Since $r\varrho - m < 0$ we have by Lemma 7

$$\sum_{s=0}^{k-r-1} D_k^{\left(k-s, \frac{r\varrho - m}{k-s}\right)} = O(x^{1+\frac{r\varrho - m}{k}}) .$$

Theorem 5 is now immediate. The following theorem results at once from Theorem 5 when we take $m = 2$.

THEOREM 6. *If $r\varrho < 2$, using the notations of Theorem 5,*

$$\Delta_k^{(r,\varrho)}(x) = -(r+1)\binom{k}{r+1} \sum_{\substack{n_1\dots n_{k-r-1} n_j^{r+1} \leqslant x \\ j=1,\dots,k-r-1}} \left(\frac{x}{n_1\dots n_{k-r-1}}\right)^{\frac{r(1+\varrho)}{r+1}} \psi_1\left(\frac{x}{n_1\dots n_{k-r-1}}\right)^{\frac{1}{r+1}} + \\ + O(x^{(k+r\varrho-2)/k}) .$$

Particular cases:

1. If we take $r = 0$, $\varrho = -1$, $k = 2$ in Theorem 6, we get

$$\Delta_2^{(0,-1)}(x) = -2 \sum_{n \leqslant x^{1/2}} \psi_1(x/n) + O(1) , \tag{30}$$

a result due to Landau [1], which was the starting point of Van der Corput's investigations of the Dirichlet's divisor problem.

2. Taking $r = 0$, $\varrho = -1$, $k = 3$, in Theorem 6, we get

$$\Delta_3^{(0,-1)}(x) = -3 \sum_{n_1^2 n_2, n_1 n_2^2 \leqslant x} \psi_1\left(\frac{x}{n_1 n_2}\right) + O(x^{1/3}) . \tag{31}$$

We have, from Theorem 6, trivially

$$\Delta_k^{(r,\varrho)}(x) = O\{D_k^{\left(r+1, \frac{r\varrho - 1}{r+1}\right)}(x)\} + O(x^{(k+r\varrho-2)/k}) \\ = O(x^{(k+r\varrho-1)/k}) \quad \text{if} \quad r\varrho < 1 , \text{ by Lemma 7} .$$

I shall return to the general problem of the order of $\Delta_k^{(r,\varrho)}(x)$ in a subsequent paper.

References

[1] E. Landau, *Gottinger Nachrichten*, 1920, pp. 13-32.
[2] A. Walfisz, *Gitterpunkte in mehrdimensionalen Kugeln*, Warszawa 1957.

Reçu par la Rédaction le 6. 6. 1962

ACTA ARITHMETICA
VIII (1963)

The lattice point problem of many-dimensional hyperboloids II

by

B. R. SRINIVASAN (Madras)

To the loving and respectful memory of Prof. Dr R. Vaidhyanathaswamy

1. In many problems in the analytic theory of numbers, it is necessary to obtain non-trivial inequalities for exponential sums of the form

$$\sum_{n} e^{2\pi i f(n)} \tag{1}$$

where $f(n)$ is a real function. An important method of obtaining such inequalities is due to Van der Corput [1]. Titchmarch ([10], [11]) has extended Van der Corput's method to two-dimensional sums of the type

$$\sum_{m,n} e^{2\pi i f(m,n)} . \tag{2}$$

We consider here sums of the type

$$\sum_{n_1,\dots,n_p} e^{2\pi i f(n_1,\dots,n_p)} \tag{3}$$

for arbitrary positive integer p and extend, step by step, Van der Corput's theory in one dimension to these p-dimensional sums. In the case $p = 1$ the present method reduces completely to Van der Corput's method. In the case $p = 2$ the present method includes (and in fact, slightly refines) Titchmarsh's method (cf. [8] also).

The method seems to be of general importance, but in each application there are considerable difficulties of detail. As a straightforward illustration, I consider here the lattice point problem of certain many-dimensional hyperboloids which I have considered elsewhere.

[1] For an account of the method and references, cf. [12].

Defining $D_k^{r,\varrho}(x)$ by

$$D_k^{(r,\varrho)}(x) = \binom{k}{r} \sum_{\substack{n_1\ldots n_{k-r} n_j^r \leqslant x \\ j=1,\ldots,k-r}} \left(\frac{x}{n_1\ldots n_{k-r}}\right)^{1+\varrho} \tag{4}$$

where x, ϱ are real $x \geqslant 1$, $\varrho > -2$; r, k integers such that $0 \leqslant r \leqslant k$; and setting

$$D_k^{(r,\varrho)}(x) = P_k^{(r,\varrho)}(x) + \Delta_k^{(r,\varrho)}(x) \tag{5}$$

where

$$\begin{aligned} P_k^{(r,\varrho)}(x) &= \frac{1}{\varrho}\binom{k}{r} \sum_{s=0}^{k-r} (-1)^s \binom{k-r}{s} u_s^{-(k-r-1)} x^{1+u_s} \times \\ &\quad \times \left\{1 + \tfrac{1}{2}u_s - u_s(u_s+1) \int_1^\infty \psi_1(y)\, y^{-u_s-2} dy\right\}^{k-r-s} \\ &\qquad \text{if } u_s = \varrho r/(r+s) \text{ and } \varrho r \neq 0; \\ &= \text{coeff of } u^{k-1} \text{ in } \frac{x^{1+u}}{u-\varrho}\left\{1 + \tfrac{1}{2}u - u(u+1)\int_1^\infty \psi_1(y)\, y^{-u-2} dy\right\}^k + \\ &\quad + \frac{x^{1+\varrho}}{\varrho^k}\left\{1 + \tfrac{1}{2}\varrho - \varrho(\varrho+1)\int_1^\infty \psi_1(y)\, y^{-\varrho-2} dy\right\}^k \quad \text{if } r = 0,\ \varrho \neq 0; \\ &= \text{coeff of } u^{k-r} \text{ in } x^{1+u}\left\{1 + \tfrac{1}{2}u - u(u+1)\int_1^\infty \psi_1(y)\, y^{-u-2} dy\right\}^k \quad \text{if } \varrho = 0, \end{aligned} \tag{6}$$

we have (cf. Theorem 6 of [9]) if $r\varrho < 2$

$$\begin{aligned} \Delta_k^{(r,\varrho)}(x) = -\binom{k}{r+1}(r+1) &\sum_{\substack{n_1\ldots n_{k-r-1} n_j^{r+1} \leqslant x \\ j=1,\ldots,k-r-1}} \left(\frac{x}{n_1\ldots n_{k-r-1}}\right)^{r(1+\varrho)/(r+1)} \times \\ &\times \psi_1\left(\left(\frac{x}{n_1\ldots n_{k-r-1}}\right)^{1/(r+1)}\right) + O\left(x^{(k+r\varrho-2)/k}\right), \end{aligned} \tag{7}$$

where

$$\psi_1(u) = u - (\text{integral part of } u) - \tfrac{1}{2}. \tag{8}$$

We concern ourselves with the problem of order of $\Delta_k^{(r,\varrho)}(x)$. We have, quite elementarily from (7), the result (cf. [9])

$$\Delta_k^{(r,\varrho)}(x) = O\left(x^{(k+r\varrho-1)/k}\right) \quad \text{if} \quad r\varrho < 1. \tag{9}$$

Let $\alpha_k^{(r,\varrho)}$ denote the lower bound of θ where

$$\Delta_k^{(r,\varrho)}(x) = O(x^\theta). \tag{10}$$

Then (9) gives

$$\alpha_k^{(r,\varrho)} \leqslant \frac{k+r\varrho-1}{k} \quad \text{if} \quad r\varrho < 1. \tag{11}$$

By the application of the present method we prove

$$\alpha_k^{(r,\varrho)} \leqslant 1 + \operatorname{Max}\left(\frac{r\varrho - \beta_{k-r}}{k}, \frac{r\varrho - \beta'_{k-r}}{r+1}\right) \quad \text{if} \quad r\varrho < 2, \tag{12}$$

where

$$\beta_{k-r} = \frac{2(k-r)(k-r+3)}{(k-r+1)(k-r+3)-1}, \frac{23}{14}, \frac{26}{17}, \frac{55}{41} \tag{13}$$

if $k \geqslant r+5$, $k = r+4$, $r+3$, $r+2$, respectively,

$$\beta'_{k-r} = \frac{2(k-r+3)}{(k-r+1)(k-r+3)-1}, \frac{16}{23} \tag{14}$$

if $k \geqslant r+3$, and $k = r+2$, respectively.

(12) is an improvement over (11) when

$$r\varrho < \frac{k\beta'_{k-r} - (r+1)}{k-(r+1)} \quad (<1).$$

In the case $\varrho = -1$, $D_k^{(r,\varrho)}(x)$ represents the number of lattice points bounded by the coordinate hyperplanes and a certain number of hyperboloids in a $k-r$ dimensional space. If further we take $r = 0$, the problem of the order of $\Delta_k^{(r,\varrho)}(x)$ is precisely the general (Piltz) divisor problem. We have from (12), (13), (14), when $r = 0$

$$\alpha_k^{(0,\varrho)} \leqslant \frac{(k-1)(k+3)-1}{(k+1)(k+3)-1}, \frac{33}{56}, \frac{25}{51}, \frac{27}{82} \tag{15}$$

if $k \geqslant 5$, $k = 4, 3, 2$, respectively.

When $k = 2$ and $\varrho = -1$, the above result is due to Van der Corput ([2], [7]). In the case $\varrho = -1$ (divisor problem case) further improvements on (15) are known (cf. Theorem 12.3 of [12]) though (15) is better than the classical estimate of Landau [5], who proved that

$$\alpha_k^{(0,-1)} \leqslant \frac{k-1}{k+1} \quad \text{for} \quad k \geqslant 2.$$

We first prove a number of lemmas. We then use them to obtain theorems on finite sums of the type

$$\sum_{n_1\ldots n_p} e^{-2\pi i}\left(\frac{x}{n_1\ldots n_p}\right)^{1/(r+1)}$$

and finally use them to obtain the required estimates for $\Delta_k^{(r,\varrho)}(x)$.

Throughout the paper, A denotes a positive absolute constant (sometimes depending only on p) not necessarily the same at each occurrence. The notations

$$f = O(g), \quad f \ll g$$

both mean that $|f| \leqslant A|g|$ for all values of the variables considered. The notation $f \gg \ll g$ means that $f \ll g$ and $g \ll f$. The letters n_i, ν_j denote integers.

