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On modular spaces of strongly summable sequences

by

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1. We shall apply the following notation: T is the linear space of sequences of real numbers with usual definitions of addition and scalar-multiplication (one might also consider sequences of complex numbers, without any essential differences), T_f — the space of "finite" sequences (i. e. sequences whose elements beginning with a certain index are all equal to zero), and T_0, T_c and T_b — the spaces of sequences convergent to zero, finite-convergent sequences and bounded sequences, respectively. The sequences will be denoted by $x = \{t_\nu\}$, $y = \{s_\nu\}, \ldots$, and x^n will denote the sequence $t_1, t_2, \ldots, t_n, 0, 0, \ldots$; e_n will mean the sequence $0, 0, \ldots, 0, 1, 0, \ldots$, having 1 at the n-th place, e_p^q — the sequence $0, 0, \ldots, 0, 1, 1, \ldots, 1, 0, \ldots$, having 1 at the p-th, $(p+1)$-th$, \ldots, (p+q-1)$-th place and zeros at other places, and finally e — the sequence $1, 1, 1, \ldots$

By *φ-function* we mean a continuous, non-decreasing function $\varphi(u)$, defined for $u \geqslant 0$, $\varphi(0) = 0$, $\varphi(u) > 0$ for $u > 0$ and $\varphi(u) \to \infty$ as $u \to \infty$. φ-functions will be denoted by $\varphi, \psi, \ldots$ and their inverse functions by $\varphi_{-1}, \psi_{-1}, \ldots$

Let $x = \{t_\nu\} \in T$. We define in T the functional

$$\varrho_\varphi(x) = \sup_n \frac{1}{n} \sum_{\nu=1}^{n} \varphi(|t_\nu|).$$

This functional is a modular in the sense of [9] over T, i. e. it satisfies the following conditions:

A1. $\varrho_\varphi(x) = 0$ is equivalent to $x = 0$,

A2. $\varrho_\varphi(-x) = \varrho_\varphi(x)$,

A3. $\varrho_\varphi(\alpha x + \beta y) \leqslant \varrho_\varphi(x) + \varrho_\varphi(y)$ for $\alpha, \beta \geqslant 0$, $\alpha + \beta = 1$.

We denote by T_φ^a the class of sequences $x = \{t_\nu\}$ for which

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\nu=1}^{n} \varphi(\lambda|t_\nu|) = 0 \quad \text{for every} \quad \lambda > 0. \tag{1.1}$$

Evidently, $T_f \subset T_0 \subset T_\varphi^a$. It is easily seen that T_φ^a is a linear space and ϱ_φ satisfies in T_φ^a the following condition (cf. [9]):

B1. $\varrho_\varphi(\lambda x) \to 0$ as $\lambda \to 0+$, $x \in T_\varphi^a$.

In particular, if $\varphi(u) = u^a$, $a > 0$, we shall write T_a instead of T_φ^a. In this case (1.1) is equivalent to

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\nu=1}^{n} |t_\nu|^a = 0; \tag{1.2}$$

thus T_φ^a becomes a field of sequences strongly summable of order a to zero. In the case of an arbitrary φ-function φ we may consider T_φ^a as a field of sequences strongly summable to zero by a method being a generalization of the classical method of strong summability of order a. Another generalization of the method of strong summability of order a will be given in section 3.

1.1. Since the modular ϱ_φ satisfies in T_φ^a the condition B1 besides the conditions A1-A3, the norm generated by ϱ_φ may be defined in T_φ^a by means of the formula

$$\|x\|_\varphi^a = \inf\{\varepsilon > 0 \colon \varrho_\varphi(x/\varepsilon) \leqslant \varepsilon\}. \tag{1.3}$$

It is easily seen that $\|\cdot\|_\varphi^a$ is a complete F-norm in T_φ^a and that the coordinates t_n of the sequence $x = \{t_\nu\}$ are continuous functionals with respect to this norm. If φ is an s-convex φ-function ($0 < s \leqslant 1$), i. e. if $\varphi(au+\beta v) \leqslant a^s\varphi(u)+\beta^s\varphi(v)$ for $u, v \geqslant 0$, $a, \beta \geqslant 0$, $a^s+\beta^s = 1$ (this implies that φ is strictly increasing, for assuming $0 < u < v$ we have $\varphi(u) \leqslant \varphi(v)u^s/v^s < \varphi(v)$), then a norm may be introduced in T_φ^a as follows:

$$\|x\|_{s\varphi}^a = \inf\{\varepsilon > 0 \colon \varrho_\varphi(x/\varepsilon^{1/s}) \leqslant 1\}. \tag{1.4}$$

The norm (1.4) is s-homogeneous and equivalent to the norm (1.3) (cf. [6], [10]). If $s = 1$, i. e. if φ is convex, then $\|\cdot\|_{1\varphi}^a$ is a homogeneous norm.

1.2. In the sequel we shall apply the following formula, if φ is s-convex:

$$\|e_p^q\|_{s\varphi}^a = \left(\varphi_{-1}\left(\frac{p+q-1}{q}\right)\right)^{-s}; \tag{1.5}$$

in particular,

$$\|e_n\|_{s\varphi}^a = (\varphi_{-1}(n))^{-s}. \tag{1.6}$$

1.3. *If $x \in T$ belongs to T_φ^a, then $\|x-x^n\|_\varphi^a \to 0$ as $n \to \infty$.*

The easy proof will be omitted.

Corollary. *The space T_φ^a is separable with respect to the norm $\|\cdot\|_\varphi^a$* (cf. [8]).

1.31. *$x \in T$ belongs to T_φ^a if and only if $\|x^n-x^m\|_\varphi^a \to 0$ as $m, n \to \infty$.*

2. *$T_\varphi^a \cap T_b = T_\psi^a \cap T_b$ for arbitrary two φ-functions φ and ψ.*

Proof. It is sufficient to show that $T_\varphi^a \cap T_b \subset T_\psi^a \cap T_b$. First, let us note that if $|t_\nu| < m\eta$ for every ν, then

$$\frac{1}{n}\sum_{\nu=1}^{n}\psi(|t_\nu|) \leqslant \psi(2\eta) + \frac{(m-1)\psi(2m\eta)}{\varphi(\eta)}\frac{1}{n}\sum_{\nu=1}^{n}\varphi(|t_\nu|). \tag{2.1}$$

Indeed, let $t_\nu \geqslant 0$ and put $t'_\nu = k\eta$ for $k\eta \leqslant t_\nu < (k+1)\eta$, $k = 0, 1, \ldots, m-1$. Then $|t'_\nu - t_\nu| < \eta$ for $\nu = 1, 2, \ldots$ and writing $\{n_\nu^k\} = \{\nu\colon k\eta \leqslant t_\nu < (k+1)\eta\}$, $\eta_i^k = 1$ if $i = n_\nu^k$ for a certain ν and $\eta_i^k = 0$ if $i \neq n_\nu^k$ for each ν, we have

$$\frac{1}{n}\sum_{\nu=1}^{n}\psi(2t'_\nu) = \frac{1}{n}\sum_{k=1}^{m-1}\sum_{i=1}^{n}\eta_i^k\psi(2k\eta) \leqslant \frac{(m-1)\psi(2m\eta)}{\varphi(\eta)}\frac{1}{n}\sum_{\nu=1}^{n}\varphi(t_\nu).$$

Hence

$$\frac{1}{n}\sum_{\nu=1}^{n}\psi(t_\nu) \leqslant \frac{1}{n}\sum_{\nu=1}^{n}\psi(2|t'_\nu - t_\nu|) + \frac{1}{n}\sum_{\nu=1}^{n}\psi(2t'_\nu)$$

$$\leqslant \psi(2\eta) + \frac{(m-1)\psi(2m\eta)}{\varphi(\eta)}\frac{1}{n}\sum_{\nu=1}^{n}\varphi(t'_\nu),$$

which proves formula (2.1).

Now, given an $x = \{t_\nu\} \in T_\varphi^a \cap T_b$, a $\lambda > 0$ and an $\varepsilon > 0$ we choose an $\eta > 0$ so that $\psi(2\eta) < \frac{1}{2}\varepsilon$ and then two integers m and n_0 so that $|\lambda t_\nu| < m\eta$ for each ν and

$$\frac{1}{n}\sum_{\nu=1}^{n}\varphi(\lambda|t_\nu|) < \frac{\varepsilon\varphi(\eta)}{2(m-1)\psi(2m\eta)}$$

for $n \geqslant n_0$. Then (2.1) implies

$$\frac{1}{n}\sum_{\nu=1}^{n}\psi(\lambda|t_\nu|) < \varepsilon$$

for $n \geqslant n_0$, whence $x \in T_\psi^a$.

