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On the differentiability of weak solutions of certain non-elliptic equations

by

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Lax [7] has given the method for studing the differentiability of weak solutions of elliptic equations of order $2m$ with the aid of Hilbert spaces H_p (p being an arbitrary integer). The purpose of this paper is to adapt the theory of Lax to some classes of non-elliptic equations. This can be made with the aid of Hilbert spaces $H_{p,q}$ (p, q are arbitrary integers) which will be defined in Chapter 1. In chapter 2 we consider the regularity properties of these spaces, when the indices are sufficiently large. In chapter 3 the differentiability theorem for certain non-elliptic equations is given. As a special case we obtain some results concerning the regularity of weak solutions of elliptic equations depending on a parameter.

1. The norms $\| \|_{p,q}$ and related Hilbert spaces

1.1. Our definition (and the definition of the spaces H_{-m} given by Lax) is based on the following theorem concerning Banach spaces:

THEOREM A. *Let X_0 and X_+ be two reflexive Banach spaces such, that*

1° X_+ is a dense subset of the space X_0,

2° $\|x\|_+ \geqslant \|x\|_0$ for all x in X_+ ($\| \|_+$ and $\| \|_0$ denotes the norms in the spaces X_+ and X_0 respectively).

Let X_0^* be a space conjugate to X_0 (that is the space of all continuous linear functionals on X_0). For $y \in X_0^*$ put

$$\|y\|_- \overset{\mathrm{df}}{=} \sup_{x \in X_+, \|x\|_+ \leqslant 1} |y(x)|$$

and let X_- be the completion of X_0^* in the norm $\| \|_-$. Then the space X_- is isometrically isomorphic with the spece X_+^* and so is the space X_+ with regard to the space X_-^*, the latter isomorphism being given by the correspondence

$$X_-^* \ni l \leftrightarrow x \in X_+,$$

when $l(y) = y(x)$ *for all* $y \in X_-$. *When we set* $b(x, y) \overset{\text{df}}{=} y(x)$ *for* $y \in X_-$ *and* $x \in X_+^a$, *then the generalized Schwarz inequality*

$$|b(x, y)| \leqslant \|x\|_+ \|y\|_-$$

holds.

This theorem can be proved by using the arguments contained in the paper of Lax [7].

1.2. The two-indices norms shall be first defined for infinitely differentiable functions in some domain Ω of the Euclidean space E^N; then we obtain the related Hilbert spaces with the aid of completion. We suppose the domain Ω to be the product of two domains: $\overset{1}{\Omega}$ of the space E^R, and $\overset{2}{\Omega}$ of the space E^S $(R+S = N)$, and we denote by $x = (x_1, \ldots, x_R)$ the point of the space E^R, and by $y = (y_1, \ldots, y_S)$ the point of the space E^S. The class of all complex-valued functions which are infinitely differentiable in Ω and whose all derivatives are square summable in Ω, will be denoted by $C_2^\infty(\Omega)$. By B we denote a linear subset of the class $C_2^\infty(\Omega)$ containing the class $C_0^\infty(\Omega)$ [1], which has the following properties:

1° for each function $\varphi \in C_0^\infty(\overset{1}{\Omega})$ or $\psi \in C_0^\infty(\overset{2}{\Omega})$ and for each $u \in B$ the functions φu and ψu are also in B,

2° for each $u \in B$ all the derivatives of u are also in B.

Let $B_{0,-}$ be the subset of the class B consisting of all functions $u(x, y)$ which vanish for $x \in \overset{1}{\Omega} - K$ and $y \in \overset{2}{\Omega}$, when K is a compact contained in $\overset{1}{\Omega}$ (depending on u). $B_{-,0}$ has the same meaning when the roles of x and y are interchanged.

In the sequel the letters m, k will denote non-negative integers and p, q — arbitrary integers. The derivative $\dfrac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \ldots \partial x_R^{\alpha_R}}$ $(|\alpha| = \alpha_1 + \ldots + \alpha_R)$ will be denoted briefly by $D_x^\alpha u$, and, analogously, $D_y^\beta u$ will denote the derivative $\dfrac{\partial^{|\beta|} u}{\partial y_1^{\beta_1} \ldots \partial y_S^{\beta_S}}$ $(|\beta| = \beta_1 + \ldots + \beta_S)$.

1.3. We first define the spaces $H_{0,q}(\Omega, B)$. Let

$$\|u\|_{0,k}^2 \overset{\text{df}}{=} \sum_{0 \leqslant |\beta| \leqslant k} \|D_y^\beta u\|_{L^2(\Omega)}^2$$

for all $u \in B$, and let $H_{0,k}(\Omega, B)$ be the completion of the class B in the norm $\| \ \|_{0,k}$. To each element u of the space $H_{0,k}(\Omega, B)$ and to each β $(0 \leqslant |\beta| \leqslant k)$ corresponds the strong derivative $D_y^\beta u$ defined as the limit

(1) $C_0^\infty(\Delta)$ (when Δ is a domain of the Euclidean space) denotes the class of all functions infinitely differentiable in Δ and having a compact support contained in Δ.

in $L^2(\Omega)$ of the sequence $\{D_y^\beta u_n\}$ when u_n belongs to B and $\|u_n - u\|_{0,k} \to 0$. The same arguments as used by Friedrichs [3] show, that the correspondence

$$H_{0,k}(\Omega, B) \ni u \to D_y^{(0,\ldots,0)} u \in L^2(\Omega)$$

is a one-to-one linear and continuous mapping, which leaves invariant the elements of B. Therefore the space $H_{0,k}(\Omega, B)$ may be treated as a subset of $L^2(\Omega)$, when each element is identified with its strong derivative of order zero. It is a Hilbert space with the scalar product

$$(u, v)_{0,k} \overset{\text{df}}{=} \sum_{0 \leqslant |\beta| \leqslant k} (D_y^\beta u, D_y^\beta v)_{L^2(\Omega)},$$

the derivatives being taken in the strong sense.

Lemma 1. *The class* $B_{0,-}$ *is dense in* $H_{0,k}(\Omega, B)$.

Proof [2]. It is sufficient to show, that an arbitrary function u belonging to B can be approximated with functions of the class B_0-, with respect to the norm $\| \ \|_{0,k}$. Let $\varphi \in C_0^\infty(\overset{1}{\Omega})$ be a function satisfying the conditions

1° $0 \leqslant \varphi(x) \leqslant 1$,

2° $\varphi(x) = 1$ for x lying in some compact Δ contained in $\overset{1}{\Omega}$, and write

$$u_1(x, y) = \varphi(x) u(x, y), \qquad (x, y) \in \Omega.$$

Then $u_1 \in B_{0,-}$ and

$$\|u - u_1\|_{0,k}^2 = \sum_{0 \leqslant |\beta| \leqslant k}' \int_\Omega |1 - \varphi(x)|^2 |D_y^\beta u(x, y)|^2 dx\,dy$$

$$\leqslant \sum_{0 \leqslant |\beta| \leqslant k} \int_{(\overset{1}{\Omega} - \Delta) \times \overset{2}{\Omega}} |D_y^\beta u(x, y)|^2 dx\,dy$$

From the square-summability of $D_y^\beta u$ follows, that the last sum may be arbitrarily small for suitable Δ, q. e. d.

