^n du}{2h_n\int_0^1 (1-t^2)^n dt},$$

it is sufficient to show that $a_n(x) \to 0$, for $2\sqrt{n}\int_0^1 (1-t^2)^n dt \to \sqrt{\pi}$. However,

$$|a_n(x)| \leqslant \sup_{|u-x|\leqslant\delta_n} |f(u)-f(x)| +$$

$$+ \frac{2\sup_{0\leqslant u\leqslant x}|f(u)| \int_0^{x-\delta_n}\left[1-\left(\frac{u-x}{h_n}\right)^2\right]^n du + |f(x)|\int_{x+\delta_n}^{h_n}\left[1-\left(\frac{u-x}{h_n}\right)^2\right]^n du}{2h_n\int_0^1 (1-t^2)^n dt}$$

$$\leqslant \sup_{|u-x|\leqslant\delta_n} |f(u)-f(x)| + \frac{2\sup_{0\leqslant u\leqslant x}|f(u)|\int_{\delta_n\sqrt{n}/h_n}^{x\sqrt{n}/h_n} e^{-t^2}dt + |f(x)|\int_{\delta_n\sqrt{n}/h_n}^{(h_n-x)\sqrt{n}/h_n} e^{-t^2}dt}{2\sqrt{n}\int_0^1 (1-t^2)^n dt}$$

for sufficiently large n. Since the function $f(t)$ is continuous at the point $x>0$ and the integral $\int_0^\infty e^{-t^2}dt$ is convergent, we obtain $a_n(x) \to 0$.

THEOREM 3. *If*

(α) $h_n/\sqrt{n} \to 0$ *as* $n\to\infty$,

(β) *the sequence* δ_n *satisfies the conditions* $\delta_n > 0$, $\delta_n \to 0$, $\delta_n\sqrt{n}/h_n \to \infty$,

(γ) *the function* $f(t)$ *is continuous in the interval* $[0,\infty)$, *then* $P_n(f;x)$ *is convergent to* $f(x)$ *almost uniformly in* $(0,\infty)$ *if and only if sequence* (2) *is convergent to zero almost uniformly in* $(0,\infty)$.

Proof. Applying identity (3), it is sufficient to prove that sequence (4) is convergent to zero almost uniformly in $(0, \infty)$. Let $x \in [a, b]$, where $a > 0$. We have

$$|a_n(x)| \leqslant \sup_{\substack{|u-x|\leqslant\delta_n \\ x\in[a,b]}} |f(u)-f(x)| + $$

$$+ \frac{2\sup\limits_{0\leqslant u\leqslant b}|f(u)| \int\limits_{\delta_n\sqrt{n}/h_n}^{b\sqrt{n}/h_n} e^{-t^2}dt + \sup\limits_{0\leqslant x\leqslant b}|f(x)| \int\limits_{\delta_n\sqrt{n}/h_n}^{(h_n-a)\sqrt{n}/h_n} e^{-t^2}dt}{2\sqrt{n}\int\limits_0^1 (1-t^2)^n dt}$$

for sufficiently large n. By the uniform continuity of $f(t)$, the sequence $a_n(x)$ is convergent to zero uniformly in $[a, b]$. The interval $[a, b]$ being arbitrary, $a_n(x) \to 0$ almost uniformly in $(0, \infty)$.

COROLLARY 1. *If*

(α) $h_n/\sqrt{n} \to 0$ *as* $n \to \infty$,

(β) $f(t)$ *is measurable in* $[0, \infty)$,

(γ) $f(t)$ *is continuous at a fixed point* $x > 0$,

(δ) *there exist constants* m *and* M *such that* $|f(t)| \leqslant Me^{mt^2}$ *for every* $t \geqslant 0$, *then* $P_n(f; x) \to f(x)$ *as* $n \to \infty$.

COROLLARY 2. *If*

(α) $h_n/\sqrt{n} \to 0$ *as* $n \to \infty$,

(β) $f(t)$ *is continuous in the interval* $[0, \infty)$,

(γ) *there exist constants* m *and* M *such that* $|f(t)| \leqslant Me^{mt^2}$ *for every* $t \geqslant 0$, *then the sequence* $P_n(f; x)$ *is convergent to* $f(x)$ *almost uniformly in* $(0, \infty)$.