Remark. It may be also deduced from formula (2.1) that if $x_i = \{t_\nu^i\} \in T_\varphi^a$ are uniformly bounded and if the sequence $\{x_i\}$ is modular convergent resp. $\|\cdot\|_\varphi^a$-norm convergent to zero for a φ-function φ, then $\{x_i\}$ is modular convergent resp. $\|\cdot\|_\psi^a$-norm convergent to zero for an arbitrary φ-function ψ.

2.1. A φ-function φ is called *non-weaker than a φ-function ψ for large u*, in symbols $\psi \overset{l}{\prec} \varphi$, if there exist constants $k, b, u_0 > 0$ such that

$$\psi(u) \leqslant b\varphi(ku) \quad \text{for} \quad u \geqslant u_0.$$

If $\psi \overset{l}{\dashv} \varphi$ and $\varphi \overset{l}{\dashv} \psi$, the functions φ and ψ are called *equivalent for large* u, in symbols $\varphi \overset{l}{\sim} \psi$. Evidently, $\varphi \overset{l}{\sim} \psi$ if and only if there are constants $a, b, k_1, k_2, u_0 > 0$ such that (cf. [5])

$$a\varphi(k_1 u) \leqslant \psi(u) \leqslant b\varphi(k_2 u) \quad \text{for} \quad u \geqslant u_0.$$

2.2. *If $T_\varphi^a \subset T_\psi^a$, then $x_i \in T_\varphi^a$, $\|x_i\|_\varphi^a \to 0$ imply $\|x_i\|_\psi^a \to 0$.*

This immediately follows from the closed graph theorem.

2.3. *The following three conditions are equivalent for φ-functions:*

(α) $\psi \overset{l}{\dashv} \varphi$,

(β) $T_\varphi^a \subset T_\psi^a$,

(γ) $\|x_i\|_\varphi^a \to 0$ *implies* $\|x_i\|_\psi^a \to 0$ *for arbitrary* $x_i \in T_f$.

Proof. (α) $\Rightarrow$ (β). Take an $x = \{t_\nu\} \in T_\varphi^a$ and an arbitrary $\lambda > 0$ and let k, b, u_0 be the constants mentioned in definition 2.1. Then we write $t'_\nu = t_\nu$ if $|t_\nu| < u_0/\lambda$ and $t'_\nu = 0$ if $|t_\nu| \geqslant u_0/\lambda$, and we put $t''_\nu = t_\nu - t'_\nu$. Obviously, $t'_\nu \in T_\varphi^a \cap T_b$, whence by 2, $\{t'_\nu\} \in T_\psi^a$. On the other hand, we have

$$\frac{1}{n}\sum_{\nu=1}^{n} \psi(\lambda|t''_\nu|) \leqslant \frac{b}{n}\sum_{\nu=1}^{n} \varphi(k\lambda|t''_\nu|)$$

for every n, whence it is easily seen that $\{t''_\nu\} \in T_\psi^a$. Hence $x = \{t'_\nu\} + \{t''_\nu\} \in T_\psi^a$.

(β) $\Rightarrow$ (γ) follows from 2.2.

(γ) $\Rightarrow$ (α). Let us suppose that $\psi \overset{l}{\dashv} \varphi$ does not hold. Given an $\varepsilon > 0$, a number u dependent on ε may be chosen satisfying the inequalities $\varphi(u) \geqslant \varepsilon$, $\psi(\varepsilon u) \geqslant 2\varepsilon^{-1}\varphi(u)$. Now, choosing an integer $n \geqslant 2$ so that $\frac{1}{2}\varepsilon n \leqslant \varepsilon(n-1) \leqslant \varphi(u) < \varepsilon n$ and writing $v = \varepsilon u$, we have

$$\varrho_\psi(v e_n) = \frac{1}{n}\psi(\varepsilon u) \geqslant \frac{2}{\varepsilon n}\varphi(u) \geqslant 1,$$

whence

$$\|v e_n\|_\psi^a \geqslant 1.$$

But, on the other hand,

$$\varrho_\varphi(\varepsilon^{-1} v e_n) = \frac{1}{n}\varphi\left(\frac{v}{\varepsilon}\right) < \varepsilon,$$

whence

$$\|v e_n\|_\varphi^a < \varepsilon.$$

Thus (γ) does not hold.

An immediate consequence of 2.2 and 2.3 is the following theorem:

2.4. *The following conditions are equivalent:*

(α) $\psi \overset{l}{\sim} \varphi$,

(β) $T_\varphi^a = T_\psi^a$,

(γ) *the norms* $\|\cdot\|_\varphi^a$, $\|\cdot\|_\psi^a$ *are equivalent in* T_f.

Similar arguments as in the proof of (γ) $\Rightarrow$ (α) in 2.3 give

2.41. *The condition $\varrho_\varphi(x_n) \to 0$ is equivalent to $\|x_n\|_\varphi^a \to 0$ for every $x_n \in T_\varphi^a$ if and only if φ satisfies condition (Δ_2) for large u*, i. e. $\varphi(2u) \leqslant k\varphi(u)$ *for* $u \geqslant u_0$ (cf. [8]).

2.5. *In order that the $\|\cdot\|_\varphi^a$-norm topology in T_φ^a be locally s-convex*([1]), *it is necessary and sufficient that $\varphi \overset{l}{\sim} \chi$, where $\chi(u) = \psi(u^s)$* and *ψ is a convex φ-function.*

Proof. In order to prove the necessity let us choose an s-convex neighbourhood U of zero in T_φ^a and a number $\delta > 0$ so that $\|x\|_\varphi^a \leqslant \delta$ implies $x \in U$ and that $x \in U$ implies $\|x\|_\varphi^a \leqslant 1$ for every $x \in T_f$. Given a number u satisfying the condition $\varphi(\delta^{-1}u) \geqslant \delta$, we choose an integer $n \geqslant 2$ so that

(2.2) $$\delta(n-1) \leqslant \varphi(\delta^{-1}u) < \delta n.$$

Let $a > 0$ satisfy the inequality $a^s \leqslant 1$. We choose a positive integer q such that

(2.3) $$\tfrac{1}{2} < q a^s \leqslant 1.$$

Since by (2.2)

$$\varrho_\varphi(\delta^{-1} u e_\nu) = \frac{1}{\nu}\varphi\left(\frac{u}{\delta}\right) \leqslant \delta \quad \text{for} \quad \nu \geqslant n,$$

we have $\|u e_\nu\|_\varphi^a \leqslant \delta$, whence $u e_\nu \in U$ for $\nu \geqslant n$. Hence (2.3) and the s-convexity of U yield

$$a u e_n^q = \sum_{\nu=n}^{n+q-1} a u e_\nu \in U,$$

but this implies $\|a u e_n^q\|_\varphi^a \leqslant 1$, whence

$$\frac{q}{n+q-1}\varphi(au) = \varrho_\varphi(a u e_n^q) \leqslant 1,$$

i. e.

$$\varphi(au) \leqslant 1 + \frac{n-1}{q} < 1 + \frac{1}{\delta}\varphi\left(\frac{u}{\delta}\right)2a^s,$$

by (2.2) and (2.3). Thus we have proved the inequality

(2.4) $$\varphi(au) \leqslant c a^s \varphi\left(\frac{u}{\delta}\right), \quad \text{where} \quad c = 1 + \frac{2}{\delta},$$

([1]) A set U is called *s-convex*, if $\alpha, \beta \geqslant 0$, $\alpha^s + \beta^s = 1$, $x, y \in U$ imply $\alpha x + \beta y \in U$; evidently, if U is s-convex and if $\alpha_1, \ldots, \alpha_n \geqslant 0$, $\alpha_1^s + \ldots + \alpha_n^s = 1$, $x_1, \ldots, x_n \in U$, then $\alpha_1 x_1 + \ldots + \alpha_n x_n \in U$. A linear topological Hausdorff space is called *locally s-convex*, if there is a base of s-convex neighbourhoods of zero in it (cf. [4], and [1], p. 163).

for all u such that $\varphi(u/\delta) \geqslant \delta$, $a^s\varphi(u/\delta) \geqslant 1$ and $0 < a^s \leqslant 1$. Now, the necessity follows from (2.4) in the same way as in [7].