We now define for $u \in L^2(\Omega)$ the norm $\|u\|_{0,-k}$ as the norm $\| \ \|_-$ described in theorem A, when $H_{0,k}(\Omega, B)$ is taken as the space X_+, and $L^2(\Omega)$ as the space X_0 and X_0^*. The corresponding space X_- is denoted by $H_{0,-k}(\Omega, B)$. From theorem A it follows that on the product $H_{0,q}(\Omega, B) \times H_{0,-q}(\Omega, B)$ the bilinear form $b_{0,q}(u, v)$ can be defined, having the property

$$b_{0,q}(u, v) = (u, v)_{L^2(\Omega)}$$

for $u, v \in L^2(\Omega)$. Because of the density of the class $C_0^\infty(\Omega)$ in $L^2(\Omega)$ it is also dense in $H_{0,-k}(\Omega, B)$.

(2) This proof has been suggested to the author by Prof. S. Łojasiewicz. The proof given previously by the author was more complicated.

1.4. Now we set

$$\|u\|_{m,q} \overset{\mathrm{df}}{=} \sum_{0\leqslant|\alpha|\leqslant m} \|D_x^\alpha u\|_{0,q}^2$$

for $u \in B$ and we define $H_{m,q}(\Omega, B)$ as the completion of the class B in the norm $\| \|_{m,q}$. An analogous reasonning as in the proof of lemma 1 shows, that $B_{-,0}$ is dense in B with respect to the norm $\| \|_{m,0}$ and therefore also with respect to the norm $\| \|_{m,-k}$. For each u belonging to $H_{m,k}(\Omega, B)$ and for each α, β $(0 \leqslant |\alpha| \leqslant m,\ 0 \leqslant |\beta| \leqslant k)$ the strong derivative $D_x^\alpha D_y^\beta u$ may be defined as the limit in $L^2(\Omega)$ of $D_x^\alpha D_y^\beta u_n$ when $\{u_n\}$ is a sequence of functions of the class B approximating u in the norm $\| \|_{m,k}$. When we identify each $u \in H_{m,k}$ with its strong derivative of order zero, the space $H_{m,k}(\Omega, B)$ can be considered as a subset of $L^2(\Omega)$ (namely the set of all functions square-summable in Ω, which have strong derivatives to the order m with respect to x and to the order k with respect to y).

LEMMA 2. *The space $H_{m,q}(\Omega, B)$ may be mapped in an one-to-one linear and continuous manner into the space $H_{0,q}(\Omega, B)$; this mapping leaves invariant the functions of the class B.*

Proof. A system $\{u^\alpha\}$ of elements of the space $H_{0,q}(\Omega, B)$ $(\alpha = \{\alpha_1, \ldots, \alpha_R\}$ $0 \leqslant |\alpha| \leqslant m)$ having the following properties corresponds to each element u of $H_{m,q}(\Omega, B)$:

1° when $\{u_n\} \subset B$ is a sequence approximating u in the norm $\| \|_{m,q}$, then $\|D_x^\alpha u_n - u^\alpha\|_{0,q} \to 0$.

2° $\|u\|_{m,q}^2 = \sum_{0\leqslant|\alpha|\leqslant m} \|u^\alpha\|_{0,q}^2$.

The mapping is given by the correspondance $u \to u^{(0,\ldots,0)}$ and it will be proved that from $u^{(0,\ldots,0)} = 0$ it follows that $u^\alpha = 0$ for $0 \leqslant |\alpha| \leqslant m$. For an arbitrary function $\varphi \in B_{0,-}$ we have after integration by parts

$$(D_x^\alpha u_n, \varphi)_{L^2(\Omega)} = (u_n, (-1)^{|\alpha|} D_x^\alpha \varphi)_{L^2(\Omega)}$$

and in the limit

$$(u^\alpha, \varphi) = (u^{(0,\ldots,0)}, (-1)^{|\alpha|} D_x^\alpha \varphi)$$

the last brackets being taken in the sense of the duality between the spaces $H_{0,q}(\Omega, B)$ and $H_{0,-q}(\Omega, B)$. From the last equality and from lemma 1 it follows, that $u^\alpha = 0$ $(0 \leqslant |\alpha| \leqslant m)$ when $u^{(0,\ldots,0)} = 0$, q. e. d.

According to lemma 2 the space $H_{m,q}(\Omega, B)$ may be treated as a subset of $H_{0,q}(\Omega, B)$ when u is identified with $u^{(0,\ldots,0)}$. Especially in the case $q = -k$ the element u^α is called *strong derivative in the norm* $\| \|_{0,-k}$ with respect to x of order α and can be denoted by D_x^α when there is no danger of misunderstanding. The spaces $H_{m,q}(\Omega, B)$ are Hilbert spaces with the scalar product

$$(u, v)_{m,q} \overset{\mathrm{df}}{=} \sum_{0\leqslant|\alpha|\leqslant m} (D_x^\alpha u, D_x^\alpha v)_{0,q};$$

in particular for $q = k$

$$(u, v)_{m,k} = - \sum_{\substack{0\leqslant|\alpha|\leqslant m\\ 0\leqslant|\beta|\leqslant k}} (D_x^\alpha D_y^\beta u, D_x^\alpha D_y^\beta v)_{L^2(\Omega)}$$

(the derivatives are taken in the strong sense).

1.5. The space $H_{-m,q}(\Omega, B)$ is defined as the space X_-, which is given by theorem A when one puts $H_{0,-q}(\Omega, B)$ as X_0, $H_{0,q}(\Omega, B)$ as X_0^* and $H_{m,-q}(\Omega, B)$ as X_+. It is isometrically isomorphic to the adjoined space of the Hilbert space $H_{m,-q}(\Omega, B)$ and therefore is a Hilbert space. A consequence of theorem A is the following

THEOREM 1. *On the product $H_{p,q}(\Omega, B) \times H_{-p,-q}(\Omega, B)$ the bilinear form $b_{p,q}$ having the following properties can be defined:*

1° $b_{p,q}(u, v) = (u, v)_{L^2(\Omega)}$ *for all p, q when u and v are in the space $L^2(\Omega)$,*

2° *the generalized Schwarz inequality*

$$|b_{p,q}(u, v)| \leqslant \|u\|_{p,q}\|v\|_{-p,-q}$$

holds for all $u \in H_{p,q}(\Omega, B)$ and $v \in H_{-p,-q}(\Omega, B)$.