Proof of corollary 2 (corollary 1 can be proved analogously). By theorem 3 it is sufficient to prove that sequence (2) is convergent to zero almost uniformly in $(0, \infty)$. Let $x \in [a, b]$, where $a > 0$. We have $2mb(h_n/\sqrt{n}) \leqslant 1$ and $m(h_n/\sqrt{n})^2 \leqslant \frac{1}{2}$ for sufficiently large n. Hence

$$|I_n(f; x)| = \left|\int\limits_{\delta_n\sqrt{n}/h_n}^{(1-x/h_n)\sqrt{n}} f\left(x+\frac{h_n}{\sqrt{n}}u\right)\left(1-\frac{u^2}{n}\right)^n du\right| \leqslant M \int\limits_{\delta_n\sqrt{n}/h_n}^{(1-x/h_n)\sqrt{n}} e^{m(x+h_n u/\sqrt{n})^2} e^{-u^2} du$$

$$\leqslant Me^{mb^2} \int\limits_{\delta_n\sqrt{n}/h_n}^{(1-a/h_n)\sqrt{n}} e^{u-\frac{1}{2}u^2} du.$$

The integral $\int\limits_0^\infty e^{u-\frac{1}{2}u^2} du$ being convergent, the sequence $I_n(f; x)$ is convergent to zero uniformly in $[a, b]$, whence almost uniformly in $(0, \infty)$.

The following question arises: are the polynomials $P_n(f; x)$ convergent for functions increasing more rapidly than e^{-t^2}? The answer depends on the choice of the sequence h_n. We consider the following example:

If the sequence h_n satisfies the conditions $h_n/\sqrt{n} \to 0$, $\limsup\limits_{n\to\infty} h_n^s/n > 0$, where $s > 2$, then $\limsup\limits_{n\to\infty} P_n(e^{t^{s+\varepsilon}}; x) = \infty$ for arbitrary $x, \varepsilon > 0$.

By identity (3) it is sufficient to show $\limsup\limits_{n\to\infty} I_n(e^{t^{s+\varepsilon}}; x) = \infty$. Since $\delta_n/h_n < \frac{1}{2}(1-x/h_n)$ for sufficiently large n, we have

$$I_n = \int\limits_{\delta_n\sqrt{n}/h_n}^{(1-x/h_n)\sqrt{n}} \left(1-\frac{t^2}{n}\right)^n \exp\left(x+\frac{h_n}{\sqrt{n}}t\right)^{s+\varepsilon} dt$$

$$\geqslant \int\limits_{\frac{1}{2}(1-x/h_n)\sqrt{n}}^{(1-x/h_n)\sqrt{n}} \left(1-\frac{t^2}{n}\right)^n \exp\left(x+\frac{h_n}{\sqrt{n}}t\right)^{s+\varepsilon} dt$$

$$\geqslant \int\limits_{\frac{1}{2}(1-x/h_n)\sqrt{n}}^{(1-x/h_n)\sqrt{n}} \left[1-\left(1-\frac{x}{h_n}\right)^2\right]^n \exp\left[x+\frac{1}{2}h_n\left(1-\frac{x}{h_n}\right)\right]^{s+\varepsilon} dt$$

$$= \frac{1}{2}\left(1-\frac{x}{h_n}\right)\sqrt{n}\exp\left\{nh_n^\varepsilon\left[\frac{h_n^s}{n}\left(\frac{x+h_n}{2h_n}\right)^{s+\varepsilon} + \frac{\lg x - \lg h_n + \lg\left(2-\frac{x}{h_n}\right)}{h_n^\varepsilon}\right]\right\}$$

$$\geqslant \frac{1}{2}\left(1-\frac{x}{h_n}\right)\sqrt{n}$$

for infinitely many n, and this proves the statement.

THEOREM 4. *If*

(α) $h_n^s/n \to 0$, *where* $s \geqslant 2$,

(β) $f(t)$ *is measurable in* $[0, \infty)$,

(γ) $f(t)$ *is continuous at a fixed point* $x > 0$,

(δ) *there exist constants* m *and* M *such that* $|f(t)| \leqslant Me^{mt^s}$ *for every* $t \geqslant 0$, *then* $P_n(f; x) \to f(x)$ *as* $n \to \infty$.