The sufficiency of 2.5 is a consequence of the following statement which is obtained applying 2.41 and 1.1:

2.51. *If $\varphi \overset{l}{\sim} \chi$, where $\chi(u) = \psi(u^s)$ and ψ is a convex φ-function, then there exists in T_φ^a an s-homogeneous norm equivalent to the norm $\|\cdot\|_\varphi^a$.*

In particular, it follows from 2.5 and 2.51 that

2.52. *If the $\|\cdot\|_\varphi^a$-norm topology in T_φ^a is locally convex, then T_φ^a with this topology is a Banach space.*

2.6. Let φ denote a convex φ-function satisfying the conditions

$$(o_1) \qquad \frac{\varphi(u)}{u} \to 0 \quad \text{as} \quad u \to 0+,$$

$$(\infty_1) \qquad \frac{\varphi(u)}{u} \to \infty \quad \text{as} \quad u \to \infty.$$

Then there exists a function φ^* complementary to φ in the sense of Young and, as is well-known, φ^* is a convex φ-function satisfying conditions (o_1), (∞_1), too.

Given an arbitrary $x = \{t_\nu\} \in T_\varphi^a$, we write

$$\|x\|_\varphi^{*a} = \sup_y \sup_n \frac{1}{n} \sum_{\nu=1}^{n} t_\nu s_\nu,$$

where $\sup\limits_y$ is taken over all $y = \{s_\nu\} \in T_{\varphi^*}^a$ satisfying the inequality $\varrho_{\varphi^*}(y) \leqslant 1$. Since, applying Young's inequality to the functions φ and φ^*, we get the inequality

$$(2.5) \qquad \frac{1}{n}\sum_{\nu=1}^{n} t_\nu s_\nu \leqslant \frac{1}{\lambda n}\sum_{\nu=1}^{n} \varphi(\lambda|t_\nu|) + \frac{1}{\lambda n}\sum_{\nu=1}^{n} \varphi^*(|s_\nu|)$$

for every $\lambda > 0$, $\|x\|_\varphi^{*a}$ is finite. It is easily seen that $\|\cdot\|_\varphi^{*a}$ is a homogeneous norm in T_φ^a and the coordinates t_n of $x = \{t_\nu\}$ are continuous functionals in this norm.

2.7. *The norm $\|\cdot\|_\varphi^{*a}$ satisfies the inequalities*

$$(2.6) \qquad \tfrac{1}{2}\|x\|_\varphi^{*a} \leqslant \|x\|_{1\varphi}^a \leqslant \|x\|_\varphi^{*a} \quad \textit{for} \quad x \in T_\varphi^a.$$

Proof. Inequality (2.6) may be obtained by a modification of a known method; we give the proof for the sake of completeness.

It follows from the definition of $\|\cdot\|_\varphi^{*a}$ that

$$(2.7) \qquad \frac{1}{n}\sum_{\nu=1}^{n} t_\nu s_\nu \leqslant \|x\|_\varphi^{*a} \quad \text{when} \quad \varrho_{\varphi^*}(y) \leqslant 1$$

and

$$(2.8) \qquad \frac{1}{n}\sum_{\nu=1}^{n} t_\nu s_\nu \leqslant \|x\|_\varphi^{*a} \varrho_{\varphi^*}(y) \quad \text{when} \quad \varrho_{\varphi^*}(y) > 1.$$

First, let us assume that $x \in T_f$, $x \neq 0$. Let $\vartheta > 1$. We choose $\bar{s}_n \geqslant 0$ in such a way that

$$\frac{|t_n|\bar{s}_n}{\vartheta\|x\|_\varphi^{*a}} = \varphi\left(\frac{|t_n|}{\vartheta\|x\|_\varphi^{*a}}\right) + \varphi^*(\bar{s}_n),$$

where $n = 1, 2, \ldots$ Hence

$$(2.9) \qquad \frac{1}{n}\sum_{\nu=1}^{n} \frac{|t_\nu|\bar{s}_\nu}{\vartheta\|x\|_\varphi^{*a}} = \frac{1}{n}\sum_{\nu=1}^{n} \varphi\left(\frac{|t_\nu|}{\vartheta\|x\|_\varphi^{*a}}\right) + \frac{1}{n}\sum_{\nu=1}^{n} \varphi^*(\bar{s}_\nu)$$

for $n = 1, 2, \ldots$ Since $t_n = 0$ for sufficiently large n, we also have $\bar{s}_n = 0$ for sufficiently large n, say $n \geqslant n_0$, and consequently, $\varrho_{\varphi^*}(\bar{y}) < \infty$, where $\bar{y} = \{\bar{s}_n\}$.

Let us suppose that $\varrho_{*\varphi}(\bar{y}) > 1$. Then, by (2.8),

$$(2.10) \qquad \frac{1}{n}\sum_{\nu=1}^{n} \frac{|t_\nu|\bar{s}_\nu}{\vartheta\|x\|_\varphi^{*a}} \leqslant \frac{1}{\vartheta}\varrho_{\varphi^*}(\bar{y})$$

for $n = 1, 2, \ldots$, whence, choosing n_0 so large that

$$\frac{1}{n_0}\sum_{\nu=1}^{n_0} \varphi^*(\bar{s}_\nu) \geqslant \frac{1}{\vartheta}\varrho_{\varphi^*}(\bar{y}),$$

we have by (2.9) and (2.10)

$$\frac{1}{n_0}\sum_{\nu=1}^{n_0} \varphi\left(\frac{|t_\nu|}{\vartheta\|x\|_\varphi^{*a}}\right) + \frac{1}{\vartheta}\varrho_{\varphi^*}(\bar{y}) \leqslant \frac{1}{\vartheta}\varrho_{\varphi^*}(\bar{y}),$$

i. e. $t_\nu = 0$ for $\nu = 1, 2, \ldots, n_0$. But this implies $\bar{s}_\nu = 0$ for $\nu = 1, 2, \ldots, n_0$ and, consequently, $\varrho_{\varphi^*}(\bar{y}) = 0$, a contradiction.

Thus we have $\varrho_{\varphi^*}(\bar{y}) \leqslant 1$, whence (2.7) and (2.9) yield

$$\frac{1}{n}\sum_{\nu=1}^{n} \varphi\left(\frac{|t_\nu|}{\vartheta\|x\|_\varphi^{*a}}\right) \leqslant \frac{1}{\vartheta}$$

for every n, i. e.

$$\varrho_\varphi\left(\frac{x}{\|x\|_\varphi^{*a}}\right) = \sup_n \frac{1}{n}\sum_{\nu=1}^{n} \varphi\left(\frac{|t_\nu|}{\|x\|_\varphi^{*a}}\right) \leqslant 1.$$

Hence by definition (1.4) with $s=1$ we obtain

(2.11) $$\|x\|^a_{1\varphi} \leqslant \|x\|^{*a}_{\varphi},$$

where $x \in T_f$. However, if $x = \{t_\nu\} \in T^a_\varphi$ does not belong to T_f, then, by 1.3 and 1.1, $\|x^n - x\|^a_{1\varphi} \to 0$ as $n \to \infty$. Moreover, by (2.5), also $\|x^n - x\|^{*a}_{\varphi} \to 0$ as $n \to \infty$, whence inequality (2.11) holds for the element x, too.