2. Lemma 1 (Lemma of partial summation). *Let $g(n_1, \dots, n_p)$ denote any numbers, real or complex such that if*

(16) $$G(n_1, \dots, n_p) = \sum_{\substack{1 \leqslant \nu_i \leqslant n_i \\ i=1,\dots,p}} g(\nu_1, \dots, \nu_p),$$

then

$$|G(n_1, \dots, n_p)| \leqslant G \quad (1 \leqslant n_i \leqslant N_i,\ i = 1, \dots, p).$$

Let $h(n_1, \dots, n_p)$ denote real numbers, $0 \leqslant h(n_1, \dots, n_p) \leqslant H$, such that the $2^p - 1$ expressions

(17) $$\Big(\prod_{i=1}^{p} \Delta_{n_i}^{a_i}\Big) h(n_1, \dots, n_p)$$

where each $a_i = 0$ or 1, the case $a_1 = \dots = a_p = 0$ being excluded, keep a fixed sign for all values of n_i considered. Here Δ_{n_i} is defined by the equation

$$\Delta_{n_i} h(n_1, \dots, n_p) = h(n_1, \dots, n_p) - h(n_1, \dots, n_{i-1}, n_i + 1, n_{i+1}, \dots, n_p).$$

Then

$$\Big|\sum_{\substack{1 \leqslant n_i \leqslant N_i \\ i=1,\dots,p}} g(n_1, \dots, n_p) h(n_1, \dots, n_p)\Big| \leqslant \tfrac{1}{2}(1 + 3^p) GH.$$

Proof. For convenience in the proof, we adopt the convention that all the functions considered denote zero if the lattice point $(n_1, \dots, n_p)$ is outside the relevant domain, i.e. $1 \leqslant n_i \leqslant N_i$, $i = 1, \dots, p$.

First we observe that

(18) $$g(n_1, \dots, n_p) = G(n_1, \dots, n_p) - \sum_i G(n_1, \dots, n_i - 1, \dots, n_p) + \sum_{i,j} G(n_1, \dots, n_i - 1, \dots, n_j - 1, \dots, n_p) - \dots + (-1)^p G(n_1 - 1, \dots, n_p - 1).$$

To prove this we count the number of times a $g(\nu_1, \dots, \nu_p)$ occurs on the right-hand side of (18) when we substitute for $G(n_1, \dots, n_p)$ etc. using (16). If r $(0 \leqslant r \leqslant p)$ of the ν_i's are less than the corresponding n_i's while the other ν_i's are equal to the corresponding n_i's, the number of times $g(\nu_1, \dots, \nu_p)$ is counted on the right-hand side of (18) is

$$1 - \binom{r}{1} + \binom{r}{2} - \dots + (-1)^r = (1-1)^r = \begin{cases} 0 & \text{if} \quad r \geqslant 1, \\ 1 & \text{if} \quad r = 0. \end{cases}$$

This proves (18).

Next,

$$S = \sum_{\substack{1 \leqslant n_i \leqslant N_i \\ i=1,\dots,p}} g(n_1, \dots, n_p) h(n_1, \dots, n_p)$$
$$= \sum_{n_1,\dots,n_p} h(n_1, \dots, n_p) \Big\{ G(n_1, \dots, n_p) - \sum_i G(n_1, \dots, n_i - 1, \dots, n_p) + \dots + (-1)^p G(n_1 - 1, \dots, n_p - 1) \Big\}.$$

by (18). Hence

$$S = \sum_{n_1,\dots,n_p} G(n_1, \dots, n_p) \Delta_{n_1} \Delta_{n_2} \dots \Delta_{n_p} h(n_1, \dots, n_p)$$
$$= G(N_1, \dots, N_p) h(N_1, \dots, N_p) + \sum_{a_1,\dots,a_p} S_{a_1,\dots,a_p}$$

where

$$S_{a_1,\dots,a_p} = \sum_{a_1,\dots,a_p} G(n_1, \dots, n_p) \Big(\prod_{j=1}^{p} \Delta_{n_j}^{a_j}\Big) h(n_1, \dots, n_p).$$

In the above sum $\sum_{a_1,\dots,a_p}$, $n_j = N_j$ if $a_j = 0$ and the sum is taken over $1 \leqslant n_j \leqslant N_j - 1$ for $j \in J$, the set of j's such that $a_j = 1$.

Now,

$$|S_{a_1,\dots,a_p}| \leqslant G \sum_{a_1,\dots,a_p} \Big| \Big(\prod_{j=1}^{p} \Delta_{n_j}^{a_j}\Big) h(n_1, \dots, n_p) \Big|$$
$$= G \Big| \prod_{j \in J} \Big(\sum_{1 \leqslant n_j \leqslant N_j - 1} \Delta_{n_j} \Big) h(n_1, \dots, n_p) \Big|_{n_j = N_j \text{ for } j \notin J}$$
$$\leqslant GH 2^{r-1} \quad \text{if } r \text{ is cardinality of } J.$$

So

$$|S| \leqslant GH + \sum_{r=1}^{p} \binom{p}{r} GH 2^{r-1} = \tfrac{1}{2}(1+3^p) GH .$$

Hence the lemma. When $p=2$, the above lemma is Lemma a of [10].

Remark 1. In particular, the above lemma holds if the condition (17) is replaced by the condition that the 2^p-1 derivatives

$$\Big\{\prod_{i=1}^{p} \Big(\frac{\partial}{\partial x_i}\Big)^{a_i}\Big\} h(x_1, \ldots, x_p)$$

are of constant sign for all values of $x_1, \ldots, x_p$ considered.

The above Remark at once follows from the observation

$$\Big(\prod_{i=1}^{p} \Delta_{n_i}^{a_i}\Big) h(n_1, \ldots, n_p)$$

$$= \int \ldots \int_{n_i+1}^{n_i} \ldots \int \Big\{\prod_{i=1}^{p} \Big(\frac{\partial}{\partial x_i}\Big)^{a_i}\Big\} h(x_1, \ldots, x_p) \prod_{i=1}^{p} (dx_i)^{a_i} .$$

Remark 2. The above lemma is still valid if the hyper-rectangle $1 \leqslant n_i \leqslant N_i$, $i = 1, \ldots, p$, is replaced by an arbitrary region D contained in the hyper-rectangle and (16) is replaced by

$$(16') \qquad G(n_1, \ldots, n_p) = \sum_{\substack{1 \leqslant \nu_i \leqslant n_i,\ i=1,\ldots,p \\ (\nu_1,\ldots,\nu_p) \in D}} g(\nu_1, \ldots, \nu_p) .$$

To see this, we have only to apply the above lemma to the new function $g^*(n_1, \ldots, n_p)$ defined as follows:

$$g^*(n_1, \ldots, n_p) = \begin{cases} g(n_1, \ldots, n_p) & \text{if} \quad (n_1, \ldots, n_p) \in D , \\ 0 & \text{if} \quad (n_1, \ldots, n_p) \notin D . \end{cases}$$

Lemma 2. *Let $f(x_1, \ldots, x_p)$ be real in a region D contained in the rectangle $a_j \leqslant x_j \leqslant b_j$, $j = 1, \ldots, p$. Then*

$$\sum_{(n_1,\ldots,n_p) \in D} e^{2\pi i f(n_1,\ldots,n_p)} \ll \frac{\prod_{j=1}^{p}(b_j-a_j)}{q_1^{1/2}} + \Bigg\{\frac{\prod_{j=1}^{p}(b_j-a_j)}{q_1} \sum_{1 \leqslant u_1 \leqslant q_1-1} \Big| \sum_{n_1,\ldots,n_p} e^{2\pi i (f(n_1+u_1, n_2, \ldots, n_p) - f(n_1, \ldots, n_p))} \Big| \Bigg\}^{1/2} ,$$

where the summation on the right-hand is taken over the lattice points $(n_1, \ldots, n_p)$ for which both

$$(n_1+u_1, n_2, \ldots, n_p) \quad \text{and} \quad (n_1, \ldots, n_p) \quad \text{lie in } D ,$$

u_1 being an integer and the only restriction on q_1 being $0 < q_1 \leqslant b_1 - a_1$.

The above lemma is due to Van der Corput (cf. Satz 1 of [3]).

Lemma 3. *Let M and N be positive integers, $u_m\ (\geqslant 0)$ and $v_n\ (>0)$ $(1 \leqslant m \leqslant M, 1 \leqslant n \leqslant N)$ denote constants. Let $A_m\ (>0)$, $B_n\ (>0)$. Then there exists a q with the properties (Q_1 and Q_2 are given non-negative numbers) $Q_1 \leqslant q \leqslant Q_2$ and*

$$\sum_{m=1}^{M} A_m q^{u_m} + \sum_{n=1}^{N} B_n q^{-v_n}$$

$$\ll \sum_{m=1}^{M} \sum_{n=1}^{N} \sqrt[u_m+v_n]{A_m^{v_n} B_n^{u_m}} + \sum_{m=1}^{M} A_m Q_1^{u_m} + \sum_{n=1}^{N} B_n Q_2^{-v_n} .$$

Proof. Consider

$$f(x) = \sum_{m=1}^{M} A_m x^{u_m} + \sum_{n=1}^{N} B_n x^{-v_n} , \qquad x \text{ real} .$$

Let

$$\Phi(x) = \max_{1 \leqslant m \leqslant M} A_m x^{u_m} , \qquad g(x) = \max_{1 \leqslant n \leqslant N} B_m x^{-v_n} .$$

Then $\Phi(x)$ is monotonic increasing, while $g(x)$ is steadily decreasing. Hence $\Phi(x) - g(x)$ is steadily increasing and there is a unique q_0 such that $\Phi(q_0) - g(q_0) = 0$. Also $\Phi(x) \gtreqless g(x)$ according as $x \gtreqless q_0$. We consider three cases.

(i) Suppose $q_0 > Q_2$. Then $\Phi(Q_2) < g(Q_2)$. So

$$f(Q_2) \leqslant M\Phi(Q_2) + Ng(Q_2) < (M+N) g(Q_2) \ll \sum_{n=1}^{N} B_n Q_2^{-v_n} .$$

(ii) Suppose $q_0 < Q_1$. Then $g(Q_1) < \Phi(Q_1)$. So

$$f(Q_1) \leqslant M\Phi(Q_1) + Ng(Q_1) < (M+N)\Phi(Q_1) \ll \sum_{m=1}^{M} A_m Q_1^{u_m} .$$

(iii) Suppose $Q_1 \leqslant q_0 \leqslant Q_2$. Then $\Phi(q_0) - g(q_0) = 0$ gives

$$A_\alpha q_0^{u_\alpha} = B_\beta q_0^{-v_\beta}$$

for some α, β such that $1 \leqslant \alpha \leqslant M, 1 \leqslant \beta \leqslant N$.