3° $\|u\|_{p,q} = \sup_{\substack{v\in H_{-p,-q}(\Omega,B)\\ \|v\|_{-p,-q}\leqslant 1}} |b_{p,q}(v, u)|$.

The correspondence

$$H_{p,q}^*(\Omega, B) \ni l \leftrightarrow u \in H_{-p,-q}(\Omega, B)$$

when

$$l(v) = b_{p,q}(v, u) \quad (v \in H_{p,q}(\Omega, B))$$

gives the isomorphic mapping of $H_{p,q}(\Omega, B)$ on $H_{-p,-q}(\Omega, B)$.

1.6. Definition 1. Let $\| \|_{(1)}$ and $\| \|_{(2)}$ be two norms of Banach type defined on a linear set X and satisfying the inequality $\|u\|_{(1)} \leqslant \|u\|_{(2)}$ for all $u \in X$. We say they are *compatible* on X ([3]), if each sequence $\{u_n\} \subseteq X$ which is fundamental in the both norms and tends to zero in the norm $\| \|_{(1)}$, tends also to zero in the norm $\| \|_{(2)}$. It is well known (see [5]), that in such a case the completion of X in the norm $\| \|_{(2)}$ can be mapped in an one-to-one linear and continuous manner in the completion of X in the norm $\| \|_{(1)}$ and this mapping leaves invariant the elements of the set X. Therefore the $\| \|_{(2)}$-completion can be treated as a dense subset of the $\| \|_{(1)}$-completion.

Let X_1 and X_2 be two Banach spaces such, that X_1 is a dense subset of X_2 and $\|u\|_{(1)} \geqslant \|u\|_{(2)}$ for all $u \in X_1$. Because of the density each linear functional on X_2 is uniquely determined by its restriction to the set X_1 and this restriction is evidently continuous in the norm $\| \|_{(1)}$, so

([3]) In Russian согласованные (see [5]).

is a linear functional on X_1. Denote by $\| \|_{(1)}^*$ and $\| \|_{(2)}^*$ the norms in corresponding adjoined spaces

$$\|l\|_{(1)}^* = \sup_{u \in X_1} \frac{|l(u)|}{\|u_{(1)}\|_1}, \qquad \|l\|_{(2)}^* = \sup_{u \in X_1} \frac{|l(u)|}{\|u\|_{(2)}}.$$

Then the inequality $\|l\|_{(1)}^* \leqslant \|l\|_{(2)}^*$ holds for all $l \in X_2^*$ and it may be proved in a simple way, that the norms $\| \|_{(1)}^*$ and $\| \|_{(2)}^*$ are compatible on X_2^*.

LEMMA 3. *For $p_1 \geqslant p_2$ and $q_1 \geqslant q_2$ the inequality*

$$\|u\|_{p_1,q_1} \geqslant \|u\|_{p_2,q_2} \tag{1}$$

holds for all $u \in B$; the norms $\| \|_{p_1,q_1}$ and $\| \|_{p_2,q_2}$ are compatible on B.

Proof. The inequality (1) follows immediately from the definition of the norms $\| \|_{p,q}$. We shall prove the compatibility of the norms. In the case when p_j and q_j $(j = 1, 2,)$ are non-negative it is evident because we identify each element of the space $H_{m,k}(\Omega, B)$ with his strong derivative of order zero. Therefore $H_{m_1,k_1}(\Omega, B)$ is a dense subset of $H_{m_2,k_2}(\Omega, B)$ $(m_1 \geqslant m_2, k_1 \geqslant k_2)$ and from the preceding remarks if follows, that the norms $\| \|_{-m_1,-k_1}$ and $\| \|_{-m_2,-k_2}$ are compatible on the class B (considered as the set of linear functionals on $H_{m_2,k_2}(\Omega, B)$). As the both spaces $H_{-m_j,-k_j}(\Omega, B)$ $(j = 1, 2)$ are the completions of B in the corresponding norms, we have the dense embedding $H_{-m_2,-k_2}(\Omega, B) \subset H_{-m_1,-k_1}(\Omega, B)$.

A similar reasonning proves that the norms $\| \|_{m_1,-k}$ and $\| \|_{m_2,-k}$ $(m_1 \geqslant m_2)$ are compatible; thus $H_{m_1,-k}(\Omega, B)$ is a dense subset of $H_{m_2,-k}(\Omega, B)$ and from this follows the compatibility of the norms $\| \|_{-m_1,k}$ and $\| \|_{-m_2,k}$ on the class B. Therefore $H_{-m_2,k}(\Omega, B)$ is also a dense subset of $H_{-m_1,k}(\Omega, B)$.

Let u_n be a sequence of functions of the class B fundamental in both norms $\| \|_{m,-k_1}$ and $\| \|_{m,-k_2}$ $(k_1 \geqslant k_2)$ and let $\|u_n\|_{m,-k_1} \to 0$ for $n \to \infty$. Then for $0 \leqslant |\alpha| \leqslant m$ the sequence $\{D_x^\alpha u_n\}$ is fundamental in the both norms $\| \|_{0,-k_1}$ and $\| \|_{0,-k_2}$, and $\|D_x^\alpha u_n\|_{0,-k_1} \to 0$. So $\|D_x^\alpha u_n\|_{0,-k_2} \to 0$, because the norms $\| \|_{0,-k_1}$ and $\| \|_{0,-k_2}$ are compatible, and therefore $\|u_n\|_{m,-k_2} \to 0$. So the norms $\| \|_{m,-k_1}$ and $\| \|_{m,-k_2}$ are compatible on B. From this follows the dense inclusion $H_{m,-k_2}(\Omega, B) \subset H_{m,-k_1}(\Omega, B)$, and as a consequence, the compatibility of the norms $\| \|_{-m,k_1}$ and $\| \|_{-m,k_2}$ on the class B. Thus the space $H_{-m,k_1}(\Omega, B)$ may be considered as a dense subset of $H_{-m,k_2}(\Omega, B)$.