Finally, we have

$$\frac{1}{n}\sum_{\nu=1}^{n} \frac{t_\nu s_\nu}{\|x\|^a_{1\varphi}} \leqslant \frac{1}{n}\sum_{\nu=1}^{n} \varphi\left(\frac{|t_\nu|}{\|x\|^a_{1\varphi}}\right) + \frac{1}{n}\sum_{\nu=1}^{n} \varphi^*(|s_\nu|)$$
$$\leqslant \varrho_\varphi\left(\frac{x}{\|x\|^a_{1\varphi}}\right) + \varrho_{\varphi^*}(y) \leqslant 2,$$

if $\varrho_{\varphi^*}(y) \leqslant 1$, whence

$$\|x\|^{*a}_{\varphi} \leqslant 2\,\|x\|^a_{1\varphi}.$$

3. In this section φ, ψ will denote convex φ-functions. For such a function φ the inverse function φ_{-1} always exists and is a concave function. Now, we define a modular

$$\varrho^b_\varphi(x) = \sum_{\nu=1}^{\infty} \varphi(|t_\nu|)$$

in T_f. By means of this modular we define in T_f a norm

(3.1) $$|||x|||_\varphi = \inf\{\varepsilon > 0 \colon \varrho^b_\varphi(x/\varepsilon) \leqslant 1\}.$$

Let us note that norm (3.1) is monotone, i. e. if $x = \{t_\nu\}$, $y = \{s_\nu\}$ and $|s_\nu| \leqslant |t_\nu|$ for every ν, then $|||y|||_\varphi \leqslant |||x|||_\varphi$. Moreover,

(3.2) $$|||e^q_p|||_\varphi = \big(\varphi_{-1}(1/q)\big)^{-1}.$$

The norm (3.1) may be applied to define some generalized strong methods of summability. Let T^b_φ be the class of sequences $x = \{t_\nu\}$ satisfying the condition

(3.3) $$\lim_{n\to\infty} \varphi_{-1}\left(\frac{1}{n}\right) |||x^n|||_\varphi = 0.$$

It is easily seen that $T_0 \subset T^b_\varphi$ and that for $\varphi(u) = u^a$, $a \geqslant 1$, we have $T^b_\varphi = T_a$ (cf. (1.2)), for in this case $\varphi_{-1}(1/n) = n^{-1/a}$, $|||x^n|||_\varphi = \big(\sum_1^n |t_\nu|^a\big)^{1/a}$. The method of strong summability defined by condition (3.3) was introduced in [11] by Taberski([2]). Evidently, T^b_φ is a linear space. We define in T^b_φ the norm

([2]) Strictly speaking, Taberski defines in [11] the norm $|||\cdot|||^*_\varphi$, the definition of which we give in (4.3), which is equivalent to the norm $|||\cdot|||_\varphi$ if φ satisfies conditions (o_1) and (∞_1).

(3.4) $$\|x\|^b_\varphi = \sup_n \varphi_{-1}\left(\frac{1}{n}\right) |||x^n|||_\varphi,$$

where $x = \{t_\nu\} \in T^b_\varphi$. T^b_φ provided with the norm $\|\cdot\|^b_\varphi$ is a Banach space and the coordinates t_n are continuous with respect to this norm.

3.1. *If $x = \{t_\nu\} \in T^b_\varphi$, then $\|x - x^n\|^b_\varphi \to 0$.*

Proof. Let $y = x - x^k$. Then $|||y^n|||_\varphi = 0$ for $n \leqslant k$ and $|||y^n|||_\varphi \leqslant |||x^n|||_\varphi$ for $n > k$ by the monotony of the norm $|||\cdot|||_\varphi$. Hence $\varphi_{-1}(1/n)\,|||y^n|||_\varphi \leqslant \varphi_{-1}(1/n)\,|||x^n|||_\varphi \to 0$ as $n \to \infty$, whence, given an $\varepsilon > 0$, $\varphi_{-1}(1/n)\,|||y^n|||_\varphi < \varepsilon$ for all n, if k is sufficiently large.

Corollary. *T^b_φ is separable.*

3.11. *$x = \{t_\nu\} \in T$ belongs to T^b_φ if and only if $\|x^n - x^m\|^b_\varphi \to 0$ as $m, n \to \infty$.*

The necessity of 3.11 follows from 3.1; the sufficiency is evident.

3.2. *In order that $T^a_\varphi \subset T^b_\psi$ it is necessary and sufficient that $\|x_n\|^a_\varphi \to 0$ implies $\|x_n\|^b_\psi \to 0$ for every sequence of elements $x_n \in T_f$.*

The necessity immediately follows from the closed graph theorem. The sufficiency is obtained from 3.11 and 1.31.

3.3. *$T^a_\varphi \subset T^b_\psi$ if and only if there exists a constant $\delta > 0$ satisfying the condition*

(3.5) $$\psi(\delta u v) \leqslant \varphi(u)\psi(v)$$

for all $u, v > 0$ such that $\varphi(u)\psi(v) \leqslant \delta$ and $\varphi(u) \geqslant 1$.

Proof. Necessity. Let $T^a_\varphi \subset T^b_\psi$. By 3.2 there is an $\eta > 0$ such that if $\|x\|^a_{1\varphi} \leqslant \eta$, then $\|x\|^b_\psi \leqslant 1$ for every $x \in T_f$, and we may suppose that $\eta \leqslant \frac{1}{2}$. Now, take $u, v > 0$ such that

(3.6) $$\varphi(u)\psi(v) \leqslant \eta, \quad \varphi(u) \geqslant 1.$$

Hence $\psi(v) \leqslant \eta$ and we may choose an integer $n \geqslant 2$ so that

(3.7) $$\frac{\eta}{n} < \psi(v) \leqslant \frac{\eta}{n-1},$$

whence $\varphi(u) < n$, by (3.6). We may choose an integer r satisfying the inequalities $1 \leqslant r < n$ so that

(3.8) $$\frac{1}{2} \leqslant \frac{r}{n}\varphi(u) < 1,$$

whence

$$u \leqslant \varphi_{-1}\left(\frac{n}{r}\right).$$

Thus, by (1.5),

$$\|u\eta e_n^r\|_{1\varphi}^a = u\eta\left(\varphi_{-1}\left(\frac{n+r-1}{r}\right)\right)^{-1} \leqslant u\eta\left(\varphi_{-1}\left(\frac{n}{r}\right)\right)^{-1} \leqslant \eta .$$

Hence by the definition of η we have

$$u\eta\psi_{-1}\left(\frac{1}{n+r-1}\right)\left(\varphi_{-1}\left(\frac{1}{r}\right)\right)^{-1} \leqslant \|u\eta e_n^r\|_\psi^b \leqslant 1,$$

whence

$$u\eta \leqslant \frac{\varphi_{-1}(1/r)}{\psi_{-1}(1/(n+r-1))}. \tag{3.9}$$

But ψ_{-1} being concave, the inequality $r<n$ gives

$$\psi_{-1}\left(\frac{1}{n+r-1}\right) > \psi_{-1}\left(\frac{1}{2n}\right) \geqslant \frac{1}{2}\psi_{-1}\left(\frac{1}{n}\right),$$

and by (3.9) we obtain

$$u\eta\psi_{-1}\left(\frac{1}{n}\right) \leqslant 2\varphi_{-1}\left(\frac{1}{r}\right),$$

whence, by (3.7) and (3.8),

$$\psi\left(\frac{1}{2}\eta uv\right) \leqslant \psi\left(\frac{1}{2}u\eta\psi_{-1}\left(\frac{\eta}{n-1}\right)\right) \leqslant \frac{1}{r} \leqslant \frac{2}{\eta}\varphi(u)\psi(v).$$

Now, choosing $\delta = \frac{1}{4}\eta^2$ we conclude the necessity in 3.3.

Sufficiency. Let $x=\{t_\nu\}\in T_\varphi^a$, $t_\nu \geqslant 0$. Assuming $\varphi(\lambda t_\nu)n^{-1}\leqslant\delta$ and $\varphi(\lambda t_\nu)\geqslant 1$, where $\delta\in(0,1)$ is given by (3.5), we have by (3.5)

$$\psi\left(\delta\lambda t_\nu\psi_{-1}\left(\frac{1}{n}\right)\right) \leqslant \varphi(\lambda t_\nu)\frac{1}{n}. \tag{3.10}$$

Given an $\varepsilon>0$, we choose $\lambda_0>0$ so large that $1/\delta\lambda_0<\frac{1}{2}\varepsilon$ and that $\varphi(\lambda_0 u)\leqslant 1$ implies $|u|<\frac{1}{2}\varepsilon$. We write $t_\nu = t_\nu' + t_\nu''$, $x'=\{t_\nu'\}$, $x''=\{t_\nu''\}$, where

$$t_\nu' = \begin{cases} t_\nu & \text{if} \quad \varphi(\lambda_0 t_\nu)\geqslant 1, \\ 0 & \text{if} \quad \varphi(\lambda_0 t_\nu) < 1. \end{cases}$$