We have

$$f(q_0) \ll (M+N)\Phi(q_0) \ll A_a q_0^{u_a} = \sqrt[u_a+v_\beta]{A_a^{v_\beta} B_\beta^{u_a}}$$

$$\ll \sum_{m=1}^{M} \sum_{n=1}^{N} \sqrt[u_m+v_n]{A_m^{v_n} B_n^{u_m}}\,.$$

In all the above cases, the Lemma is true. Hence, the Lemma.

The above Lemma is Lemma 4 of [7]. Since we are using the Lemma often in the sequal, we have reproduced its short proof here. In the case $Q_1 = 0$, $Q_2 = \infty$ (so that the parameter q is unrestricted except that is positive) the Lemma is due to Van der Corput (Hilfssatz 4 of [1]).

Remark 1. The constant involved in the majorisation $\ll$ depends only on M and N (in fact $\leqslant M+N$) and so is absolute if M and N are absolute constants.

Remark 2. The inequality above in Lemma 3 corresponds to the best possible choice of q in the range $Q_1 \leqslant q \leqslant Q_2$, i.e. the above inequality is stronger than (i.e. implies) any other inequality obtainable by considering any q in $Q_1 \leqslant q \leqslant Q_2$.

Proof of the above Remark 2 is easy and is given in [7]. Remark 1 is obvious.

3. Throughout the following lemmas, we suppose that D is a finite region in a p-dimensional Euclidean space and that any line parallel to any of the coordinate axes meets it in $O(1)$ straight line segments, and the same is true for the interesections of D with regions of the type $f_{x_i} \leqslant$ const and $f_{x_i} \geqslant$ const, $i = 1, \ldots, p$, where $f(x_1, \ldots, x_p)$ is a real function defined over D such that the transformation $y_i = f_{x_i}$, $i = 1, \ldots, p$, is one-one over D. We suppose further that any line parallel to any of the coordinate axes meets the surface got by equating to zero any of the second order partial derivatives of f in $O(1)$ points.

The conditions regarding the regions and the function f are in particular satisfied if D is bounded by $O(1)$ algebraic surfaces of bounded degrees and the surfaces $f_{x_i} = $ const, $f_{x_i x_j} = 0$ are also algebraic and of bounded degree.

LEMMA 4. *Let*

$$\left|\frac{\partial(f_{x_{i_1}}, \ldots, f_{x_{i_s}})}{\partial(x_{i_1}, \ldots, x_{i_s})}\right| \geqslant A r_{i_1} r_{i_2} \ldots r_{i_s} > 0 \quad \textit{in} \quad D$$

$$(1 \leqslant i_1 < i_2 < \ldots < i_s \leqslant p,\ 1 \leqslant s \leqslant p),$$

where the r's are independent of the x's. Then

$$\int_D \ldots \int e^{if(x_1,\ldots,x_p)} dx_1 \ldots dx_p \ll \frac{1}{\sqrt{r_1 \ldots r_p}}\,.$$

Proof. We shall prove this by induction on p. When $p = 1$, the above lemma is well known (Lemma 4.4 of [12]). We have

$$I = \int_D \ldots \int e^{if} dx_1 \ldots dx_p$$

$$= \int_{\substack{D, |f_{x_j}| \leqslant \sqrt{r_j} \\ j=1,\ldots,p}} \ldots \int e^{if} + \sum_{j=1}^{p} \int_{\substack{D \cap D_j \\ |f_{x_j}| > \sqrt{r_j}}} \ldots \int e^{if}$$

$$= J + \sum_{j=1}^{p} J_j\,, \qquad \text{say,}$$

where D_j is given by $|f_{x_s}| \leqslant \sqrt{r_s}$ for $1 \leqslant s < j$. Now,

$$\operatorname{re} J_j = \int_{D \cap D_j, |f_{x_j}| > \sqrt{r_j}} \ldots \int \cos f \, dx_1 \ldots dx_p$$

$$= \int_{D \cap D_j, |f_{x_j}| > \sqrt{r_j}} \ldots \int \frac{d(\sin f)}{f_{x_j}} dx_1 \ldots dx_{j-1} dx_{j+1} \ldots dx_p$$

$$= \int_{D \cap D_j, f_{x_j} > \sqrt{r_j}} \ldots \int + \int_{D \cap D_j, f_{x_j} < \sqrt{r_j}} \ldots \int$$

$$= \sum_{|f_{x_j}| > \sqrt{r_j}} \frac{1}{f_{x_j}(\xi_1, \ldots, \xi_p)} \int_{D_j'} \ldots \int \sin f \, dx_1 \ldots dx_{j-1} dx_{j+1} \ldots dx_p$$

by successive application of the second mean value theorem, since $1/f_{x_j}$ has $O(1)$ maxima nad minima on any line parallel to one of the coordinate axes. By the above and by the hypotheses on the regions, the sum here contains $O(1)$ terms (the ξ's and D_j''s in each of the terms are not necessarily the same). Also D_j' satisfy the general conditions satisfied by D. Hence, by the induction hypothesis,

$$\operatorname{re} J_j = O\left(\frac{1}{\sqrt{r_j}} \cdot \frac{1}{\sqrt{r_1 \ldots r_{j-1} r_{j+1} \ldots r_p}}\right) = O\left(\frac{1}{\sqrt{r_1 \ldots r_p}}\right).$$

Similarly

$$\operatorname{im} J_j = O\left(\frac{1}{\sqrt{r_1 \ldots r_p}}\right) \quad \text{and so} \quad J_j = O\left(\frac{1}{\sqrt{r_1 \ldots r_p}}\right).$$

We make the transformation $y_j = f_{x_j}$, $j = 1, \ldots, p$ in J. We then have

$$J = \int_{\substack{D, |f_{x_j}| \leqslant \sqrt{r_j} \\ j=1,\ldots,p}} \ldots \int e^{if} dx_1 \ldots dx_p$$

$$= \int_{D'} \ldots \int e^{i\Phi(y_1,\ldots,y_p)} \left|\frac{\partial(f_{x_1}, \ldots, f_{x_p})}{\partial(x_1, \ldots, x_p)}\right| dy_1 \ldots dy_p$$

$$= O\left(\frac{\sqrt{r_1 \ldots r_p}}{r_1 \ldots r_p}\right) = O\left(\frac{1}{\sqrt{r_1 \ldots r_p}}\right);$$

since D' is contained in the hyper-rectangle $|y_j| \leqslant \sqrt{r_j}$, $j = 1, \ldots, p$. Hence the lemma.

LEMMA 5. *Let $f(x_1, \ldots, x_p)$ posses continuous partial derivatives up to the third order in D. Again let*

$$\left|\frac{\partial(f_{x_{i_1}}, \ldots, f_{x_{i_s}})}{\partial(x_{i_1}, \ldots, x_{i_s})}\right| \geqslant A r_{i_1}^2 \ldots r_{i_s}^2 > 0 \qquad (1 \leqslant i_1 < i_2 < \ldots < i_s \leqslant p, 1 \leqslant s \leqslant p)$$

and

$$|f_{x_i x_j}| \leqslant A r_i r_j, \qquad |f_{x_i x_j x_k}| \leqslant A R r_i r_j r_k \qquad \text{throughout } D.$$

Further, let $f_{x_j}(c_1, \ldots, c_p) = 0$ for $j = 1, \ldots, p$. Let α_j and β_j be the values of f_{x_j} at the end-points of the largest segment of the curve $f_{x_1} = \ldots = f_{x_{j-1}} = f_{x_{j+1}} = \ldots = f_{x_p} = 0$ which contains $(c_1, \ldots, c_p)$ and lies entirely within D for $j = 1, \ldots, p$. Then if m is the number of changes of sign in the sequence

$$+1, \frac{\partial(f_{x_1}, \ldots, f_{x_j})}{\partial(x_1, \ldots, x_j)}, \qquad j = 1, \ldots, p;$$

then

$$\int_D \cdots \int e^{if} dx_1 \ldots dx_p - (2\pi)^{p/2} \frac{e^{\frac{i\pi}{4}(p-2m) + if(c_1, \ldots, c_p)}}{\left|\frac{\partial(f_{x_1}, \ldots, f_{x_p})}{\partial(x_1, \ldots, x_p)}\right|_{(c_1, \ldots, c_p)}^{1/2}}$$
$$\ll \frac{1}{r_1 \ldots r_p}\left\{R^{1/(p+4)} + \sum_{j=1}^{p} r_j\left(\frac{1}{|\alpha_j|} + \frac{1}{|\beta_j|}\right)\right\}.$$

Proof. We write

$$I = \int_D \cdots \int e^{if} dx_1 \ldots dx_p$$
$$= \int_{\substack{D, |f_{x_j}| \leqslant \delta r_j \\ j=1, \ldots, p}} \cdots \int e^{if} dx_1 \ldots dx_p + \sum_{j=1}^{p} \int_{\substack{D \cap D_j \\ |f_{x_j}| > \delta r_j}} \cdots \int e^{if} dx_1 \ldots dx_p$$
$$= J + \sum_{j=1}^{p} J_j \qquad \text{(say)},$$

where D_j is given by $|f_{x_s}| \leqslant \delta r_s$ for $1 \leqslant s < j$ and δ is a positive constant to be chosen later. Here we assume that the region $|f_{x_j}| \leqslant \delta r_j$, $j = 1, \ldots, p$, is entirely contained within D, that is

$$\text{(19)} \qquad \delta \leqslant \operatorname{Min}_j \left(\frac{|\alpha_j|}{r_j}, \frac{|\beta_j|}{r_j}\right).$$

By repeated application of the second mean value theorem (as in the proof of Lemma 4) and by Lemma 4, we find

$$\text{(20)} \qquad J_j = O\left(\frac{1}{\delta r_1 \ldots r_p}\right).$$

We put in the integral J, $y_j = f_{x_j}$, $j = 1, \ldots, p$, and

$$\Phi(y_1, \ldots, y_p) = \sum_{j=1}^{p} x_j y_j - f(x_1, \ldots, x_p).$$

Then $\Phi_{y_j} = x_j$, $j = 1, \ldots, p$, and Φ has continuous third partial derivatives in the transform Δ of the region D, by the transformation $y_j = f_{x_j}$, $1 \leqslant j \leqslant p$. Then we have

$$\text{(21)} \qquad J = \int_{-\delta r_1}^{\delta r_1} \cdots \int_{-\delta r_p}^{\delta r_p} e^{i(\sum_1^p y_j \Phi_{y_j} - \Phi)} \left|\frac{\partial(\Phi_{y_1}, \ldots, \Phi_{y_p})}{\partial(y_1, \ldots, y_p)}\right| dy_1 \ldots dy_p.$$