THEOREM 5. *If*

(α) $h_n^s/n \to 0$, *where* $s \geqslant 2$,

(β) $f(t)$ *is continuous in the interval* $[0, \infty)$,

(γ) *there exist constants* m *and* M *such that* $|f(t)| \leqslant Me^{mt}$ *for every* $t \geqslant 0$, *then the sequence* $P_n(f; x)$ *is convergent to* $f(x)$ *almost uniformly in* $(0, \infty)$.

Proof of theorem 5 (theorem 4 can be proved similarly). By theorem 3 it is sufficient to show almost uniform convergence of sequence (2). Let $x \in [a, b]$, where $a > 0$. We have

$$|I_n(f;x)| \leqslant M \int_{\delta_n\sqrt{n}/h_n}^{(1-x/h_n)\sqrt{n}} \exp\left[m\left(x+\frac{h_n}{\sqrt{n}}t\right)^s - t^2\right]dt$$

$$\leqslant M \int_{\delta_n\sqrt{n}/h_n}^{(1-a/h_n)\sqrt{n}} \exp\left\{t^2\left[m\left(\frac{b}{(\delta_n\sqrt{n}/h_n)^{2/s}}+\frac{h_n}{n^{1/s}}\left(1-\frac{a}{h_n}\right)^{1-2/s}\right)^s-1\right]\right\}dt$$

$$\leqslant M \int_{\delta_n\sqrt{n}/h_n}^{(1-a/h_n)\sqrt{n}} \exp\left(-\frac{1}{2}t^2\right)dt$$

for sufficiently large n, since the expression in brackets is less than $-\frac{1}{2}$. The interval $[a, b]$ being arbitrary, the sequence I_n tends to zero almost uniformly in $(0, \infty)$.

THEOREM 6. *If*

(α) $h_n^s/n \to 0$, *where* $s \geqslant 2$,

(β) $f(t)$ *is measurable in* $[0, \infty)$,

(γ) *the* $2k$-*th* ($k \geqslant 1$) *derivative* $f^{(2k)}(x)$ *exists at a fixed point* $x > 0$,

(δ) *there exist constants* m *and* M *such that* $|f(t)| \leqslant Me^{mt^s}$ *for every* $t \geqslant 0$, *then*

$$\lim_{n\to\infty}\left(\frac{\sqrt{n}}{h_n}\right)^{2k}\left\{P_n[f(t);x]-\sum_{r=0}^{2k-1}\frac{f^{(r)}(x)}{r!}P_n[(t-x)^r;x]\right\} = \frac{\Gamma(k+\frac{1}{2})}{\Gamma(\frac{1}{2})}\,\frac{f^{(2k)}(x)}{(2k)!}.$$

Proof. By the assumption (γ),

$$f(t) = f(x)+\frac{f'(x)}{1!}(t-x)+\ldots+\frac{f^{(2k)}(x)}{(2k)!}(t-x)^{2k}+(t-x)^{2k}\eta(t-x), \tag{5}$$

where $\lim_{u\to 0}\eta(u) = 0$. Next, by the assumption (δ), there is a constant $L(x)$ such that $|\eta(u)| \leqslant L(x)\exp[m(x+u)^s]$ for every $u \geqslant -x$. Applying (5) we obtain

$$\left(\frac{\sqrt{n}}{h_n}\right)^{2k}\left\{P_n[f(t);x]-\sum_{r=0}^{2k-1}\frac{f^{(r)}(x)}{r!}P_n[(t-x)^r;x]\right\} \tag{6}$$

$$= \frac{f^{(2k)}(x)}{(2k)!}\left(\frac{\sqrt{n}}{h_n}\right)^{2k}P_n[(t-x)^{2k};x]+\left(\frac{\sqrt{n}}{h_n}\right)^{2k}P_n[(t-x)^{2k}\eta(t-x);x].$$