Then we have

$$\psi_{-1}(1/n)\,|||x^n|||_\psi \leqslant \psi_{-1}(1/n)\,|||(x')^n|||_\psi + \psi_{-1}(1/n)\,|||(x'')^n|||_\psi . \tag{3.11}$$

The norm $|||\cdot|||_\psi$ being monotone, we have

$$|||(x'')^n|||_\psi \leqslant |||\tfrac{1}{2}\varepsilon e^n|||_\psi = \tfrac{1}{2}\varepsilon\left(\psi_{-1}(1/n)\right)^{-1},$$

i. e.

$$\psi_{-1}(1/n)\,|||(x'')^n|||_\psi \leqslant \tfrac{1}{2}\varepsilon \quad \text{for} \quad n=1,2,\ldots \tag{3.12}$$

Since $x'\in T_\varphi^a$, there is an integer n_0 such that

$$\frac{1}{n}\sum_{\nu=1}^{n}\varphi(\lambda_0 t_\nu') \leqslant \delta \quad \text{for} \quad n\geqslant n_0; \tag{3.13}$$

hence $\varphi(\lambda_0 t_\nu')n^{-1}\leqslant\delta$ for $1\leqslant\nu\leqslant n$, and since $\varphi(\lambda_0 t_\nu')\geqslant 1$ if $t_\nu'\neq 0$, we have

$$\sum_{\nu=1}^{n}\psi\left(\delta\lambda_0 t_\nu'\psi_{-1}(1/n)\right) \leqslant \frac{1}{n}\sum_{\nu=1}^{n}\varphi(\lambda_0 t_\nu') \leqslant \delta \leqslant 1$$

by (3.10) and (3.13). However, by definition (3.1) of the norm $|||\cdot|||_\psi$, this gives

$$|||(x')^n|||_\psi \leqslant \frac{1}{\delta\lambda_0\psi_{-1}(1/n)} < \frac{\varepsilon}{2\psi_{-1}(1/n)} \quad \text{for} \quad n\geqslant n_0. \tag{3.14}$$

Finally, inequalities (3.11), (3.12), and (3.14), give

$$\psi_{-1}(1/n)\,|||x^n|||_\psi \leqslant \varepsilon \quad \text{for} \quad n\geqslant n_0,$$

i. e. $x\in T_\psi^b$.

3.4. *$T_\psi^b \subset T_\varphi^a$ if and only if there exists a constant $\delta>0$ satisfying the condition*

$$\psi\left(\frac{1}{\delta}uv\right) \geqslant \varphi(u)\psi(v) \tag{3.15}$$

for all $u,v>0$ such that $uv\leqslant\delta$ and $u\geqslant 1$.

Proof. Necessity. We suppose that $T_\psi^b\subset T_\varphi^a$. Then there is an $\eta>0$ such that $\|x\|_\psi^b\leqslant\eta$ implies $\|x\|_{1\varphi}^a\leqslant 1$ for every $x\in T_f$, and we may suppose that $\eta\leqslant 1$. Now take $u,v>0$ satisfying the condition

$$uv\leqslant\eta\psi_{-1}(1), \quad u\geqslant\eta. \tag{3.16}$$

Then $v\leqslant\psi_{-1}(1)$ and there exists an integer $n\geqslant 2$ such that

$$\psi_{-1}\left(\frac{1}{n}\right) < v \leqslant \psi_{-1}\left(\frac{1}{n-1}\right). \tag{3.17}$$

Hence, by (3.16),

$$u\psi_{-1}\left(\frac{1}{n}\right) < \eta\psi_{-1}(1), \quad \psi\left(\frac{1}{\eta}u\psi_{-1}\left(\frac{1}{n}\right)\right) < 1.$$

Thus there exists a positive integer r such that

$$\frac{1}{2} \leqslant r\psi\left(\frac{1}{\eta}u\psi_{-1}\left(\frac{1}{n}\right)\right) < 1. \tag{3.18}$$

Let us remark that $r \leqslant n$. Indeed, supposing $r > n$, we should get

$$n\psi\left(\frac{1}{\eta} u\psi_{-1}\left(\frac{1}{n}\right)\right) < 1,$$

whence $u < \eta$, which is a contradiction.

Now, taking $n \leqslant j < n+r$, we have by (3.2)

$$(3.19) \qquad |||(ue_n^r)^j|||_\psi = u\left(\psi_{-1}\left(\frac{1}{j-n+1}\right)\right)^{-1}.$$

However, (3.18) yields

$$\frac{1}{\eta} u\psi_{-1}\left(\frac{1}{n}\right) \leqslant \psi_{-1}\left(\frac{1}{r}\right),$$

whence

$$\frac{1}{\eta} u\psi_{-1}\left(\frac{1}{j}\right) \leqslant \frac{1}{\eta} u\psi_{-1}\left(\frac{1}{n}\right) \leqslant \psi_{-1}\left(\frac{1}{r}\right) \leqslant \psi_{-1}\left(\frac{1}{j-n+1}\right).$$

Consequently, by (3.19) we obtain

$$\psi_{-1}\left(\frac{1}{j}\right) |||(ue_n^r)^j|||_\psi = \frac{u\psi_{-1}(1/j)}{\psi_{-1}(1/(j-n+1))} \leqslant \eta,$$

i. e.

$$\|ue_n^r\|_\psi^b \leqslant \eta.$$

But according to the choice of η, this implies

$$\|ue_n^r\|_{1\varphi}^b \leqslant 1.$$

Hence we obtain by (1.5)

$$u\left(\varphi_{-1}\left(\frac{2n}{r}\right)\right)^{-1} \leqslant u\left(\varphi_{-1}\left(\frac{n+r-1}{r}\right)\right)^{-1} = \|ue_n^r\|_{1\varphi}^a \leqslant 1$$

and this yields

$$(3.20) \qquad \varphi(u) \leqslant \frac{2n}{r}.$$

Now, from (3.17), we obtain $\psi(v) \leqslant 1/(n-1) \leqslant 2/n$ for $n \geqslant 2$, whence, by (3.20),

$$\frac{1}{2r} = \frac{1}{8}\cdot\frac{2n}{r}\cdot\frac{2}{n} \geqslant \frac{1}{8}\varphi(u)\psi(v).$$

But, by (3.18) and (3.17), the value on the left-hand side of these inequalities is less than $\psi(\eta^{-1}u\psi_{-1}(1/n)) \leqslant \psi(\eta^{-1}uv)$, whence

$$\varphi(u)\psi(v) \leqslant 8\psi\left(\frac{1}{\eta} uv\right) \leqslant \psi\left(\frac{8}{\eta} uv\right)$$

for $uv \leqslant \eta\psi_{-1}(1)$, $u \geqslant \eta$, and the necessity of 3.4 is proved with $\delta = \eta\min\left(\frac{1}{8}, \psi_{-1}(1)\right)$.

Sufficiency. Let $x = \{t_\nu\} \in T_\psi^b$, $t_\nu \geqslant 0$, and let the number $\delta \in (0,1)$ be given by (3.15). Assuming that $\lambda t_\nu \psi_{-1}(1/n) \leqslant \delta$ and $\lambda t_\nu \geqslant 1$ we have, by (3.15),

$$(3.21) \qquad \psi\left(\frac{1}{\delta}\lambda t_\nu \psi_{-1}\left(\frac{1}{n}\right)\right) \geqslant \frac{1}{n}\varphi(\lambda t_\nu).$$

We choose now two arbitrary positive numbers λ and $\varepsilon \leqslant \psi(1)$ and we take a $\lambda_0 \geqslant \lambda$, $\lambda_0 \geqslant 1$, such that $\lambda_0 u \leqslant 1$ implies $\varphi(u) \leqslant \varepsilon$. We write $t_\nu = t'_\nu + t''_\nu$, $x' = \{t'_\nu\}$, $x'' = \{t''_\nu\}$, where

$$t'_\nu = \begin{cases} t_\nu & \text{if} \quad \lambda_0\lambda t_\nu \geqslant 1, \\ 0 & \text{if} \quad \lambda_0\lambda t_\nu < 1. \end{cases}$$

Then $\lambda_0\lambda t''_\nu < 1$, whence $\varphi(\lambda t''_\nu) \leqslant \varepsilon$ and

$$(3.22) \qquad \frac{1}{n}\sum_{\nu=1}^{n}\varphi(\lambda t''_\nu) \leqslant \varepsilon.$$