So the lemma is proved and we have also verified

THEOREM 2. *For $p_1 \geqslant p_2$ and $q_1 \geqslant q_2$ the space $H_{p_1,q_1}(\Omega, B)$ may be treated as a dense subset of the space $H_{p_2,q_2}(\Omega, B)$. The inequality (1) holds for all $u \in H_{p_1,q_1}(\Omega, B)$.*

Let $p_1 \geqslant p$ and $q_1 \geqslant q$. According to what has been stated above we have the embeddings

$$H_{p_1,q_1}(\Omega, B) \subset H_{p,q}(\Omega, B), \qquad H_{-p,-q}(\Omega, B) \subset H_{-p_1,-q_1}(\Omega, B).$$

Let $u \in H_{p_1,q_1}(\Omega, B)$, $v \in H_{-p,-q}(\Omega, B)$ and let $\{u_n\}$ and $\{v_n\}$ be the corresponding approximating sequences of smooth functions

$$\|u_n - u\|_{p_1,q_1} \to 0$$

$$\|v_n - v\|_{-p,-q} \to 0.$$

From theorem 1 and lemma 3 it follows that $b_{p,q}(u, v) = \lim_{n \to \infty} (u_n, v_n)_{L^2(\Omega)} = b_{p_1,q_1}(u, v)$; so for fixed $v \in H_{-p,-q}(\Omega, B)$ the form $b_{p_1,q_1}(-, v)$ is a restriction to the space $H_{p_1,q_1}(\Omega, B)$ of the form $b_{p,q}(-, v)$ (evidently the roles of u and v may be interchanged). Therefore we can omit the index and in the sequel we shall write simply (u, v) instead of $b_{p,q}(u, v)$.

From the definition of the norms $\| \|_{p,q}$ can be obtained in a simple manner

LEMMA 4. *The inequality*

$$\|u\|_{p,q} \geqslant \|D_x^\alpha D_y^\beta u\|_{p-|\alpha|,q-|\beta|} \tag{2}$$

holds for

$$u \in \begin{cases} B & \text{when } |\alpha| \leqslant p, |\beta| \leqslant q, \\ B_{0,-} & \text{when } |\alpha| > p, |\beta| \leqslant q, \\ B_{-,0} & \text{when } |\alpha| \leqslant p, |\beta| > q, \\ C_0^\infty(\Omega) & \text{when } |\alpha| > p, |\beta| > q. \end{cases}$$

When Ω is the N-dimensional cube and B is the class of all functions which belong to $C^\infty(E^N)$ and are periodic, with Ω as the period-parallelogram, then the inequality (2) holds for all $u \in B$ without any restriction concerning the support.

2. Some differentiability properties of the spaces $H_{m,k}(\Omega, B)$

2.1. The present chapter contains some inequalities concerning the norms $\| \|_{m,k}$, which are similar to the inequalities for the norms $\| \|_m$ obtained by Ehrling [1]. From these inequalities follows and analogue to the Sobolev Lemma for the spaces $H_{m,k}(\Omega, B)$.

We make the following assumptions concerning the domains $\overset{j}{\Omega}$ $(j = 1, 2)$ (see [1] and [8]):

1° The boundary $\partial\overset{1}{\Omega}$ of $\overset{1}{\Omega}$ is a set-theoretical union of a finite number of pieces T each of which can be represented in a suitable chosen system of coordinates by the equation

$$x_R = f(x_1, \ldots, x_{R-1})$$

when the point $(x_1, \ldots, x_{R-1})$ varies over a closed domain of the space E^{R-1} and the function f satisfies the Lipschitz condition.

2° When the coordinate system is chosen as in point 1° and the positive direction of the x_R-axis is oriented outside $\overset{1}{\Omega}$, there is some positive constant h, such that for every point $x \in T$ the segment $(x_1, \ldots, x_{R-1}) = \text{const}$, $f(x_1, \ldots, x_{R-1}) > x_R > f(x_1, \ldots, x_{R-1}) - h$ belongs to $\overset{1}{\Omega}$.

3° $\partial\overset{1}{\Omega}$ is the boundary of $\partial\overset{1}{\Omega} \cup \overset{2}{\Omega}$.

4° there exists an R-dimensional spherical sector Σ with a positive radius and a positive spherical angle, so that each point $x \in \overline{\overset{1}{\Omega}}$ is the vertex of a sector Σ_x contained in $\overline{\overset{1}{\Omega}}$ and congruent with Σ.

5° $\overline{\overset{1}{\Omega}}$ is the union of a finite number of regions each of which is defined in a suitable system of coordinates by the inequalities

$$0 \leqslant x_i \leqslant d_i \quad (i = 1, 2, \ldots, R-1),$$
$$0 \leqslant x_R \leqslant g(x_1, \ldots, x_{R-1}),$$

where d_i are some constants and g is a continuous function with positive lower bound.

6° $\overset{2}{\Omega}$ satisfied conditions 1°-5°, the point $x \in E^R$ being replaced by $y \in E^S$.

Let $P^m(\Omega)$ be the class of all functions u having the following property: each derivative (in the ordinary sense) $D^\alpha u$ $(0 \leqslant |\alpha| \leqslant m)$ exists and is continuous everywhere in Ω, and can be extended to a continuous function in $\overline{\Omega}$. $P^{m,n}(\Omega)$ has a similar meaning, when the derivatives $D^\alpha u$ are replaced by $D_x^\alpha D_y^\beta u$ $(0 \leqslant |\alpha| \leqslant m,\ 0 \leqslant |\beta| \leqslant n)$. We put by definition $P^\infty(\Omega) = \bigcap_{m=0}^{\infty} P^m(\Omega)$.

From the inequalities proved by Ehrling in [1] follows in a simple manner

Lemma 5. *Put for* $u \in C_2^\infty(\Omega)$

$$|u|_{k,l}^2 \overset{\text{df}}{=} \sum_{\substack{|\alpha|=k \\ |\beta|=l}} \|D_x^\alpha D_y^\beta u\|_{L^2(\Omega)}^2.$$

There are positive constants A and t_0 (depending on Ω, m, n) such that the inequality

$$(3) \qquad |u|_{k,l}^2 \leqslant A t^{\frac{k+l}{m+n}-1} \left(t|u|_{0,0}^2 + t^{\frac{n}{m+n}} |u|_{m,0}^2 + t^{\frac{m}{m+n}} |u|_{0,n}^2 + |u|_{m,n}^2 \right)$$
$$(0 \leqslant k \leqslant m;\ 0 \leqslant l \leqslant n)$$

holds for $u \in P^\infty(\Omega)$ and $t \geqslant t_0$.

With the aid of similar estimates, as used by Ehrling [1], can be also proved

Lemma 6. *There exists a positive constant A (depending on Ω, $|\alpha|$, $|\beta|$) such that for $u \in P^\infty(\Omega)$*

$$(4) \qquad \sup_{(x,y) \in \overline{\Omega}} |D_x^\alpha D_y^\beta u(x, y)| \leqslant A \|u\|_{\left[\frac{R}{2}\right]+|\alpha|+1, \left[\frac{S}{2}\right]+|\beta|+1},$$

$$(5) \qquad \sup_{y \in \overline{\overset{2}{\Omega}}} \int_{\overset{1}{\Omega}} |D_x^\alpha D_y^\beta u(x, y)|^2 dx \leqslant A \|u\|_{|\alpha|, \left[\frac{S}{2}\right]+|\beta|+1}^2.$$

In the inequality (5) *the roles of x and y may be interchanged.*

2.2. We suppose now that B is a subset of $P^\infty(\Omega)$. The following two lemmas show that the functions belonging to the space $H_{m,k}(\Omega, B)$ with sufficiently large indices have some regularity properties analogous to those given by Sobolev's Lemma in the case of the space H_m.