Now put $a_{ij} = \Phi_{y_i y_j}(0, \ldots, 0)$ and

$$2\chi(y_1, \ldots, y_p) = \sum_{1 \leqslant i \leqslant p} a_{ii} y_i^2 + 2 \sum_{1 \leqslant i \leqslant j \leqslant p} a_{ij} y_i y_j.$$

We have

$$\text{(22)} \qquad \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{i\chi} dy_1 \ldots dy_p = \int_{-\delta r_1}^{\delta r_1} \cdots \int_{-\delta r_p}^{\delta r_p} e^{i\chi} + \sum_{j=1}^{p} \int_{D_j, |y_j| > \delta r_j} \cdots \int e^{i\chi},$$

where D_j is given by $|y_s| \leqslant \delta r_s$ for $1 \leqslant s < j$ and $|y_s| \leqslant \infty$ for $s > j$. Again

$$2\chi = q_1 y_1^2 + q_2(y_2 + a_{21} y_1)^2 + \ldots + q_p(y_p + a_{p,p-1} y_{p-1} + \ldots + a_{p,1} y_1)^2$$
$$= \sum_{j=1}^{p} q_j z_j^2 \qquad \text{(say)},$$

where

$$q_j = \left(\frac{\partial(\Phi_{y_j}, \ldots, \Phi_{y_p})}{\partial(y_j, \ldots, y_p)} \bigg/ \frac{\partial(\Phi_{y_{j+1}}, \ldots, \Phi_{y_p})}{\partial(y_{j+1}, \ldots, y_p)}\right)_{(0,0,\ldots,0)}.$$

Now the matrices $(\Phi_{y_i y_j})$ and $(f_{x_i x_j})$ are inverse matrices since

$$\sum_j \Phi_{y_i y_j} f_{x_j x_k} = \sum_j \frac{\partial x_i}{\partial y_j} \frac{\partial y_j}{\partial x_k} = \frac{\partial x_i}{\partial x_k} = \delta_{ik}$$

(the Kronecker delta).

And so we find

$$q_j = \left(\frac{\partial(f_{x_1}, \ldots, f_{x_{j-1}})}{\partial(x_1, \ldots, x_{j-1})} \bigg/ \frac{\partial(f_{x_1}, \ldots, f_{x_j})}{\partial(x_1, \ldots, x_j)}\right)_{(c_1, \ldots, c_p)} = \frac{\Delta_{j-1}}{\Delta_j} \qquad \text{(say)}.$$

Now

$$\int_{D_1, |y_1| > \delta r_1} \cdots \int e^{i\chi} dy_1 \ldots dy_p = \int_{D_1', |z_1| > \delta r_1} \cdots \int e^{\frac{i}{2}\sum_{j=1}^{p} q_j z_j^2} dz_1 \ldots dz_p \tag{23}$$

$$= O\left(\frac{1}{|q_1|\delta r_1} \cdot \frac{1}{\sqrt{|q_2 \ldots q_p|}}\right)$$

$$= O\left(\frac{1}{\delta r_1}\sqrt{|f_{x_1^2}|}\sqrt{\left|\frac{\partial(f_{x_1}, \ldots, f_{x_p})}{\partial(x_1, \ldots, x_p)}\right|_{(c_1, \ldots, c_p)}}\right)$$

$$= O\left(\frac{r_1 \ldots r_p}{\delta}\right)$$

by repeated application of the second mean value theorem, and by Lemma 4. Similarly, for every $j = 2, \ldots, p$,

$$\int_{D_j, |y_j| > \delta r_j} \cdots \int e^{i\chi} dy_1 \ldots dy_p = O\left(\frac{r_1 \ldots r_p}{\delta}\right). \tag{24}$$

Next

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{i\chi} dy_1 \ldots dy_p = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{\frac{i}{2}(\sum_{j=1}^{p} q_j z_j^2)} dz_1 \ldots dz_p \tag{25}$$

$$= \prod_{j=1}^{p}\left(2\int_0^{\infty} e^{i\frac{q_j}{2}z_j^2} dz_j\right)$$

$$= (2\pi)^{p/2} \frac{e^{\frac{i\pi}{4}(p-2m)}}{\sqrt{|q_1 \ldots q_p|}}$$

$$= (2\pi)^{p/2} e^{\frac{i\pi}{4}(p-2m)} \left|\frac{\partial(f_{x_1}, \ldots, f_{x_p})}{\partial(x_1, \ldots, x_p)}\right|^{1/2}_{(c_1, \ldots, c_p)}$$

where m is the number of $q_j < 0$ for $j = 1, \ldots, p$, i.e., m is the number of changes of sign in the sequence

$$+1, \frac{\partial(f_{x_1}, \ldots, f_{x_j})}{\partial(x_1, \ldots, x_j)}, \quad j = 1, \ldots, p.$$

From (20) to (25), we have

$$I = (2\pi)^{(p/2)} \frac{e^{\frac{i\pi}{4}(p-2m) + if(c_1, \ldots, c_p)}}{\left|\frac{\partial(f_{x_1}, \ldots, f_{x_p})}{\partial(x_1, \ldots, x_p)}\right|^{1/2}_{(c_1, \ldots, c_p)}} + I' + I'' + O\left(\frac{1}{\delta r_1 \ldots r_p}\right) \tag{26}$$

where

$$I' = \int_{-\delta r_1}^{\delta r_1} \cdots \int_{-\delta r_p}^{\delta r_p} e^{i(\sum_1^p y_j \Phi_{y_j} - \Phi)} \left\{\left|\frac{\partial(\Phi_{y_1}, \ldots, \Phi_{y_p})}{\partial(y_1, \ldots, y_p)}\right| - \left|\frac{\partial(\Phi_{y_1}, \ldots, \Phi_{y_p})}{\partial(y_1, \ldots, y_p)}\right|_{(0, \ldots, 0)}\right\} dy_1 \ldots dy_p,$$

$$I'' = \left|\frac{\partial(\Phi_{y_1}, \ldots, \Phi_{y_p})}{\partial(y_1, \ldots, y_p)}\right|_{(0, \ldots, 0)} \int_{-\delta r_1}^{\delta r_1} \cdots \int_{-\delta r_p}^{\delta r_p} dy_1 \ldots dy_p \{e^{i(\sum_{j=1}^{p} y_j \Phi_{y_j} - \Phi)} - e^{i(\chi - \Phi(0, \ldots, 0))}\}.$$

Now

$$\Phi_{y_i y_j} = \frac{\text{cofactor of } f_{x_i x_j} \text{ in } H(f)}{H(f)} \ll \sum \frac{r_1^2 \ldots r_p^2}{r_1^2 \ldots r_p^2} \cdot \frac{1}{r_i r_j} \ll \frac{1}{r_i r_j}$$

(where $H(f)$ denotes the Hessian of f),

$$\frac{\partial}{\partial x_i}\left\{\frac{\partial(f_{x_1}, \ldots, f_{x_p})}{\partial(x_1, \ldots, x_p)}\right\} \ll R r_i r_1^2 \ldots r_p^2,$$

and

$$\frac{\partial}{\partial y_i}\left(\frac{\partial(\Phi_{y_1}, \ldots, \Phi_{y_p})}{\partial(y_1, \ldots, y_p)}\right) = -\sum_j \Phi_{y_i y_j} \frac{\frac{\partial}{\partial x_j}\left(\frac{\partial(f_{x_1}, \ldots, f_{x_p})}{\partial(x_1, \ldots, x_p)}\right)}{|H(f)|^2}$$

$$\ll \sum_j \frac{1}{r_i r_j} \cdot \frac{R r_j r_1^2 \ldots r_p^2}{r_1^4 \ldots r_p^4} \ll \frac{R}{r_1^2 \ldots r_p^2 r_i},$$

$$\Phi_{y_i y_j y_k} = \frac{\partial}{\partial y_k}\left(\frac{\text{cofactor of } f_{x_i x_j} \text{ in } H(f)}{H(f)}\right)$$

$$= \frac{1}{H(f)} \sum_l \Phi_{y_k y_l} \frac{\partial}{\partial x_l}\{\text{cofactor of } f_{x_i x_j} \text{ in } H(f)\} + (\text{cof. of } f_{x_i x_j}) \frac{\partial}{\partial y_k}\{H(\Phi)\}$$

$$\ll \frac{1}{r_1^2 \ldots r_p^2} \sum_l \frac{1}{r_k r_l} \cdot \frac{R r_1^2 \ldots r_p^2 r_l}{r_i r_j} + \frac{r_1^2 \ldots r_p^2}{r_i r_j} \cdot \frac{R}{r_1^2 \ldots r_p^2} \cdot \frac{1}{r_k}$$

$$\ll \frac{R}{r_i r_j r_k}.$$

Now

$$I' \ll \int_{-\delta r_1}^{\delta r_1} \cdots \int_{-\delta r_p}^{\delta r_p} \sum_{i=1}^{p} |y_i| \left|\frac{\partial}{\partial y_i}\left(\frac{\partial(\Phi_{y_1}, \ldots, \Phi_{y_p})}{\partial(y_1, \ldots, y_p)}\right)\right|_{(\xi_1, \ldots, \xi_p)} dy_1 \ldots dy_p$$

(by the continuity of the third partial derivatives of Φ)

$$\ll \delta^p r_1 \ldots r_p \sum_{i=1}^{p} \delta r_i \frac{R}{r_1^2 \ldots r_p^2 r_i} \ll \delta^{p+1} \frac{R}{r_1 \ldots r_p} \tag{27}$$

and

$$(28)\quad I'' \ll \frac{1}{r_1^2\dots r_p^2}\int\limits_{-\delta r_1}^{\delta r_1}\dots\int\limits_{-\delta r_p}^{\delta r_p}\left|\sum_{j=1}^{p} y_j\Phi_{y_j}-\Phi-\chi+\Phi(0,\dots,0)\right|dy_1\dots dy_p$$

$$\ll \frac{1}{r_1^2\dots r_p^2}\int\limits_{-\delta r_1}^{\delta r_1}\dots\int\limits_{-\delta r_p}^{\delta r_p}\left|\sum_{i,j,k}|y_iy_jy_k|\,|\Phi_{y_iy_jy_k}|_{\xi_1,\dots,\xi_p}\right|dy_1\dots dy_p$$

$$\ll \frac{1}{r_1^2\dots r_p^2}\delta^p r_1\dots r_p\sum_{i,j,k}\delta^3 r_ir_jr_k\cdot\frac{R}{r_ir_jr_k}$$

$$\ll \delta^{p+3}\frac{R}{r_1\dots r_p}.$$

From (26), (27) and (28) we have

$$I-(2\pi)^{p/2}\frac{e^{\frac{i\pi}{4}(p-2m)+if(c_1,\dots,c_p)}}{\left|\dfrac{\partial(f_{x_1},\dots,f_{x_p})}{\partial(x_1,\dots,x_p)}\right|^{1/2}_{(c_1,\dots,c_p)}}\ll\frac{1}{r_1\dots r_p}\left(\frac{1}{\delta}+R\delta^{p+1}+R\delta^{p+3}\right)$$

provided δ satisfies the conditions (19).