Now we take $\eta = \delta\psi_{-1}(\varepsilon)\left(\lambda_0\lambda\psi_{-1}(1)\right)^{-1}$. Since $x' \in T_\psi^b$, there is an integer n_0 such that

$$\psi_{-1}(1/n)|||(x')^n|||_\psi < \eta$$

for $n \geqslant n_0$. The norm $|||\cdot|||_\psi$ being monotone we thus obtain by (3.2)

$$t'_\nu \frac{\psi_{-1}(1/n)}{\psi_{-1}(1)} = \psi_{-1}(1/n)|||t'_\nu e_\nu|||_\psi \leqslant \psi_{-1}(1/n)|||(x')^n|||_\psi < \eta \leqslant \frac{\delta\psi_{-1}(\varepsilon)}{\lambda_0\lambda\psi_{-1}(1)};$$

hence $\lambda_0\lambda t'_\nu\psi_{-1}(1/n) \leqslant \delta\psi_{-1}(\varepsilon) \leqslant \delta$, whence, by (3.21),

$$\frac{1}{n}\varphi(\lambda t'_\nu) \leqslant \frac{1}{n}\varphi(\lambda_0\lambda t'_\nu) \leqslant \psi\left(\frac{1}{\delta}\lambda_0\lambda t'_\nu\psi_{-1}\left(\frac{1}{n}\right)\right) \leqslant \varepsilon$$

for $1 \leqslant \nu \leqslant n$, $n \geqslant n_0$. Thus

$$(3.23) \qquad \frac{1}{n}\sum_{\nu=1}^{n}\varphi(\lambda t'_\nu) \leqslant \varepsilon \quad \text{for} \quad n \geqslant n_0.$$

Finally, by (3.22), (3.23),

$$\frac{1}{n}\sum_{\nu=1}^{n}\varphi(\lambda t_\nu) \leqslant 2\varepsilon \quad \text{for} \quad n \geqslant n_0,$$

i. e. $x \in T_\varphi^a$.

3.5. *If $T_\varphi^b = T_\varphi^a$, then $\varphi(u) \overset{l}{\sim} (\psi(1/u))^{-1}$.*

Proof. We take a number $\delta > 0$ introduced by 3.3 and 3.4; we may always suppose that $\delta < \psi(1)$. Let $u \geqslant 1/\delta$. Then $\delta u \geqslant 1$ and $\delta u \cdot u^{-1} = \delta$, whence we may apply (3.15) with δu and u^{-1} in place of u and v, respectively. We obtain

$$\psi(1) \geqslant \varphi(\delta u)\psi(1/u). \tag{3.24}$$

Moreover, taking $u \geqslant u_0 = \max(1/\delta, \psi(1)\varphi_{-1}(1)/\delta^2)$ we get $\varphi(\delta^2 u\psi^{-1}(1)) \geqslant 1$ and, by the convexity of φ and by (3.24),

$$\varphi\left(\frac{\delta^2 u}{\psi(1)}\right)\psi\left(\frac{1}{u}\right) \leqslant \frac{\delta}{\psi(1)}\varphi(\delta u)\psi\left(\frac{1}{u}\right) \leqslant \delta,$$

whence we may apply (3.5) with $\delta^2 u\psi^{-1}(1)$ and u^{-1} instead of u and v, respectively. We obtain

$$\psi\left(\frac{\delta^3}{\psi(1)}\right) \leqslant \varphi\left(\frac{\delta^2}{\psi(1)}u\right)\psi\left(\frac{1}{u}\right) \leqslant \frac{\delta}{\psi(1)}\varphi(\delta u)\psi\left(\frac{1}{u}\right),$$

i. e.

$$\varphi(\delta u)\psi\left(\frac{1}{u}\right) \geqslant \frac{1}{\delta}\psi(1)\psi\left(\frac{\delta^3}{\psi(1)}\right). \tag{3.25}$$

Formulae (3.24) and (3.25) give

$$\frac{1}{\psi(1)}\varphi(\delta u) \leqslant \frac{1}{\psi(1/u)} \leqslant \delta\left(\psi(1)\psi\left(\frac{\delta^3}{\psi(1)}\right)\right)^{-1}\varphi(\delta u)$$

for $u \geqslant u_0$, i. e. $\varphi(u) \overset{l}{\sim} (\psi(1/u))^{-1}$ (cf. 2.1).

Corollary. *There exist φ-functions φ such that $T_\varphi^b \neq T_\varphi^a$.*

Indeed, taking $\varphi(u) = e^u - 1 - u$ it is easily shown that the condition $\varphi(u) \overset{l}{\sim} (\varphi(1/u))^{-1}$ does not hold, whence $T_\varphi^b \neq T_\varphi^a$.

The above corollary may be strengthened as follows:

3.51. *If there exists a $\sigma_0 > 0$ such that*

$$\overline{\lim_{u\to\infty}} \frac{\varphi(u)}{u^\sigma} = \infty \tag{3.26}$$

for every $\sigma \geqslant \sigma_0$, where $\varphi(u)$ is a convex φ-function, then for every convex φ-function ψ, $T_\psi^b \neq T_\varphi^a$.

In particular, if $\varphi(u) = e^u - 1 - u$, then $T_\psi^b \neq T_\varphi^a$ for every convex φ-function ψ.

Proof. Let us suppose that $T_\varphi^a \subset T_\psi^b$ and that condition (3.26) holds. Take $u > \max(1/\delta, \varphi_{-1}(1))$, where $\delta > 0$ is given by 3.3 and

write $a = \delta u$. Choose $v_0 = \psi_{-1}(\delta/\varphi(u))$ and take $0 < v \leqslant v_0$. Then $\varphi(u) > 1$ and $\varphi(u)\psi(v) \leqslant \delta$, whence, by 3.4,

$$\psi(av) \leqslant c\psi(v), \tag{3.27}$$

where $c = \varphi(u)$. Now we choose an integer $n \geqslant 0$ so that

$$v_0 a^{-n-1} < v \leqslant v_0 a^{-n}. \tag{3.28}$$

Then, applying (3.27) successively, we obtain

$$c^{-n-1}\psi(v_0) \leqslant \psi(v). \tag{3.29}$$

In order to estimate c^{-n-1} we write $s = (n+1)\lg c/\lg a$. Substituting $n+1$ from this equation into (3.28), we get

$$\left(\frac{v}{av_0}\right)^{\lg c/\lg a} \leqslant c^{-n-1} = a^{-s} < \left(\frac{v}{v_0}\right)^{\lg c/\lg a},$$

whence, by (3.29),

$$\left(\frac{v}{av_0}\right)^{\lg c/\lg a}\psi(v_0) \leqslant \psi(v),$$

i. e.

$$\frac{1}{a^\sigma}\frac{\psi(v_0)}{v_0^\sigma} \leqslant \frac{\psi(v)}{v^\sigma} \quad \text{for} \quad v \leqslant v_0,$$

where $\sigma = \lg c/\lg a$. This shows that

$$\lim_{v\to 0+}\frac{\psi(v)}{v^\sigma} > 0 \tag{3.30}$$

for the above chosen σ (see [5]). Now, σ may be chosen arbitrarily large, for

$$\sigma = \frac{\lg c}{\lg a} = \frac{\lg\varphi(u)}{\lg\delta + \lg u},$$

and it follows from (3.26) that

$$\overline{\lim_{u\to\infty}}\frac{\lg\varphi(u)}{\lg u} = \infty.$$

We suppose now that $T_\psi^b = T_\varphi^a$ and that (3.26) holds. Then, by 3.5, there are constants $k, u_0 > 0$ such that $\varphi(\delta u)\psi(1/u) \leqslant k$ for $u \geqslant u_0$, whence

$$\frac{\varphi(\delta u)}{(\delta u)^\sigma}\cdot\frac{\psi(1/u)}{(1/u)^\sigma} \leqslant \frac{k}{\delta^\sigma} \quad \text{for} \quad u \geqslant u_0,\ \sigma > 0.$$

Applying (3.30) it is easily seen that

$$\overline{\lim_{u\to\infty}}\frac{\varphi(u)}{u^\sigma} < \infty,$$

where σ may be chosen arbitrarily large, which is a contradiction to (3.26).

It may be still asked whether either $T_\varphi^b \subset T_\varphi^a$ or $T_\varphi^a \subset T_\varphi^b$ always holds. As the following examples show, the answer to these two questions is negative.