Lemma 7. *Let $u \in H_{m,k}(\Omega, B)$ $(m > R/2,\ k > S/2)$. There exists a function $u_1 \in P^{m-\left[\frac{R}{2}\right]-1, k-\left[\frac{S}{2}\right]-1}(\Omega)$ such that the equalities*

$$D_x^\alpha D_y^\beta u(x, y) = D_x^\alpha D_y^\beta u_1(x, y) \qquad \left(0 \leqslant |\alpha| < m - \frac{R}{2},\ 0 \leqslant |\beta| < k - \frac{S}{2}\right)$$

hold almost everywhere in Ω. So the space $H_{m,k}(\Omega, B)$ may be treated as a subset of $P^{m-\left[\frac{R}{2}\right]-1, k-\left[\frac{S}{2}\right]-1}(\Omega)$.

Proof. According to the remarks of the section 1.6. and to lemma 6 it is sufficient to show, that the norms $|||\ |||_{m-[R/2]-1, k-[S/2]-1}$ and $\|\ \|_{m,k}$ are compatible on B, where

$$|||\ |||_{m,k} \overset{\text{df}}{=} \sup_{\substack{(x,y) \in \overline{\Omega} \\ 0 \leqslant |\alpha| \leqslant m \\ 0 \leqslant |\beta| \leqslant k}} |D_x^\alpha D_y^\beta u(x, y)|.$$

Let $\{u_n\} \subset B$ be a sequence fundamental in the both norms and tending to zero in the norm $\|\ \|_{m-[R/2]-1, k-[S/2]-1}$. Because of the completness of the space $H_{m,k}(\Omega, B)$ it is a square summable function u such that $\|u_n - u\|_{m,k} \underset{n\to\infty}{\to} 0$. But

$$\Big|\int_{\Omega} |u_n(x, y)|\,dx\,dy - \int_{\Omega} |u(x, y)|\,dx\,dy\Big| \leqslant \int_{\Omega} |u_n(x, y) - u(x, y)|\,dx\,dy$$

and the integral on the right is not greater than $|\Omega|^{1/2}\|u_n - u\|_{m,k}$. Therefore

$$\int_{\Omega} |u(x, y)|\,dx\,dy = 0$$

and so $u(x, y)$ vanish almost everywhere in Ω, q. e.d.

As a consequence of inequality (5) we obtain (with $A_1 = |\overset{1}{\Omega}|^{1/2}A^{1/2}$)

$$\sup_{y\in\overset{2}{\Omega}} \int_{\overset{1}{\Omega}} |D_x^\alpha D_y^\beta u(x, y)|\,dx \leqslant A_1 \|u\|^2_{|\alpha|, [S/2]+|\beta|+1}$$

for $u \in P^\infty(\Omega)$. A similar reasonning as in the proof of lemma 7 shows that the norms $|||\ |||_{m,k-[S/2]-1}$ and $\|\ \|_{m,k}$ are compatible on B, where

$$|||u|||_{m,k} \overset{\text{df}}{=} \sup_{\substack{y\in\overset{2}{\Omega}\\ 0\leqslant|\alpha|\leqslant m\\ 0\leqslant|\beta|\leqslant k}} \int_{\overset{1}{\Omega}} |D_x^\alpha D_y^\beta u(x, y)|\,dx.$$

From this it follows

LEMMA 8. *Let* $u \in H_{m,k}(\Omega, B)$ $(m \geqslant 0,\ k > S/2)$. *For each* α $(0 \leqslant |\alpha| \leqslant m)$ *there exists a function* $u^\alpha \in P^{k-[S/2]-1}(\overset{2}{\Omega})$ *such that the equalities*

$$\int_{\overset{1}{\Omega}} D_a^x D_y^\beta u(x, y)\,dx = D_y^\beta u^\alpha(y) \qquad (0 \leqslant |\alpha| \leqslant m,\ 0 \leqslant |\beta| < k - S/2)$$

hold almost everywhere in $\overset{2}{\Omega}$. *So for* $0 \leqslant |\alpha| \leqslant m$ *the functions*

$$\int_{\overset{1}{\Omega}} D_x^\alpha u(x, -)\,dx$$

may be treated as belonging to the class $P^{k-[S/2]-1}(\overset{2}{\Omega})$ *and the derivation of order* β *with respect to* y $(0 \leqslant |\beta| < k - S/2)$ *can be made under the sign of integral, when this last derivative is taken in the strong sense.*

3. Application of the spaces $H_{p,q}(\Omega, B)$ to the study of weak solutions of some non-elliptic equations

3.1. Let Λ be the class of all differential operators defined in Ω, which can be expressed in the form

$$(6)\qquad Lu = \sum_{\substack{|\alpha|=|\alpha'|=m\\ |\beta|=|\beta'|=n}} (-1)^{m+n} D_x^\alpha D_y^\beta (a_{\alpha\alpha'\beta\beta'} D_x^{\alpha'} D_y^{\beta'} u) + \sum_{\substack{0\leqslant|\alpha|,|\alpha'|\leqslant m\\ 0\leqslant|\beta|,|\beta'|\leqslant n\\ |\alpha|+|\alpha'|+|\beta|+|\beta'|<2(m+n)}} (-1)^{|\alpha|+|\beta|} D_x^\alpha D_y^\beta (b_{\alpha\alpha'\beta\beta'} D_x^{\alpha'} D_x^{\beta'} u) \qquad (m, n \geqslant 0)$$

for sufficiently differentiable u and which satisfies the following assumptions concerning the coefficients:

1° $a_{\alpha\alpha'\beta\beta'}$ and $b_{\alpha\alpha'\beta\beta'}$ are complex-valued functions infinitely differentiable in Ω and having all derivatives bounded in Ω,

2° $a_{\alpha\alpha'\beta\beta'}(x, y) = \overline{a}_{\alpha'\alpha\beta'\beta}(x, y)$ for $(x, y) \in \Omega$,

3° let ζ be the system of complex numbers $\zeta_{\alpha,\beta}(|\alpha| = m,\ |\beta| = n)$; when one puts

$$Q(x, y; \zeta) \overset{\text{df}}{=} \sum_{\substack{|\alpha|=|\alpha'|=m\\ |\beta|=|\beta'|=n}} a_{\alpha\alpha'\beta\beta'}(x, y)\zeta_{\alpha,\beta}\overline{\zeta_{\alpha',\beta'}}$$

there exists a positive constant d such that the inequality

$$Q(x, y; \zeta) \geqslant d \sum_{\substack{|\alpha|=m\\ |\beta|=n}} |\zeta_{\alpha,\beta}|^2$$

holds everywhere in Ω.