Now by Lemma 3, there exists a δ satisfying (19) such that

$$\frac{1}{\delta}+R\delta^{p+1}+R\delta^{p+3}\ll R^{1/(p+2)}+R^{1/(p+4)}+\operatorname*{Max}_j\left(\frac{r_j}{|\alpha_j|},\frac{r_j}{|\beta_j|}\right)$$

$$\ll R^{1/(p+2)}+R^{1/(p+4)}+\sum_{j=1}^{p}r_j\left(\frac{1}{|\alpha_j|}+\frac{1}{|\beta_j|}\right).$$

Hence we have, choosing this δ,

$$I-(2\pi)^{p/2}\frac{e^{\frac{i\pi}{4}(p-2m)+if(c_1,\dots,c_p)}}{\left|\dfrac{\partial(f_{x_1},\dots,f_{x_p})}{\partial(x_1,\dots,x_p)}\right|^{1/2}_{(c_1,\dots,c_p)}}$$

$$\ll\frac{1}{r_1\dots r_p}\left\{R^{1/(p+2)}+R^{1/(p+4)}+\sum_{j=1}^{p}r_j\left(\frac{1}{|\alpha_j|}+\frac{1}{|\beta_j|}\right)\right\}.$$

If $R<1$, $R^{1/(p+2)}<R^{1/(p+4)}$ and so we have Lemma 5. If $R\geqslant 1$, Lemma 5 follows trivially from Lemma 4. This completes the proof of Lemma 5.

4. Proposition. (The general Fourier summation formula.) *Let D be a finite region in the p-dimensional Euclidean space such that any line parallel to any of the coordinate axes meets it in a finite number of line segments. Further, let D have no lattice points on the boundary and let $\partial x_i/\partial x_j$ $(1\leqslant i,j\leqslant p)$ be continuous on each of the bounding surfaces of D. If $\Phi(x_1,\dots,x_p)$ is any real function with continuous first order partial derivatives in D, then*

$$(29)\quad \sum_{(n_1,n_2,\dots,n_p)\in D}\Phi(n_1,\dots,n_p)$$

$$=\sum_{\nu_1=-\infty}^{\infty}\dots\sum_{\nu_p=-\infty}^{\infty}\int\limits_D\dots\iint\Phi(x_1,x_2,\dots,x_p)\,e^{-2\pi i(\nu_1x_1+\dots+\nu_px_p)}dx_1\dots dx_p.$$

In the case $p=1$, the above formula can be proved, under the conditions stated, from Euler's summation formula (cf. p. 13 of [12]). Proof of the p-dimensional formula follows by repeated application of the one-dimensional formula. In the one-dimensional case, modifications in the sum on the left-hand have to be made if the end-points of the range of summation are integers. Analogous modifications have to be made in the left-hand side of (29) if there are lattice points on the boundary of D. To avoid this, we have stipulated the condition that there are no lattice points on the boundary. In any particular case, it is always easy to choose a domain equivalent to the given domain, for which this condition is also satisfied.

Henceforward, D always satisfies the conditions stipulated above, in the statement of the Fourier summation formula.

Lemma 6. *Let $f(x_1,\dots,x_p)$ be real, with second order partial derivatives in D. Let f_{x_j} have $O(1)$ maxima and minima on any straight line parallel to a coordinate axis in D. Further, let $\alpha_j=\operatorname*{Min}_D f_{x_j}$, $\beta_j=\operatorname*{Max}_D f_{x_j}$ and η_j be any real constants, $0<\eta_j<1$ for $j=1,\dots,p$.*

Let

$$\left|\frac{\partial(f_{x_{i_1}},\dots,f_{x_{i_s}})}{\partial(x_{i_1},\dots,x_{i_s})}\right|\geqslant Ar_{i_1}\dots r_{i_s}>0\qquad(1\leqslant i_1<\dots<i_s\leqslant p,\ 1\leqslant s<p)$$

throughout D, where the r's are independent of the x's. Then

$$\sum_D e^{2\pi i f(n_1,\dots,n_p)}-\sum_{\substack{\alpha_j-\eta_j<\nu_j<\beta_j+\eta_j\\ j=1,\dots,p}}\int\limits_D\dots\int e^{2\pi i\left(f(x_1,\dots,x_p)-\sum_{j=1}^{p}\nu_jx_j\right)}dx_1\dots dx_p$$

$$\ll\sum\prod_{j\in J}\frac{\beta_j-\alpha_j+1}{\sqrt{r_j}}\prod_{j'\in J'}\log(\beta_{j'}-\alpha_{j'}+2)$$

where the sum is taken over every partition J, J' of the set of integers $1,2,\dots,p$, J' being non-null.

Proof. We may suppose without loss of generality that

$$\eta_j - 1 \leqslant a_j < \eta_j\,, \quad j = 1, \ldots, p\,,$$

so that $\nu_j \geqslant 0$, $j = 1, \ldots, p$. For if k_j is an integer such that

$$\eta_j - 1 \leqslant a_j - k_j = a_j' < \eta_j\,, \quad j = 1, \ldots, p\,,$$

and

$$g(x_1, \ldots, x_p) = f(x_1, \ldots, x_p) - k_1x_1 - k_2x_2 - \ldots - k_px_p\,,$$

we have to prove

$$\sum_D e^{2\pi i g(n_1, \ldots, n_p)} - \sum_{\substack{a_j'-\eta_j<\nu_j-k_j<\beta_j'+\eta_j\\ j=1,\ldots,p}} \int\limits_D \cdots \int e^{2\pi i\{g(x_1,\ldots,x_p) - \sum_{j=1}^{p} (\nu_j-k_j)x_j\}} dx_1 \ldots dx_p$$
$$\ll \sum \prod_{j \in J} \left(\frac{\beta_j' - a_j' + 1}{\sqrt{r_j}}\right) \prod_{j' \in J'} \log(\beta_{j'}' - a_{j'}' + 2)\,,$$

i.e. the same formula for $g(x_1, \ldots, x_p)$.

We have, by the Fourier summation formula,

$$S = \sum_D e^{2\pi i f(n_1, \ldots, n_p)} = \sum_{\nu_1=-\infty}^{\infty} \ldots \sum_{\nu_p=-\infty}^{\infty} \int\limits_D \cdots \int e^{2\pi i (f(x_1,\ldots,x_p) - \sum_1^p \nu_j x_j)} dx_1 \ldots dx_p\,.$$

Call the range of integers $a_j - \eta_j < \nu_j < \beta_j + \eta_j$ or equivalently the range of integers $0 \leqslant \nu_j < \beta_j + \eta_j$ by R_j and let R_j' denote the complement of R_j in the set of all integers, for $j = 1, \ldots, p$. We have

$$(30) \qquad S = \sum S_{J,J'}$$

where

$$S_{J,J'} = \sum_{\substack{\nu_j \in R_j, j \in J\\ \nu_{j'}' \in R_{j'}', j' \in J'}} \int\limits_D \cdots \int e^{2\pi i (f - \sum_1^p \nu_j x_j)} dx_1 \ldots dx_p$$

and the sum is taken over *every* partition J, J' of the set of integers $1, 2, \ldots, p$.

We first consider the simple case when J' consists of unity alone. Assuming $\nu_1 \in R_1'$, so that, in particular, $\nu_1 \neq 0$, we have

$$\int\limits_D \cdots \int e^{2\pi i (f - \sum_1^p \nu_j x_j)} dx_1 \ldots dx_p = \sum_{O(1)} \int_{a_p}^{b_p} e^{-2\pi i \nu_p x_p} dx_p \int_{a_{p-1}}^{b_{p-1}} \ldots \int_{a_1}^{b_1} e^{2\pi i (f - \nu_1 x_1)} dx_1$$

and

$$\int_{a_1}^{b_1} e^{2\pi i (f - \nu_1 x_1)}\, dx_1 = \left(\frac{e^{2\pi i \overline{f - \nu_1 x_1}}}{-2\pi i \nu_1}\right)_{a_1}^{b_1} + \int_{a_1}^{b_1} \frac{f_{x_1}}{2\pi i \nu_1 (f_{x_1} - \nu_1)}\, d(e^{2\pi i \overline{f - \nu_1 x_1}})\,.$$

Now if $\nu_1 < 0$ or $\nu_1 \geqslant \beta_1 + \eta_1$, and f_{x_1} is monotonic, $\pm f_{x_1}/(f_{x_1} - \nu_1)$ is bounded and monotonic in the same sense as f_{x_1}. Hence if $\nu_1 \in R_1'$, we have

$$(31) \quad \int\limits_D \cdots \int e^{2\pi i (f - \sum_1^p \nu_j x_j)} dx_1 \ldots dx_p$$
$$= \sum_{O(1)} \left\{\left(\frac{e^{-2\pi i \nu_1 x_1}}{-2\pi i \nu_1} \int_{a_p}^{b_p} \ldots \int_{a_2}^{b_2} e^{2\pi i (f - \sum_2^p \nu_j x_j)} dx_2 \ldots dx_p\right)_{x_1=a_1}^{x_1=b_1} + \right.$$
$$\left. + \sum_{O(1)} \left(\frac{f_{x_1}}{2\pi i \nu_1 (f_{x_1} - \nu_1)}\right)_{\xi_1, \ldots, \xi_p} \int_{\xi_2'}^{\xi_2''} \ldots \int_{\xi_p'}^{\xi_p''} \begin{matrix}\cos\\ i \sin\end{matrix} \left(f - \sum_1^p \nu_j x_j\right) dx_2 \ldots dx_p\right\}$$

by repeated application of the second mean value theorem, since the domain of integration can be split up into $O(1)$ domains in each of which f_{x_1} is monotonic separately in each of the variables.