3.52. (a) *If* $\varphi_1(u) = (1+u)\lg(1+u) - u$, *then there are* $x \in T_{\varphi_1}^a$ *which do not belong to* $T_{\varphi_1}^b$.

(b) *If* $\varphi_2(u) = e^u - 1 - u$, *then there are* $x \in T_{\varphi_2}^b$ *which do not belong to* $T_{\varphi_2}^a$.

Proof. (a) Let us choose $\delta > 0$ as in 3.3 and let $\varphi_1(u) \geqslant 1$, $\varphi_1(u)\varphi_1(1/u) \leqslant \delta$. Then, supposing $T_{\varphi_1}^a \subset T_{\varphi_1}^b$, we have $\varphi_1(\delta) \leqslant \varphi_1(u)\varphi_1(1/u)$, by (3.5). Hence $\varphi_1(u)\varphi_1(1/u) \geqslant \min(\delta, \varphi_1(\delta))$ for $\varphi_1(u) \geqslant 1$. However, this is impossible, for $\varphi_1(u) < u^{3/2}$ and $\varphi_1(1/u) < 1/u^2$ for sufficiently large u.

(b) Supposing $T_{\varphi_2}^b \subset T_{\varphi_2}^a$ and applying 3.4 with $v = \delta/u$, $u \geqslant 1$, we obtain $\varphi_2(u)\varphi_2(\delta/u) \leqslant \varphi_2(1)$. However, this is impossible, for $\varphi_2(u) > \frac{1}{6}u^3$ and $\varphi_2(\delta/u) > \delta^2/2u^2$.

4. A sequence $x = \{t_\nu\}$ will be called φ_a-resp. φ_b- *strongly summable to a number* $t = t(x)$ if $\{t_\nu - t\}$ belongs to T_φ^a resp. T_φ^b. Let us remark that the φ_a-method of summability is defined for an arbitrary φ-function, while the φ_b-method only for convex φ-functions. φ_b-methods of summability were introduced by assumptions (o_1), (∞_1) in [11]; it follows from 3.51 that in the general case they are not equivalent to the φ_a-methods. However, if $\varphi(u) = u^\alpha$, $\alpha \geqslant 1$, both methods coincide (we have then strong summability of order α). It is readily shown that the method φ_a as well as the method φ_b is permanent. We shall denote the field of summability of the method φ_a by $T(\varphi_a)$.

4.1. *For every φ-function φ the field $T(\varphi_a)$ provided with the F-norm $\|\cdot\|_\varphi^a$ defined in* (1.3) *is a complete normed linear space, and the coordinates t_n are continuous with respect to the norm $\|\cdot\|_\varphi^a$.*

Proof. It is easily seen that the modular $\varrho_\varphi(x)$ satisfies condition B1 for every $x \in T(\varphi_a)$. It remains to show that $T(\varphi_a)$ is complete (cf. [9]).

Assuming $x_n = \{t_\nu^n\} \in T(\varphi_a)$ and $\varrho_\varphi(\lambda(x_p - x_q)) \to 0$ as $p, q \to \infty$ for every $\lambda > 0$, we get $t_\nu^n \to t_\nu$ as $n \to \infty$ for $\nu = 1, 2, \ldots$ and $\varrho_\varphi(\lambda(x_n - x)) \to 0$ for every $\lambda > 0$, where $x = \{t_\nu\}$. It is sufficient to show that $x \in T(\varphi_a)$. Take an $\varepsilon > 0$ and let $t^n = t(x_n)$. Then $\{t_\nu^n - t^n\} \in T_\varphi^a$ for $n = 1, 2, \ldots$ and we have

$$\varphi(\tfrac{1}{3}|t^p - t^q|) \leqslant \frac{1}{n}\sum_{\nu=1}^{n} \varphi(|t_\nu^p - t^p|) + \frac{1}{n}\sum_{\nu=1}^{n} \varphi(|t_\nu^p - t_\nu^q|) + \frac{1}{n}\sum_{\nu=1}^{n} \varphi(|t_\nu^q - t^q|) < \varepsilon$$

for p, q sufficiently large and a certain $n = n(p, q)$; hence the sequence $\{t^n\}$ is convergent, say $t^n \to t$. Now, given $\lambda > 0$, we have

$$\frac{1}{n}\sum_{\nu=1}^{n} \varphi(\lambda|t_\nu - t|) \leqslant \varrho_\varphi(3\lambda(x_p - x)) + \frac{1}{n}\sum_{\nu=1}^{n} \varphi(3\lambda|t_\nu^p - t^p|) + \varphi(3\lambda|t^p - t|) < \varepsilon$$

for a fixed p dependent on ε and for sufficiently large n. Thus $\{t_\nu - t\} \in T_\varphi^a$, i. e. $x \in T(\varphi_a)$.

Continuity of the coordinates t_n with respect to the norm $\|\cdot\|_\varphi^a$ is obvious.

4.11. *The generalized limit $t(x)$ is defined uniquely and it is a distributive and modular-continuous functional in the space $T(\varphi_a)$, i. e. if $\varrho_\varphi(\lambda(x_n - x)) \to 0$ for a $\lambda > 0$, where $x_n \in T(\varphi_a)$, $x \in T(\varphi_a)$, then $t(x_n) \to t(x)$.*

The proof follows the same lines as that of 4.1.

4.2. *If the field of summability $T(\varphi_a)$ of a φ_a-method is a B_0-space by a B_0-norm $\|\cdot\|^\circ$ such that the coordinate t_n are continuous with respect to $\|\cdot\|^\circ$, then $\varphi \overset{l}{\sim} \psi$, where ψ is a convex φ-function.*

Proof. By the assumption and by 4.1 the norms $\|\cdot\|^\circ$ and $\|\cdot\|_\varphi^a$ are complete in $T(\varphi_a)$ and the coordinates are continuous in each of these norms. Hence, by the closed graph theorem, the norms $\|\cdot\|^\circ$ and $\|\cdot\|_\varphi^a$ are equivalent in $T(\varphi_a)$. Since T_φ^a is a closed linear subspace of $T(\varphi_a)$ in the norm $\|\cdot\|_\varphi^a$, the norms $\|\cdot\|^\circ$ and $\|\cdot\|_\varphi^a$ are also equivalent in T_φ^a and it is sufficient to apply 2.5 with $s = 1$.

In particular, it follows from 4.2 that

4.21. *If there exists a matrix-method of summability whose field is identical with the field of a φ_a-method, then $\varphi \overset{l}{\sim} \psi$, where ψ is a convex φ-function.*

This is a generalization of a Kuttner's theorem [3] (cf. [12]).

4.3. *If a convex φ-function satisfies conditions (o_1) and (∞_1), then the inverse φ_{-1} of φ and the inverse φ_{-1}^* of the function φ^* complementary to φ in the sense of Young satisfy the inequalities*

$$u \leqslant \varphi_{-1}(u)\varphi_{-1}^*(u) \leqslant 2u \quad \text{for} \quad u \geqslant 0.$$

This theorem is known (cf. [2], p. 25). For the sake of completeness we give here a proof which makes no use of the integral representation of the function φ.

Proof. Given a $u > 0$, we choose $v > 0$ in such a way that

$$\frac{uv}{u\varphi_{-1}^*(u)} = \frac{1}{u}\varphi\left(\frac{u}{\varphi_{-1}^*(u)}\right) + \frac{1}{u}\varphi^*(v). \tag{4.1}$$

We have

$$\frac{uv}{u\varphi_{-1}^*(u)} \leqslant 1 + \frac{1}{u}\varphi^*(v). \tag{4.2}$$

Indeed, if $u^{-1}\varphi^*(v) \leqslant 1$, then $v \leqslant \varphi_{-1}^*(u)$ and $uv/u\varphi_{-1}^*(u) \leqslant 1$, and if $u^{-1}\varphi^*(v) > 1$, then, by convexity of φ^*, $u^{-1}\varphi^*(v/u^{-1}\varphi^*(v)) \leqslant 1$ and $uv/u\varphi_{-1}^*(u) \leqslant u^{-1}\varphi^*(u)$.