The expression on the right of (6) shall be called *canonical form* of the operator L. From assumption 3° it follows that the operators of class Λ are not elliptic in Ω. In the special case $n = 0$ operator L has the canonical form

$$(6\text{a})\qquad Lu = \sum_{|\alpha|=|\alpha'|=m} (-1)^m D_x^\alpha (a_{\alpha\alpha'} D_x^{\alpha'} u) + \sum_{\substack{0\leqslant|\alpha|,|\alpha'|\leqslant m\\ |\alpha|+|\alpha'|\leqslant 2m}} (-1)^{|\alpha|} D_x^\alpha (b_{\alpha\alpha'} D_x^{\alpha'} u).$$

It is elliptic in $\overset{1}{\Omega}$ and its coefficients depend on the parameter $y \in \overset{2}{\Omega}$. So the study of operators belonging to Λ gives us some informations about the elliptic operators depending on a parameter.

Denote by Δ_x the differential operator $I - \sum_{i=1}^{R} \partial^2/\partial x_i^2$; Δ_y has the same meaning, when x is replaced by y. Simple calculations show that

$$(\Delta_x)^r u = \sum_{0\leqslant|\mu|\leqslant r} (-1)^{|\mu|} \binom{r}{|\mu|} D_x^{2\mu} u,$$

and similarly for $(\Delta_y)^r$ $(r = 1, 2, \ldots)$. From the definition of class Λ follows in a simple manner

LEMMA 9. *When* $L \in \Lambda$, *the formal adjoined operator* L^+ *and all the products of* L *with* Δ_x *and* Δ_y *are also in* Λ.

3.2. In the following we apply the Hilbert spaces defined in chapter 1 to the study of the weak solution of equation

$$Lu = v, \tag{7}$$

when L belongs to Λ and v is an element of some space $H_{p,q}$. Our procedure is similar to the method used by Lax [7] in the case of an elliptic operator. We start with some energetic inequalities, which are analogous to the well known inequality for elliptic operators given by Gårding [4].

LEMMA 10. *Let L be an operator of class Λ with $m, n > 0$; so each of the differential expressions $\Delta_x^r \Delta_y^s L u$, $\Delta_x^r L \Delta_y^s u$, $\Delta_y^s L \Delta_x^r u$, $L\Delta_x^r \Delta_y^s u$, $(0 \leqslant r \leqslant r_0,\ 0 \leqslant s \leqslant s_0,\ u \in P^\infty(\Omega))$ can be brought to the canonical form*

$$\sum_{\substack{|\alpha|=|\alpha'|=m\\ |\beta|=|\beta'|=n\\ |\mu|=r\\ |\nu|=s}} (-1)^{m+n+r+s} D_x^{\alpha+\mu} D_y^{\beta+\nu}(a_{\alpha\alpha'\beta\beta'} D_x^{\alpha'+\mu} D_y^{\beta'+\nu} u) + \tag{8}$$
$$+ \sum_{\substack{0\leqslant|\alpha|,|\alpha'|\leqslant m+r\\ 0\leqslant|\beta|,|\beta'|\leqslant n+s\\ |\alpha|+|\alpha'|+|\beta|+|\beta'|<2(m+n+r+s)}} (-1)^{|\alpha|+|\beta|} D_x^{\alpha} D_y^{\beta}(c_{\alpha\alpha'\beta\beta'} D_x^{\alpha'} D_y^{\beta'} u).$$

Denote by $I_{r,s}(u)$ the corresponding Dirichlet integral and let Ω satisfy the assumptions of section 2.1. There are some positive constants t_1, c (depending on Ω, L, r_0, s_0) such that the inequality

$$|I_{r,s}(u)| \geqslant c\|u\|_{m+r,n+s}^2 \quad (u \in P^\infty(\Omega),\ 0 \leqslant r \leqslant r_0,\ 0 \leqslant s \leqslant s_0) \tag{9}$$

holds, when the functions $\operatorname{Re} b_{\alpha\alpha 00}$ $(|\alpha| = m)$, $\operatorname{Re} b_{00\beta\beta}$ $(|\beta| = n)$ and $\operatorname{Re} b_{0000}$ have a lower bound in Ω exceeding t_1.

Proof. Let $I_{r,s}^1(u)$ be the Dirichlet integral corresponding to the first sum in (8). According to the condition 3° (section 3.1) we have

$$I_{r,s}^1(u) = d\,|u|_{m+r,n+s}^2. \tag{10}$$

The second sum can be presented in the form

$$\sum_{\substack{0\leqslant|\mu|\leqslant r\\ 0\leqslant|\nu|\leqslant s\\ |\alpha|=m}}^{(1)} (-1)^{m+|\mu|+|\nu|} \binom{r}{|\mu|}\binom{s}{|\nu|} D_x^{\alpha+\mu} D_y^{\nu}(\operatorname{Re} b_{\alpha\alpha 00} D_x^{\alpha+\mu} D_y^{\nu} u) + \tag{11}$$
$$+ \sum_{\substack{0\leqslant|\mu|\leqslant r\\ 0\leqslant|\nu|\leqslant s\\ |\beta|=n}}^{(2)} (-1)^{n+|\mu|+|\nu|} \binom{r}{|\mu|}\binom{s}{|\nu|} D_x^{\mu} D_y^{\beta+\nu}(\operatorname{Re} b_{00\beta\beta} D_x^{\mu} D_y^{\beta+\nu} u) +$$
$$+ \sum_{\substack{0\leqslant|\mu|\leqslant r\\ 0\leqslant|\nu|\leqslant s}}^{(3)} (-1)^{|\mu|+|\nu|} \binom{r}{|\mu|}\binom{s}{|\nu|} D_x^{\mu} D_y^{\nu}(\operatorname{Re} b_{0000} D_x^{\mu} D_y^{\nu} u) +$$
$$+ \sum_{\substack{0\leqslant|\alpha|,|\alpha'|\leqslant m+r\\ 0\leqslant|\beta|,|\beta'|\leqslant n+s\\ |\alpha|+|\alpha'|+|\beta|+|\beta'|<2(m+n+r+s)}}^{(4)} (-1)^{|\alpha|+|\beta|} D_x^{\alpha} D_y^{\beta}(d_{\alpha\alpha'\beta\beta'} D_x^{\alpha'} D_y^{\beta'} u),$$