Now we have

$$(32) \qquad \left|\sum_{\nu_1 \in R_1'} \frac{e^{-2\pi i \nu_1 x_1}}{2\pi i \nu_1}\right| \leqslant \frac{1}{2} + \sum_{1 \leqslant \nu_1 < \beta_1 + \eta_1} \frac{1}{2\pi \nu_1} \ll \log(\beta_1 - a_1 + 2)\,,$$

$$(33) \qquad \sum_{\nu_1 \in R_1'} \left|\frac{f_{x_1}}{\nu_1 (f_{x_1} - \nu_1)}\right|_{(\xi_1, \xi_2, \ldots, \xi_p)} \ll \sum_{\nu_1=1}^{\infty} \frac{\beta_1}{\nu_1(\nu_1 + \beta_1)} + \sum_{\nu_1 \geqslant \beta_1 + \eta_1} \frac{\beta_1}{\nu_1(\nu_1 - \beta_1)}$$
$$\ll \log(\beta_1 - a_1 + 2)\,.$$

From (31), (32), (33) and Lemma 4, we find

$$\sum_{\substack{\nu_1 \in R_1'\\ \nu_j \in R_j, j=2,\ldots,p}} \int\limits_D \cdots \int e^{2\pi i (f - \sum_1^p \nu_j x_j)} dx_1 \ldots dx_p \ll \log(\beta_1 - a_1 + 2) \prod_{j=2}^{p} \frac{\beta_j - a_j + 1}{\sqrt{r_j}}\,.$$

In an exactly similar manner, we can prove the more general result

$$(34) \qquad S_{J,J'} \ll \prod_{j \in J} \frac{\beta_j - a_j + 1}{\sqrt{r_j}} \prod_{j' \in J'} \log(\beta_{j'} - a_{j'} + 2)$$

for every partition J, J' of the set of integers $1, 2, \ldots, p$, J' being *non-null*.

From (30) and (34) follows Lemma 6.

Lemma 7. *Under the conditions of Lemma 6, let Δ_η be defined as the smallest region containing the lattice points $(\nu_1, \ldots, \nu_p)$ such that*

$$D \cap |f_{x_1} - \nu_1| < \eta_1 \cap \ldots \cap |f_{x_p} - \nu_p| < \eta_p \neq \Phi\,.$$

Then we have

$$\sum_{D} e^{2\pi i f(n_1,\ldots n_p)} - \sum_{(\nu_1,\ldots,\nu_p)\in\Delta_\eta} \underset{\substack{D,|f_{x_j}-\nu_j|<\eta_j\\ j=1,\ldots,p}}{\int\ldots\int} e^{2\pi i(f(x_1,\ldots,x_p)-\sum_{j=1}^{p}\nu_j x_j)}dx_1\ldots dx_p$$

$$\ll \sum \prod_{j\in J}\frac{\beta_j-\alpha_j+1}{\sqrt{r_j}}\prod_{j'\in J'}\log(\beta_{j'}-\alpha_{j'}+2)$$

where the sum is taken over every partition J, J' of the set of integers $1, 2, \ldots, p$, J' being non-null.

Proof. We have, using the notations of the the proof of Lemma 6,

$$(35)\quad \sum_{\substack{\nu_j\in R_j\\ j=1,\ldots,p}} \underset{D}{\int\ldots\int} e^{2\pi i(f-\sum_1^p \nu_j x_j)}dx_1\ldots dx_p -$$

$$- \sum_{(\nu_1,\ldots,\nu_p)\in\Delta_\eta} \underset{\substack{D,|f_{x_j}-\nu_j|<\eta_j\\ j=1,\ldots,p}}{\int\ldots\int} e^{2\pi i(f-\sum_1^p \nu_j x_j)}dx_1\ldots dx_p = \sum_{\nu_j\in R_j, j=1,\ldots,p}(I_1+\ldots+I_p)$$

where

$$I_j = \underset{\substack{D\cap D_j\\ |f_{x_j}-\nu_j|\geqslant\eta_j}}{\int\ldots\int} e^{2\pi i(f-\sum_1^p\nu_j x_j)}dx_1\ldots dx_p\,,$$

D_j being given by $|f_{x_s}-\nu_s|<\eta_s$ for $1\leqslant s<j$.

We can now assume without loss of generality that

$$\eta_j\leqslant\alpha_j<\eta_j+1\,,\qquad j=1,\ldots,p\,,$$

so that $\nu_j>0$ if $\nu_j\in R_j$, $j=1,\ldots,p$. We have then, as in the proof of Lemma 6,

$$\sum_{\nu_j\in R_j, j=1,\ldots,p} I_1 = \sum_{O(1)}\sum_{\substack{\nu_j\in R_j\\ j=1,2,\ldots,p}}\left\{\frac{e^{-2\pi i\nu_1 x_1}}{-2\pi i\nu_1}\int_{a_p}^{b_p}\ldots\int_{a_2}^{b_2} e^{2\pi i(f-\sum_2^p\nu_j x_j)}dx_2\ldots dx_p\right\}_{x_1=a_1}^{x_1=b_1}$$

$$+\sum_{O(1)}\left(\frac{f_{x_1}}{2\pi i\nu_1(f_{x_1}-\nu_1)}\right)_{\xi_1,\ldots,\xi_p}\int_{\xi_2'}^{\xi_2''}\ldots\int_{\xi_p'}^{\xi_p''}\begin{matrix}\cos\\ i\sin\end{matrix}\left(f-\sum_1^p\nu_j x_j\right)dx_2\ldots dx_p\Bigg\}\,,$$

since the domain can be split up into $O(1)$ domains in each of which $\pm f_{x_1}/(f_{x_1}-\nu_1)$ is monotonic separately in each of the variables and bounded because $|f_{x_1}-\nu_1|\geqslant\eta_1$ throughout.

Hence we find, as in the proof of Lemma 6,

$$\sum_{\nu_j\in R_j, j=1,\ldots,p} I_1 \ll \log(\beta_1-\alpha_1+2)\prod_{j=2}^{p}\frac{\beta_j-\alpha_j+1}{\sqrt{r_j}}\,.$$

A similar result holds for every other term on the right-hand side of (35) and Lemma 7 now follows from 35 and Lemma 6.

5. Theorem 1. *Let $f(x_1,\ldots,x_p)$ possess continuous second order partial derivatives in the hyper-rectangle D', $a_j<x_j\leqslant b_j$, $j=1,\ldots,p$, containing D. Let*

$$|f_{x_i x_j}|\leqslant A\lambda_{ij}\qquad(1\leqslant i,j\leqslant p)$$

and

$$\left|\frac{\partial(f_{x_{i_1}}\ldots f_{x_{i_r}})}{\partial(x_{i_1},\ldots,x_{i_r})}\right|\geqslant A\lambda_{i_1i_1}\ldots\lambda_{i_ri_r}>0\quad \text{for}\quad 1\leqslant i_1<i_2\ldots<i_r\leqslant p,\ 1\leqslant r\leqslant p\,,$$

throughout D', where λ's are positive numbers independent of x, satisfying the relations $(b_j-a_j)\lambda_{ij}\ll(b_i-a_i)\lambda_{ii}$ and $b_j\geqslant a_j+1$. Then

$$S=\sum_{(n_1,\ldots,n_p)\in D} e^{2\pi i f(n_1,\ldots,n_p)}\ll\prod_{j=1}^{p}\{(b_j-a_j)\lambda_{jj}^{1/2}+\lambda_{jj}^{-1/2}\}\,.$$

Proof. We have from Lemmas 6 and 4

$$S\ll\prod_{j=1}^{p}\frac{\beta_j-\alpha_j+1}{\sqrt{\lambda_{jj}}}+\sum\prod_{j\in J}\frac{\beta_j-\alpha_j+1}{\sqrt{\lambda_{jj}}}\cdot\prod_{j'\in J'}\log(\beta_{j'}-\alpha_j'+2)\,.$$

Now

$$\beta_j-\alpha_j\ll\sum_{i=1}^{p}(b_i-a_i)\lambda_{ij}\ll(b_j-a_j)\lambda_{jj}$$

and

$$\begin{aligned}\log(\beta_j-\alpha_j+2)&\ll\log\{(b_j-a_j)\lambda_{jj}+2\}\\ &\ll\{(b_j-a_j)\lambda_{jj}+2\}^{1/2}\\ &=\lambda_{jj}^{1/2}(b_j-a_j+2/\lambda_{jj})^{1/2}\\ &\ll\lambda_{jj}^{1/2}(b_j-a_j+1/\lambda_{jj})\qquad\text{since } b_j-a_j\geqslant 1\,.\end{aligned}$$

Theorem 1 is now immediate.

Theorem 2 (The general Van der Corput transformation.) *Let $f(x_1,\ldots,x_p)$ possess continuous third order partial derivatives in the hyper-rectangle D', $a_j<x_j\leqslant b_j$, $j=1,\ldots,p$, where $b_j\geqslant a_j+1$. Let*

$$\left|\frac{\partial(f_{x_{i_1}},\ldots,f_{x_{i_s}})}{\partial(x_{i_1},\ldots,x_{i_s})}\right|\geqslant A\frac{r^{2s}}{\prod_{j=1}^{s}(b_{i_j}-a_{i_j})^2}>0\qquad\begin{pmatrix}1\leqslant i_1<\ldots<i_s\leqslant p\,,\\ 1\leqslant s\leqslant p\end{pmatrix}$$

and

$$|f_{x_ix_j}| \leqslant A\,\frac{r^2}{(b_i-a_i)(b_j-a_j)}\,, \qquad |f_{x_ix_jx_k}| \leqslant A\,\frac{Rr^3}{(b_i-a_i)(b_j-a_j)(b_k-a_k)}$$

throughout $D' \supset D$.