Easy transformations yield

$$\left(\frac{\sqrt{n}}{h_n}\right)^{2k}P_n[(t-x)^{2k};x] = n^k\frac{B(k+\frac{1}{2};n+1)}{B(\frac{1}{2};n+1)}-r_n(x), \tag{7}$$

where

$$r_n(x) = \frac{\int_{x\sqrt{n}/h_n}^{\sqrt{n}} t^{2k}(1-t^2/n)^n\,dt+\int_{(1-x/h_n)\sqrt{n}}^{\sqrt{n}} t^{2k}(1-t^2/n)^n\,dt}{2\sqrt{n}\int_0^1(1-t^2)^n\,dt}.$$

However, we have for sufficiently large n

$$0 \leqslant \int_{x\sqrt{n}/h_n}^{\sqrt{n}} u^{2k}\left(1-\frac{u^2}{n}\right)^n du \leqslant \int_{x\sqrt{n}/h_n}^{\sqrt{n}} u^{2k}e^{-u^2}du$$

and

$$0 \leqslant \int_{(1-x/h_n)\sqrt{n}}^{\sqrt{n}} u^{2k}\left(1-\frac{u^2}{n}\right)^n du \leqslant \int_{(1-x/h_n)\sqrt{n}}^{\sqrt{n}} u^{2k}e^{-u^2}du.$$

This proves that $r_n(x) \to 0$, for the integral $\int_0^\infty u^{2k}e^{-u^2}du$ is convergent. Applying the well-known relation between the functions B and Γ and the functional equation of the function Γ we obtain

$$n^k\frac{B(k+\frac{1}{2};n+1)}{B(\frac{1}{2};n+1)} = n^k\frac{\Gamma(k+\frac{1}{2})\Gamma(n+1)}{\Gamma(n+1+k+\frac{1}{2})}\,\frac{\Gamma(n+1+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(n+1)}$$

$$= n^k\frac{\frac{1}{2}(\frac{1}{2}+1)\ldots(\frac{1}{2}+n)}{(k+\frac{1}{2})(k+\frac{1}{2}+1)\ldots(k+\frac{1}{2}+n)}$$

$$= \frac{n!\,n^{k+\frac{1}{2}}}{(k+\frac{1}{2})(k+\frac{1}{2}+1)\ldots(k+\frac{1}{2}+n)}\,\frac{\frac{1}{2}(\frac{1}{2}+1)\ldots(\frac{1}{2}+n)}{n!\,n^{1/2}} \to \frac{\Gamma(k+\frac{1}{2})}{\Gamma(\frac{1}{2})}$$

for $\Gamma(z) = \lim_{n\to\infty}\frac{n!\,n^z}{z(z+1)\ldots(z+n)}$.

By (7) we get

$$\lim_{n\to\infty}\left(\frac{\sqrt{n}}{h_n}\right)^{2k}P_n[(t-x)^{2k};x] = \frac{\Gamma(k+\frac{1}{2})}{\Gamma(\frac{1}{2})}. \tag{8}$$

Let δ_n satisfy the conditions $\delta_n > 0$, $\delta_n \to 0$, $\delta_n\sqrt{n}/h_n \to \infty$. We have

$$\left|\left(\frac{\sqrt{n}}{h_n}\right)^{2k} P_n[(t-x)^{2k}\eta(t-x);x]\right|$$

$$\leqslant \left(\frac{\sqrt{n}}{h_n}\right)^{2k} \frac{\left(\int\limits_{x-\delta_n}^{x+\delta_n} + \int\limits_{0}^{x-\delta_n} + \int\limits_{x+\delta_n}^{h_n}\right)(t-x)^{2k}|\eta(t-x)|\left[1-\left(\frac{t-x}{h_n}\right)^2\right]^n dt}{2h_n\int\limits_0^1(1-t^2)^n dt}$$

$$\leqslant \frac{\sup\limits_{|u|\leqslant\delta_n}|\eta(u)| \int\limits_{-\delta_n\sqrt{n}/h_n}^{\delta_n\sqrt{n}/h_n} u^{2k}e^{-u^2}du}{2\sqrt{n}\int\limits_0^1(1-t^2)^n dt} +$$