when the coefficients $d_{\alpha\alpha'\beta\beta'}$ do not depend on the functions $\operatorname{Re} b_{\alpha\alpha 00}$ $(|\alpha| = m)$, $\operatorname{Re} b_{00\beta\beta}$ $(|\beta| = n)$, $\operatorname{Re} b_{0000}$ (they depend only on the derivatives of these functions of order at most r with respect to x, and at most s with respect to y). Denoting by $I_r^{j,s}(u)$ the Dirichlet integral corresponding to the sum $\Sigma^{(j)}$ in (11) $(j = 1, 2, 3, 4)$ we obtain

$$I_{r,s}^1(u) \geqslant t_1\Big(|u|_{m+r,0}^2 + \sum_{\substack{|\alpha|=m\\ 0\leqslant|\mu|<r\\ 0\leqslant|\nu|\leqslant s}} \|D_x^{\alpha+\mu} D_y^{\nu} u\|_{L^2(\Omega)}^2 + \sum_{\substack{|\alpha|=m\\ |\mu|=r\\ 0<|\nu|\leqslant s}} \|D_x^{\alpha+\mu} D_y^{\nu} u\|_{L^2(\Omega)}^2\Big).$$

From this and from similar estimates for $I_{r,s}^2(u)$ and $I_{r,s}^3(u)$ follows

$$I_{r,s}^1(u) + I_{r,s}^2(u) + I_{r,s}^3(u) \geqslant t_1(|u|_{m+r,0}^2 + |u|_{0,n+s}^2 + |u|_{0,0}^2). \tag{12}$$

The remaining integral $I_{r,s}^4(u)$ can be estimated with the aid of inequality (3)

$$I_{r,s}^4(u) \leqslant \sum_{\substack{0\leqslant|\alpha|,|\alpha'|\leqslant m+r\\ 0\leqslant|\beta|,|\beta'|\leqslant n+s\\ |\alpha|+|\alpha'|+|\beta|+|\beta'|<2(m+n+r+s)}} \sup_{(x,y)\in\Omega} |d_{\alpha\alpha'\beta\beta'}(x, y)|\, \|D_x^{\alpha} D_y^{\beta} u\|_{L^2(\Omega)} \|D_x^{\alpha'} D_y^{\beta'} u\|_{L^2(\Omega)} \tag{13}$$
$$\leqslant \varphi(t)\big(t|u|_{0,0}^2 + t|u|_{m+r,0}^2 + t|u|_{0,n+s}^2 + |u|_{m+r,n+s}^2\big) \quad (t \geqslant t_0),$$

when $\varphi(t) \to 0$ as $t \to \infty$. Suppose $t_1 \geqslant t_0$; so from (10), (12) and (13) we obtain for $t \geqslant t_1$

$$|I_{r,s}(u)| \geqslant \big(d - \varphi(t)\big)|u|_{m+r,n+s}^2 + t_1\big(1 - \varphi(t)\big)(|u|_{0,0}^2 + |u|_{m+r,0}^2 + |u|_{0,n+s}^2)$$

Let

$$\varphi(t) \leqslant \min\Big(\frac{d}{2}, \frac{1}{2}\Big)$$

for $t \geqslant t_1'$. So for $t \geqslant \max(t_1, t_1')$

$$|I_{r,s}(u)| \geqslant \frac{d}{2}|u|_{m+r,n+s}^2 + \frac{t_1}{2}(|u|_{0,0}^2 + |u|_{m+r,0}^2 + |u|_{0,n+s}^2)$$

and according to the estimate (3) we get the inequality (9), q. e. d.

Using similar arguments the following two lemmas can be proved:

LEMMA 11. *Let L be an operator of class Λ with $m > 0$, $n = 0$ (so it is an elliptic operator depending on a parameter and its canonical form is given by formula (6a)). We suppose that Ω satisfies all the assumptions of section 2.1 and we denote by $I_r(u)$ the Dirichlet integral corresponding to the canonical form of the operator $\Delta_x^r L$ or $L\Delta_x^r$. There are some positive constants t_2, c (depending on Ω, L, r_0) such that*

$$|I_r(u)| \geqslant c\|u\|_{m+r,0}^2 \quad (u \in P^\infty(\Omega),\ 0 \leqslant r \leqslant r_0), \tag{14}$$

when the function $\operatorname{Re} b_{00}$ has lower bound in Ω exceeding t_2.

LEMMA 12. *We suppose that all the assumptions of lemma* 11 *are true; let* $I_{r,s}(u)$ $(0 \leqslant r \leqslant r_0,\ 0 \leqslant s \leqslant s_0)$ *have the same meaning as in lemma* 10. *So there are positive constants* t_3, c (*also depending on* Ω, L, r_0, s_0) *such that*

$$(15)\qquad |I_{r,s}(u)| \geqslant c\|u\|^2_{m+r,s} \qquad (u \in P^\infty(\Omega),\ 0 \leqslant r \leqslant r_0,\ 0 \leqslant s \leqslant s_0)$$

when the functions $\operatorname{Re} a_{\alpha\alpha}$ $(|\alpha| = m)$ *and* $\operatorname{Re} b_{00}$ *have a lower bound in* Ω *exceeding* t_3.

Remark. Simple calculation shows that the inequalities (9), (14) and (15) are true in the case $L = \Delta_x^m \Delta_y^n$ $(m, n \geqslant 0)$. More generally, when $a_{\alpha\alpha'\beta\beta'} = b_{\alpha\alpha'\beta\beta'} = 0$ for $\alpha \neq \alpha'$ or $\beta \neq \beta'$ and the remaining coefficients have a positive lower bound in Ω, L satisfies the energetic inequality

$$(16)\qquad |I(u)| \geqslant c\|u\|^2_{m,n},$$

although the assumption that some coefficients are large may be not satisfied. ($I(u)$ denotes the Dirichlet integral corresponding to the canonical form of L).

The inequalities (9), (14) and (15) can be brought to a different form when we suppose that the coefficients $a_{\alpha\alpha'\beta\beta'}$ and $b_{\alpha\alpha'\beta\beta'}$ (or $a_{\alpha\alpha'}$ and $b_{\alpha\alpha'}$) and the function u satisfy such boundary conditions that after the integration by parts the boundary integrals vanish. We obtain then the estimate

$$(17)\qquad |(L_{r,s}u, u)| \geqslant c\|u\|^2_{m+r,n+s} \qquad (u \in P^\infty(\Omega),\ 0 \leqslant r \leqslant r_0,\ 0 \leqslant s \leqslant s_0),$$

when $L_{r,s}$ denotes some of the operators $\Delta_x^r\Delta_y^s L$, $\Delta_x^r L \Delta_y^s$, $\Delta_y^s L \Delta_x^r$, $L\Delta_x^r\Delta_y^s$.