Let $(x_{1\nu}, \ldots, x_{p\nu})$ *be defined by*

$$f_{x_i}(x_{1\nu}, \ldots, x_{p\nu}) = \nu_i\,, \qquad (\nu_1, \ldots, \nu_p) \in \Delta\,,$$

where Δ *is the transform of the region* D *by the transformation* $y_i = f_{x_i}$, $i = 1, \ldots, p$. *Then, if* m *is the number of changes of sign in the sequence*

$$+1\,,\ \frac{\partial(f_{x_1}, \ldots, f_{x_j})}{\partial(x_1, \ldots, x_j)}\,, \qquad j = 1, \ldots, p\,,$$

we have

$$\sum_{(n_1,\ldots,n_p)\in D} e^{2\pi i f(n_1,\ldots,n_p)} - e^{\frac{i\pi}{4}(p-2m)} \sum_{(\nu_1,\ldots,\nu_p)\in\Delta} \frac{e^{2\pi i\{f(x_{1\nu},\ldots,x_{p\nu}) - \sum_{j=1}^{p}\nu_j x_{j\nu}\}}}{\left|\frac{\partial(f_{x_1}, \ldots, f_{x_p})}{\partial(x_1, \ldots, x_p)}\right|^{1/2}_{(x_{1\nu},\ldots,x_{p\nu})}}$$

$$\ll R^{\frac{1}{p+4}} r^p + \sum_{j=1}^{p} \prod_{s\neq j}\left(r + \frac{b_s-a_s}{r}\right)\cdot\left\{\frac{b_j-a_j}{r} + \log\left(\frac{r^2}{b_j-a_j}+2\right)\right\}.$$

Proof. In the following proof $\alpha_{j\nu}$ and $\beta_{j\nu}$ $(\alpha_{j\nu} < \beta_{j\nu})$ denote the values of f_{x_j} at the end-points of the largest segment of the curve $f_{x_i} = \nu_i$, $i = 1, \ldots, j-1, j+1, \ldots, p$, which lies entirely within D. Also

$$\alpha_j = \operatorname*{Min}_{D} f_{x_j}\,, \qquad \beta_j = \operatorname*{Max}_{D} f_{x_j}\,, \qquad j = 1, \ldots, p\,.$$

The region Δ_η is defined as in Lemma 7. The region Δ_1 is defined as the largest region contained in Δ containing lattice points $(\nu_1, \ldots, \nu_p)$ such that the hyper-rectangle $|y_j - \nu_j| \leqslant 1$, $j = 1, \ldots, p$, is entirely contained in Δ.

We have, by Lemma 5,

$$(36) \qquad \sum_{(\nu_1,\ldots,\nu_p)\in\Delta_1} \underset{\substack{D,\ |f_{x_j}-\nu_j|<\eta_j\\ j=1,\ldots,p}}{\int\!\!\ldots\!\!\int} e^{2\pi i(f - \sum_1^p \nu_j x_j)}\,dx_1\ldots dx_p -$$

$$- e^{\frac{i\pi}{4}(p-2m)} \sum_{(\nu_1,\ldots,\nu_p)\in\Delta_1} \frac{e^{2\pi i(f(x_{1\nu},\ldots,x_{p\nu}) - \sum_1^p \nu_j x_{j\nu})}}{\left|\frac{\partial(f_{x_1}, \ldots, f_{x_p})}{\partial(x_1, \ldots, x_p)}\right|^{1/2}_{(x_{1\nu},\ldots,x_{p\nu})}}$$

$$\ll \prod_{j=1}^{p} \frac{(\beta_j-\alpha_j)}{r/(b_j-a_j)}\left\{R^{\frac{1}{p+4}} + \sum_{j=1}^{p}\frac{r}{(b_j-a_j)(\beta_j-\alpha_j)} \sum\sum_{\alpha_{j\nu}+1<\nu_j<\beta_{j\nu}-1}\left(\frac{1}{\beta_{j\nu}-\nu_j} + \frac{1}{\nu_j-\alpha_{j\nu}}\right)\right\}$$

$$\ll \prod_{j=1}^{p} \frac{(\beta_j-\alpha_j)(b_j-a_j)}{r}\left\{R^{\frac{1}{p+4}} + \sum_{j=1}^{p}\frac{r}{(b_j-a_j)(\beta_j-\alpha_j)}\log(\beta_j-\alpha_j+2)\right\}.$$

Now

$$(37) \qquad \beta_j - \alpha_j \ll \sum_{i=1}^{p}(b_i-a_i)\frac{r^2}{(b_i-a_i)(b_j-a_j)} \ll \frac{r^2}{b_j-a_j}\,.$$

Also, the error term introduced by replacing the domain of summation Δ_1 by Δ_η and Δ respectively in the first and second terms of the left-hand side of (36) is

$$\ll \sum_{j=1}^{p}\frac{(b_1-a_1)(b_2-a_2)\ldots(b_p-a_p)}{r^p}\prod_{s\neq j}(\beta_s-\alpha_s+1)$$

$$(38) \qquad \ll \sum_{j=1}^{p}\frac{b_j-a_j}{r}\prod_{s\neq j}\left(r + \frac{b_s-a_s}{r}\right).$$

As in the proof of Theorem 1,

$$\log(\beta_j - \alpha_j + 2) \ll r + \frac{b_j-a_j}{r}$$

and so

$$(39) \qquad \sum_{\substack{J\cup J'=(1,2,\ldots,p)\\ J'\neq\Phi}} \prod_{j\in J}\frac{\beta_j-\alpha_j+1}{r/(b_j-a_j)}\prod_{j'\in J'}\log(\beta_{j'}-\alpha_{j'}+2)$$

$$\ll \sum_{j=1}^{p}\log(\beta_j-\alpha_j+2)\prod_{s\neq j}\left(r+\frac{b_s-a_s}{r}\right).$$

Theorem 2 now follows from (36) to (39) and Lemma 7.

THEOREM 3. *Let* f *possess continuous partial derivatives up to* $(k+2)$*-th order in the hyper-rectangle* D', $a_j < x_j \leqslant b_j$, $j = 1, \ldots, p$, *containing* D. *Let* $k = k_1 + \ldots + k_p \geqslant 1$ *and* $k_j \geqslant 0$. *Let* $g = f_{x_1^{k_1}\ldots x_p^{k_p}}$ *and*

$$\left|\frac{\partial(g_{x_{i_1}}, \ldots, g_{x_{i_s}})}{\partial(x_{i_1}, \ldots, x_{i_s})}\right| \geqslant A\lambda_{i_1i_1}\ldots\lambda_{i_si_s} > 0 \qquad (1 \leqslant i_1 < i_2 < \ldots < i_s \leqslant p,\ 1 \leqslant s \leqslant p)$$

$$|g_{x_ix_j}| \leqslant A\lambda_{ij} \qquad (1 \leqslant i, j \leqslant p)$$

in D', *where* λ*'s are positive numbers independent of* x, *satisfying the conditions* $(b_j-a_j)\lambda_{ij} \ll (b_i-a_i)\lambda_{ii}$. *Further, let*

$$b_j \geqslant a_j + 1, \qquad K = 2^k, \qquad K_j = 2^{k_1+\ldots+k_j} \quad \textit{for} \quad j = 1, \ldots, p.$$

Then

$$\frac{S}{\prod_{j=1}^{p}(b_j-a_j)} = \frac{1}{\prod_{1}^{p}(b_j-a_j)}\sum_{D} e^{2\pi i f(n_1,\ldots,n_p)}$$

$$\ll {\sum_{\mu=1}^{p}}'(b_\mu - a_\mu)^{-\frac{1}{K_\mu}} + \omega^{-\frac{1}{2K-2}} + a_{(p+1)/2}^{\frac{1}{K}}\,\omega^{-\frac{1}{2K}} +$$

$$+\sum_{0\leqslant s\leqslant p/2}\left(a_s a_{(p+1)/2}^{p-2s}\right)^{\frac{1}{K(p+1-2s)}}+\sum_{0\leqslant s\leqslant p/2} a_s^{\frac{1}{(K-1)(p-2s)+K}}+$$

$$+\,\omega^{-\frac{1}{K}}\sum_{p/2+1\leqslant s\leqslant p} a_s^{\frac{1}{K}}+\sum_{\substack{0\leqslant s\leqslant p/2,\\ p/2+1<s'\leqslant p}}\left(a_s a_{s'}^{p/2-1}\right)^{\frac{1}{K(p/2-s+1)}},$$

where the dash in the μ-sum denotes that the sum is taken over those μ for which $k_\mu>0$, and

$$\omega=\prod_{j=1}^{p}(b_j-a_j)^{k_j}, \tag{40}$$

$$a_s=\text{coeff of } x^s \text{ in } \prod_{j=1}^{p}\lambda_{jj}^{1/2}\left\{1+\frac{x}{(b_j-a_j)\lambda_{jj}}\right\} \quad \text{if} \quad s\neq p/2+1,$$

$$a_{p/2+1}=\prod_{j=1}^{p}\left\{\log(b_j-a_j)^{k_j}\text{coeff of } x^{p/2+1} \text{ in } \prod_{j=1}^{p}\lambda_{jj}^{1/2}\left\{1+\frac{x}{(b_j-a_j)\lambda_{jj}}\right\}\right\}. \tag{41}$$

Proof. Let

$$x_j'=x_j+u_{1j}t_{k_1+\ldots+k_{j-1}+1}+\ldots+u_{k_jj}t_{k_1+\ldots+k_j} \qquad (1\leqslant j\leqslant p)$$

where the u's are positive.