$$+L(x)\frac{e^{mx^s}\int\limits_{\delta_n\sqrt{n}/h_n}^{x\sqrt{n}/h_n} u^{2k}e^{-u^2}du + \int\limits_{\delta_n\sqrt{n}/h_n}^{(1-x/h_n)\sqrt{n}} u^{2k}\exp[m(x+h_n u/\sqrt{n})^s - u^2]du}{2\sqrt{n}\int\limits_0^1(1-t^2)^n dt}$$

$$\leqslant \sup_{|u|\leqslant\delta_n}|\eta(u)|\int\limits_{-\infty}^{+\infty}u^{2k}e^{-u^2}du + L(x)\Bigg\{e^{mx^s}\int\limits_{\delta_n\sqrt{n}/h_n}^{x\sqrt{n}/h_n}u^{2k}e^{-u^2}du+$$

$$+\int\limits_{\delta_n\sqrt{n}/h_n}^{(1-x/h_n)\sqrt{n}} u^{2k}\exp\left\{u^2\left[m\left(\frac{x}{(\delta_n\sqrt{n}/h_n)^{2/s}}+\frac{h_n}{n^{1/s}}\left(1-\frac{x}{h_n}\right)^{1-2/s}\right)^s-1\right]\right\}du$$

for sufficiently large n. Since $\lim\limits_{u\to 0}\eta(u)=0$ and the expression in brackets is less than $-\frac{1}{2}$ for sufficiently large n, we have

$$\lim_{n\to\infty}\left(\frac{\sqrt{n}}{h_n}\right)^{2k} P_n[(t-x)^{2k}\eta(t-x);x] = 0. \tag{9}$$

Applying (8) and (9) we obtain the theorem from (6).

References

[1] L. C. Hsu, *The polynomial approximation of continuous functions defined on* $(-\infty,\infty)$, Czechoslovak Mat. Journal 9 (1959), p. 574-578.

[2] — *Approximation of non-bounded continuous functions by certain sequences of linear positive operators or polynomials*, Studia Mathematica 21 (1961), p. 37-43.

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STUDIA MATHEMATICA T. XXI. (1962)

Entire functions in B_0-algebras

by

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A B_0-algebra is a completely metrizable locally convex topological algebra over the real or complex scalars. We shall also assume that the algebras in question possess unit elements.

The topology in a B_0-algebra R may be introduced by means of a denumerable sequence of pseudonorms satisfying

$$\|x\|_i \leqslant \|x\|_{i+1}, \quad i=1,2,\ldots, \tag{1}$$

and

$$\|xy\|_i \leqslant \|x\|_{i+1}\|y\|_{i+1} \tag{2}$$

(see [13], theorem 24). A sequence x_n tends to x_0 if and only if $\lim\limits_{n\to\infty}\|x_n-x_0\|_i = 0$, $i=1,2,\ldots$ The basis of neighbourhoods of zero in R is of the form $\{K_i(1/n)\}$ $(i,n=1,2,\ldots)$, where $K_i(r)=\{x\in R\colon \|x\|_i<r\}$. Any sub-sequence of the sequence $\{\|x\|_i\}$ also satisfies (1) and (2) and gives in R the same topology.

A B_0-algebra R is called *m-convex* if there exists an equivalent system of pseudonorms satisfying

$$\|xy\|_i \leqslant \|x\|_i\|y\|_i, \quad i=1,2,\ldots \tag{3}$$

The concept of an m-convex B_0-algebra, first introduced by Arens [2], was then considered in detail by Michael in [7]. A B_0-algebra is m-convex if and only if there exists a fundamental system $\{U\}$ of neighbourhoods of 0 which are *idempodent* (i. e. such that $UU\subset U$, where $XY=\{z\in R\colon z=xy,\ x\in X,\ y\in Y\}$, X, Y — arbitrary subsets of R), or if there exists an equivalent system of pseudonorms such that multiplication is continuous with respect to each one [7]. In [7] it is also shown that if U is an idempotent subset of R, then so are its convex hull $\mathrm{conv}\,U$ and its closure $\overline{U}$.

If R is an m-convex B_0-algebra and $\varphi(z)=\sum\limits_{n=0}^{\infty}a_nz^n$ is an entire function of complex variable z, then for every $x\in R$ the series $\varphi(x)=\sum\limits_{0}^{\infty}a_nx^n$