3.3. In this and in next section we suppose that Ω is the N-dimensional cube defined by inequalities $|x_i| < a$ $(i = 1, \ldots, R)$, $|y_j| < a$ $(j = 1, \ldots, S)$. Let B_p be the class of all functions infinitely differentiable in the whole space E^N and periodic with Ω as the period-parallelogram. Our purpose is a study of periodic weak solutions (see definition 2) of equation (7) with the aid of the spaces $H_{p,q}(\Omega, B_p)$. We begin with the following differential inequality:

LEMMA 13. *Let L be an operator of class Λ with coefficients* $a_{\alpha\alpha'\beta\beta'}$ *and* $b_{\alpha\alpha'\beta\beta'}$ (*or* $a_{\alpha\alpha'}$ and $b_{\alpha\alpha'}$) *belonging to* B_p. *We suppose that the inequality* (17) *is true. So*

$$(18)\qquad \|u\|_{p,q} \leqslant c\|Lu\|_{p-2m,q-2n} \qquad (u \in B_p)$$

(c *denotes some positive constant depending on* Ω, L, r_0, s_0).

Proof. We suppose, for example, that $p \leqslant m$, $q \geqslant n$ (the remaining cases may be treated similarly). Let $p = m-r$, $q = n+s$ $(0 \leqslant r \leqslant r_0$, $0 \leqslant s \leqslant s_0)$ and let u be an arbitrary function of class B_p. By means of

Fourier expansion $u_1 \in B_p$ can be constructed such that $\Delta_x^r u_1 = u$. From Lemma 4 it follows that

$$(19)\qquad \|u\|_{m-r,n+s} \leqslant c_1\|u_1\|_{m+r,n+s}.$$

Applying inequality (17) we get

$$c_2\|u_1\|^2_{m-r,n+s} \leqslant |(\Delta_y^s Lu, u_1)|.$$

After integration by parts it follows from this, in virtue of lemma 4 and theorem 1, that

$$(20)\qquad \|u_1\|^2_{m+r,n+s} \leqslant c_3\|Lu\|_{-m-r,-n+s}\|u_1\|_{m+r,n+s}.$$

From (19) and (20) follows estimate (18), q. e. d.

LEMMA 14. *Under the assumptions of lemma* 13 *the set* Γ *of all functions* Lv (*when* $v \in B_p$) *is dense in every space* $H_{p,q}(\Omega, B_p)$.

Proof. According to theorem 2 it is sufficient to examine the case $p \geqslant -m$, $q \geqslant -n$. Let l be a linear functional on $H_{-m,-n}(\Omega, B_p)$ vanishing on Γ. From theorem 1 follows the existence of $u_0 \in H_{m,n}(\Omega, B)$ such that

$$l(z) = (z, u_0) \qquad \big(z \in H_{-m,-n}(\Omega, B_p)\big).$$

Consider the bilinear form

$$b(v, u) \stackrel{\text{df}}{=} (Lv, u) \qquad (u \in H_{m,n}(\Omega, B_p),\ v \in B_p).$$

Because of the estimate

$$|b(v, u)| \leqslant \|u\|_{m,n}\|Lv\|_{-m,-n} \leqslant c\|u\|_{m,n}\|v\|_{m,n}$$

it can be enlarged to the continuous bilinear form on the whole space $H_{m,n}(\Omega, B_p)$, and according to our supposition

$$b(v, u_0) = 0 \qquad \big(v \in H_{m,n}(\Omega, B_p)\big);$$

in particular

$$(21)\qquad b(u_0, u_0) = 0.$$

From inequality (17) it follows in the limit that

$$|b(u_0, u_0)| \geqslant c\|u_0\|^2_{m,n};$$

therefore $u_0 = 0$ and according to theorem 1 the functional l has the norm zero, also vanish identically on $H_{-m,-n}(\Omega, B_p)$. Thus we have proved that Γ is dense in $H_{-m,-n}(\Omega, B_p)$.

Let now f_0 be an arbitrary function of class B_p and ε a positive number. When we apply what has been just proved to the operator $\Delta_x^r\Delta_y^s L$ it follows that there exists a function $f \in B_p$ such that

$$\|\Delta_x^r\Delta_y^s f_0 - \Delta_x^r\Delta_y^s Lf\|_{-m-r,-n-s} < \varepsilon.$$

From lemma 13 applied to the operator $\Delta_x^r \Delta_y^s$ it follows that

$$\|f_0 - Lf\|_{-m+r,-n+s} \leqslant c\,\|\Delta_x^r \Delta_y^s (f_0 - Lf)\|_{-m-r,-n-s},$$

and therefore Γ is dense in the space $H_{-m+r,-n+s}(\Omega, B_p)$ $(r, s \geqslant 0)$, q. e. d.

3.4. Definition 2. Let u, v be two elements of a space $H_{p_0,q_0}(\Omega, B_p)$ and let L be a differential operator with coefficients belonging to B_p. We say that u is the *periodic weak solution* of the equation

$$Lu = v \tag{7}$$

if the equality $(u, L^+\varphi) = (v, \varphi)$ holds identically for $\varphi \in B_p$.

The following theorem is analogous to the differentiability theorem of Lax [7] for elliptic equations.

THEOREM 3. *Let Ω the N-dimensional cube and L an operator of class Λ satisfying inequality* (17) *with coefficients belonging to B_p. We suppose that u is the periodic weak solution of equation* (7) *lying in a (sufficiently large) space $H_{p_0,q_0}(\Omega, B)_p$. When v is an element of $H_{p,q}(\Omega, B_p)$, then u is in $H_{p+2m,q+2n}(\Omega, B_p)$.*

Proof. From the generalized Cauchy inequality we obtain applying lemma 13 to the operator L^+ (when we suppose that r_0 and s_0 are sufficiently large)

$$|(L^+\varphi, u)| \leqslant c\|v\|_{p,q}\|L^+\varphi\|_{-p-2m,-q-2n}.$$

So the linear functional $l(\psi) \stackrel{\mathrm{df}}{=} (\psi, u)$ is bounded on the dense subset Γ of the space $H_{-p-2m,-q-2n}(\Omega, B_p)$ and therefore can be prolonged uniquely to the linear functional on the whole space. From theorem 1 it follows that u belongs to $H_{p+2m,q+2n}(\Omega, B_p)$, q. e. d.

It follows from theorem 3 and lemmas 7 and 8 that u has some differentiability properties in the classical sense, when the numbers $p+2m$ and $q+2n$ are non-negative and at least one of them is sufficiently large. In the special case $n = 0$, from theorem 3 follows the differentiability of periodic weak solutions of elliptic equations depending on a parameter (according to the remarks in section 3.1).

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