From (4.1) and (4.2) it follows

$$\frac{1}{u}\varphi\left(\frac{u}{\varphi_{-1}^*(u)}\right) \leqslant 1 = \frac{1}{u}\varphi\left(\frac{1}{(\varphi_{-1}(u))^{-1}}\right),$$

whence, by the monotony of φ,

$$(\varphi_{-1}(u))^{-1} \leqslant \frac{1}{u}\varphi_{-1}^*(u),$$

and, consequently,

$$u \leqslant \varphi_{-1}(u)\varphi_{-1}^*(u).$$

On the other hand, applying Young's inequality to the values $\varphi_{-1}(u)$, $\varphi_{-1}^*(u)$ we get the inequality

$$\varphi_{-1}(u)\varphi_{-1}^*(u) \leqslant 2u.$$

4.4. Let φ be a convex φ-function satisfying (o_1) and (∞_1) and let φ^* be as in 4.3. We define a norm in T_f as follows:

$$|||x|||_\varphi^* = \sup_y \sum_{\nu=1}^{\infty} t_\nu s_\nu, \tag{4.3}$$

where the supremum is taken over all sequences $y = \{s_\nu\}$ satisfying the inequality $\Sigma\varphi(|s_\nu|) \leqslant 1$. It is well-known that $|||\cdot|||_\varphi^*$ is a homogeneous norm satisfying the inequalities

$$\frac{1}{2}|||x|||_\varphi^* \leqslant |||x|||_\varphi \leqslant |||x|||_\varphi^* \tag{4.4}$$

for all $x \in T_f$.

The following formula holds (cf. [11]):

$$|||e^n|||_\varphi^* = n\varphi_{-1}^*(1/n). \tag{4.5}$$

Indeed, we have

$$\varphi^*\left(\frac{1}{n}\sum_{\nu=1}^{n}|s_\nu|\right) \leqslant \frac{1}{n}\sum_{\nu=1}^{n}\varphi^*(|s_\nu|) \leqslant \frac{1}{n},$$

whence

$$\sum_{\nu=1}^{n}|s_\nu| \leqslant n\varphi_{-1}^*\left(\frac{1}{n}\right),$$

and, on the other hand, taking $s_\nu = \varphi_{-1}^*(1/n)$ for $\nu \leqslant n$, we get

$$\sum_{\nu=1}^{n}\varphi^*(|s_\nu|) = 1, \qquad \sum_{\nu=1}^{n}s_\nu = n\varphi_{-1}^*\left(\frac{1}{n}\right).$$

4.5. *Every method $T(\varphi_b)$, where φ is a convex φ-function satisfying the conditions (o_1) and (∞_1), is equivalent to a permanent row-finite matrix-method of summability $A = (a_{kl})$.*

In the case when $\varphi(u) = u^a$, $a \geqslant 1$, this theorem is proved by Zeller [12].

Proof. For every positive integer n, a finite number of sequences $s_i^j(n)$, $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, p(n)$ may be found such that

$$\sum_{\nu=1}^{n}\varphi^*(|s_\nu^j(n)|) \leqslant 1$$

and

$$\frac{1}{2}|||x|||_\varphi^* \leqslant \left|\sum_{\nu=1}^{n}t_\nu s_\nu^j(n)\right| \leqslant |||x|||_\varphi^* \tag{4.6}$$

for every sequence $x: t_1, t_2, \ldots, t_n, 0, 0, \ldots$

If $x = \{t_\nu\} \in T_\varphi^b$, then we have by (4.6)

$$\frac{1}{2}\varphi_{-1}\left(\frac{1}{n}\right)|||x^n|||_\varphi^* \leqslant \varphi_{-1}\left(\frac{1}{n}\right)\left|\sum_{\nu=1}^{n}t_\nu s_\nu^j(n)\right| \leqslant \varphi_{-1}\left(\frac{1}{n}\right)|||x^n|||_\varphi^*. \tag{4.7}$$

Applying (4.7) with $x = e$, (4.5) and 4.3 with $u = 1/n$ we get

$$\frac{1}{2} \leqslant \frac{1}{2}n\varphi_{-1}\left(\frac{1}{n}\right)\varphi_{-1}^*\left(\frac{1}{n}\right) \leqslant \varphi_{-1}\left(\frac{1}{n}\right)\left|\sum_{\nu=1}^{n}s_\nu^j(n)\right| \leqslant n\varphi_{-1}\left(\frac{1}{n}\right)\varphi_{-1}^*\left(\frac{1}{n}\right) \leqslant 2. \tag{4.8}$$

Let $a_\nu^j(n) = \sigma^j(n)\varphi_{-1}(1/n)s_\nu^j(n)$, where $\sigma^j(n)$ are chosen in such a manner that

$$\sum_{\nu=1}^{n}a_\nu^j(n) = 1 \quad \text{for} \quad n = 1, 2, \ldots; j = 1, 2, \ldots, p(n). \tag{4.9}$$

Inequalities (4.8) imply $\frac{1}{2} \leqslant |\sigma^j(n)| \leqslant 2$.

We form now a matrix $A = (a_{kl})$ as follows:

$$\begin{array}{l}
a_1^1(1), 0, 0, \ldots\ldots\ldots\ldots \\
\ldots\ldots\ldots\ldots\ldots\ldots \\
a_1^{p(1)}(1), 0, 0, \ldots\ldots \\
\ldots\ldots\ldots\ldots\ldots\ldots \\
a_1^1(n), \ldots, a_n^1(n), 0, 0, \ldots\ldots \\
\ldots\ldots\ldots\ldots\ldots\ldots \\
a_1^{p(n)}(n), \ldots, a_n^{p(n)}(n), 0, 0, \ldots \\
\ldots\ldots\ldots\ldots\ldots\ldots
\end{array}$$

It is easily seen that the method A satisfies the required conditions. Indeed, inequalities (4.7), (4.4) and definition (3.3) of the space T_φ^b imply that every sequence $x \in T_\varphi^b$ is A-summable to zero and, conversely, every sequence A-summable to zero belongs to T_φ^b. In particular, sequences convergent to zero are A-summable to zero; moreover, by (4.9), the sequence $1, 1, \ldots$ is A-summable to 1. Thus A is a permanent method and, moreover, a sequence is A-summable to t if and only if it is φ_b-summable to t.

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Les intégrales de fonctions presque-périodiques et les sections de séries de Fourier

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R. Doss a démontré que, f étant une fonction presque-périodique (p. p.) de Bohr et $\sum_{n=1}^{\infty} a_n e^{i\lambda_n t}$ sa série de Fourier, si les points $\alpha < \beta$ sont à distance positive du spectre $\{\lambda_n\}$, alors la série partielle $\sum_{\alpha<\lambda_n<\beta} a_n e^{i\lambda_n t}$ constitue le développement de Fourier d'une fonction de Bohr [3]. Doss ajoute qu'un résultat analogue subsiste pour les fonctions p. p. Stepanoff et p. p. Weyl. Nous nous proposons de généraliser ce théorème en atténuant les conditions et en admettant une notion plus générale de presque-périodicité. L'auteur tient à remercier M. J. -P. Kahane de ses remarques et de ses utiles conseils au cours de la rédaction de ce travail.

THÉORÈME 1. *Si $f(t) \sim \sum_n a_n e^{i\lambda_n t}$ est une fonction p. p. Bohr, Stepanoff* (S), *Weyl* (W) *ou Besicovitch* (B) *et si $\varphi(t) \in L^2(-\infty, \infty)$ est une fonction continue paire ou impaire, dont la transformée de Fourier*

$$\hat{\varphi}(u) = \underset{n\to\infty}{\mathrm{l.i.m.}} \frac{1}{2\pi} \int_{-n}^{n} \varphi(t) e^{-itu} dt$$

est integrable (L) *dans $(-\infty, \infty)$, alors la série $\sum_n a_n \varphi(\lambda_n) e^{i\lambda_n t}$ représente le développement de Fourier d'une fonction p. p. de Bohr, Stepanoff, Weyl ou Besicovitch respectivement.*

Démonstration. Si f est p. p. Bohr, p. p. S ou p. p. W, la fonction

$$\Phi(t) = \int_{-\infty}^{\infty} f(t-u) \hat{\varphi}(u) du \tag{1}$$

(bien définie pour presque tout t) est du même type respectivement, ce qui est facile à vérifier, puisque $\hat{\varphi} \in L(-\infty, \infty)$, en partant de la définition intrinsèque des classes examinées, c'est-à-dire d'une définition