Then set

$$\Delta f=\int_0^1\ldots\int_0^1 f_{t_1\ldots t_k}(x_1',x_2',\ldots,x_p')\,dt_1\ldots dt_k$$

$$=U\int_0^1\ldots\int_0^1 g(x_1',\ldots,x_p')\,dt_1\ldots dt_k$$

where

$$U=u_{11}\ldots u_{k_11}u_{12}\ldots u_{k_pp}.$$

If now

$$x_{jr}=x_j+u_{1j}t_{k_{j-1}'+1,r}+\ldots+u_{k_j,j}t_{k_j',r} \qquad (1\leqslant r\leqslant s)$$

$$k_j'=k_1+\ldots+k_j \quad (1\leqslant j\leqslant p), \qquad k_0'=0,$$

then

$$\frac{\partial(\Delta f_{x_{i_1}},\ldots,\Delta f_{x_{i_s}})}{\partial(x_{i_1},\ldots,x_{i_s})}$$

$$=\int_0^1\ldots\int_0^1\frac{\partial\left(f_{x_{i_1}t_{11}\ldots t_{k_1}}(x_{11},\ldots,x_{p1}),\ldots,f_{x_{i_s}t_{1s}\ldots t_{ks}}(x_{1s},\ldots,x_{ps})\right)}{\partial(x_{i_1},\ldots,x_{i_s})}\,dt_{11}\ldots dt_{ks}$$

$$=U^s\int_0^1\ldots\int_0^1\frac{\partial\left(g_{x_{i_1}}(x_{11},\ldots,x_{p1}),\ldots,g_{x_{i_s}}(x_{is},\ldots,x_{ps})\right)}{\partial(x_{i_1},\ldots,x_{i_s})}\,dt_{11}\ldots dt_{ks}.$$

Hence, by the given conditions of Theorem 3, the conditions of Theorem 1 are satisfied for the functions Δf with $U\lambda_{ij}$ in the place of λ_{ij} and so we have by Theorem 1

$$S'=\sum e^{2\pi i\Delta f(n_1,\ldots,n_p)}\ll\prod_{j=1}^{p}\{(b_j-a_j)(U\lambda_{jj})^{1/2}+(U\lambda_{jj})^{-1/2}\}. \tag{42}$$

And, by successive applications (k_j times w.r.t the variable n_j) of Lemma 2 to the sum S, we find

$$S\ll\prod_{j=1}^{p}(b_j-a_j)\{q_1^{-1/2}+q_2^{-1/2^2}+\ldots+q_k^{-1/2^k}\}+ \tag{43}$$
$$+\left\{\prod_{j=1}^{p}(b_j-a_j)\right\}^{1-1/K}\left\{\frac{1}{q_1\ldots q_k}\sum_{u_{ij}}|S'|\right\}^{1/K}$$

where

$$0<q_\beta\leqslant b_\alpha-a_\alpha \quad \text{for} \quad \beta=k_1+\ldots+k_{\alpha-1}+1,\ldots,k_1+\ldots+k_\alpha, \quad \alpha=1,\ldots,p,$$

and u_{ij} in $\sum_{u_{ij}}|S'|$ runs through

$$1\leqslant u_{ij}=u_\alpha'\leqslant q_\alpha-1 \qquad (\alpha=1,\ldots,k),$$

u_α' being the αth u in the sequence $u_{11},\ldots,u_{k_11};u_{12},\ldots,u_{k_22},\ldots,u_{k_pp}$. From (41), (42) and (43) we deduce, putting $\varrho_s=q_1\ldots q_s$ $(1\leqslant s\leqslant k)$,

$$\frac{S}{\prod_{j=1}^{p}(b_j-a_j)}\ll\varrho_1^{-1/2}+\varrho_2^{-1/2^2}\varrho_1^{1/2^2}+\ldots+\varrho_k^{-1/2^k}\varrho_{k-1}^{1/2^k}+ \tag{44}$$
$$+\sum_{0\leqslant s\leqslant p/2}\{a_s\varrho_k^{p/2-s}\}^{1/K}+\varrho_k^{-1/K}\sum_{p/2+1\leqslant s\leqslant p}a_s^{1/K}$$

provided the ϱ's satisfy the conditions

$$\frac{\varrho_2}{b_1-a_1}\leqslant\varrho_1\leqslant b_1-a_1, \qquad \frac{\varrho_{\beta+1}}{b_\alpha-a_\alpha}\leqslant\varrho_\beta \qquad (1<\beta<k) \tag{45}$$

where $\beta=k_1+\ldots+k_{\alpha-1}+1,\ldots,k_1+\ldots+k_\alpha$, $\alpha=1,\ldots,p$,

$$0<\varrho_k\leqslant\omega. \tag{46}$$

Now it can easily be shown by induction, by appealing to Lemma 3, that there exist $\varrho_1,\ldots,\varrho_{k-1}$ satisfying (45) such that

$$\varrho_1^{-1/2}+\varrho_2^{-1/2^2}\varrho_1^{1/2^2}+\ldots+\varrho_k^{-1/2^k}\varrho_{k-1}^{1/2^k}\ll\sum_{\mu=1}^{p}{}'(b_\mu-a_\mu)^{-1/K_\mu}+\varrho_k^{-1/(2K-2)}. \tag{47}$$

Again by Lemma 3, there exists a ϱ_k satisfying (46) such that

$$(48)\quad \varrho_k^{-1/(2K-2)} + \sum_{0\leqslant s<p/2+1} \{a_s\varrho_k^{p/2-s}\}^{1/K} + \varrho_k^{-1/K} \sum_{p/2+1\leqslant s'\leqslant p} a_{s'}^{1/K}$$
$$\ll \omega^{-1/(2K-2)} + \omega^{-1/K} \sum_{p/2+1\leqslant s'\leqslant p} a_{s'}^{1/K} + a_{(p+1)/2}^{1/K}\omega^{-1/2K} +$$
$$+ \sum_{0\leqslant s\leqslant p/2} \{a_s a_{(p+1)/2}^{p-2s}\}^{1/K(p+1-2s)} + \sum_{0\leqslant s\leqslant p/2} a_s^{1/((K-1)(p-2s)+K)} +$$
$$+ \sum_{\substack{0\leqslant s\leqslant p/2\\ p/2+1\leqslant s'\leqslant p}} (a_s a_{s'}^{p/2-s})^{1/K(p/2-s+1)}.$$

Choosing $\varrho_1, \ldots, \varrho_k$ as above, we have Theorem 3 from (44), (47) and (48).

THEOREM 4. *Under the hypotheses of Theorem 3,*

$$\frac{S}{\prod_{j=1}^{p}(b_j-a_j)} \ll \sum_{\mu=1}^{p}{}' (b_\mu-a_\mu)^{-\frac{1}{K_\mu}} + \sum_{0\leqslant s<p/2+1} a_s^{\frac{1}{K}} a_0^{\frac{1}{K}\cdot\frac{K+2s(K-1)}{K+p(K-1)}} +$$
$$+ \omega^{-\frac{1}{2K-2}} a_0^{-\frac{1}{p(K-1)+K}} + a_0^{\frac{1}{K}\cdot\frac{K+(p+2)(K-1)}{K+p(K-1)}} \sum_{p/2+1\leqslant s\leqslant p} a_s^{\frac{1}{K}}.$$

Proof. We choose $\varrho_1, \ldots, \varrho_{k-1}$ as in the proof of Theorem 3, but we choose

$$\varrho_k = a^{-\frac{2(K-1)}{p(K-1)+K}} \quad \text{so that} \quad \varrho_k^{-\frac{1}{2K-2}} = a_0^{\frac{1}{K}} \varrho_k^{\frac{p}{2K}}.$$

Then we have from (44) and (47)

$$\frac{S}{\prod_{j=1}^{p}(b_j-a_j)} \ll \sum_{\mu=1}^{p}{}' (b_\mu-a_\mu)^{-\frac{1}{K_\mu}} +$$
$$+ \sum_{0\leqslant s<p/2+1} a_s^{\frac{1}{K}} a_0^{\frac{1}{K}\cdot\frac{K+2s(K-1)}{K+p(K-1)}} + a_0^{\frac{1}{K}\cdot\frac{K+(p+2)(K-1)}{K+p(K-1)}} \sum_{p/2+1\leqslant s\leqslant p} a_s^{\frac{1}{K}}$$

provided

$$a_0^{-\frac{2(K-1)}{p(K-1)+K}} \ll \omega.$$

Thus we have Theorem 4 in this case. In the contrary case Theorem 4 is trivial. Hence the theorem.

Remark. Evidently, in view of the remarks after Lemma 3, Theorem 3 gives the best possible inequality that can be obtained by a judicious choice of the parameters and hence is in particular stronger than Theorem 4. In the case $p = 1$, Theorems 3 and 4 reduce to the following Theorems.

THEOREM 3′. *Suppose $f(x)$ to be real and has derivatives up to $(k+2)$-th order $(k\geqslant 1)$ in (a, b). Let*

$$0 < \lambda_{k+2} \leqslant f^{(k+2)}(x) \leqslant A\lambda_{k+2} \quad \text{or} \quad \lambda_{k+2} \leqslant -f^{(k+2)}(x) \leqslant A\lambda_{k+2}$$

throughout (a, b). Let $b \geqslant a+1$, $K = 2^k$. Then

$$\sum_{a<n\leqslant b} e^{2\pi i f(n)} \ll (b-a)\lambda_{k+2}^{1/(4K-2)} + (b-a)^{1-(k+2)/2K}\lambda_{k+2}^{-1/2K} + (b-a)^{1-1/2K}.$$

THEOREM 4′. *Under the hypotheses of Theorem 3′*

$$\sum_{a<n\leqslant b} e^{2\pi i f(n)} \ll (b-a)\lambda_{k+2}^{1/(4K-2)} + (b-a)^{1-1/K}\lambda_{k+2}^{-1/(4K-2)}.$$

Theorem 3′ is Van der Corput's inequality (Satz 4 of [3]) for one-dimensional exponential sums, while Theorem 4′ is the Titchmarsh inequality (Theorem 5.13 of [12]). In view of the above remark it follows in particular that Van der Corput's inequality is stronger than Titchmarsh's inequality. A direct proof of this statement is given in [7].

6. The lattice point-problem. We require the following further lemmas.

LEMMA 8. *For arbitrary $M>0$ and for any function g,*

$$(49)\quad \sum_{(n_1,\ldots,n_p)\in D} \psi_1\big(g(n_1, \ldots, n_p)\big)$$
$$\ll \frac{s|D|}{M} + \sum_{m=1}^{\infty}\Big|\sum_{(n_1,\ldots,n_p)\in D} e^{-2\pi i m g(n_1,\ldots,n_p)}\Big| \operatorname{Min}\Big(\frac{1}{m}, \frac{M^s}{m^{s+1}}\Big)$$

where ψ_1 is the function defined in (8), $|D|$ is the volume of the region D and s is any positive integer.

Proof. We have

$$M^s\int_0^{\pm 1/M}\cdots\int_0^{\pm 1/M} \psi_1\Big(g(n_1, \ldots, n_p) + \sum_{j=1}^{s} y_j\Big)\prod_{j=1}^{s} dy_j$$
$$= \sum_{m=-\infty}^{\infty}{}' \frac{e^{-2\pi i m g(n_1,\ldots,n_p)}}{2\pi i m}\Big(\int_0^{\pm 1/M} M e^{-2\pi i m y}\,dy\Big)^s = \sum_{m=-\infty}^{\infty}{}' c_m e^{-2\pi i m g(n_1,\ldots,n_p)}$$

where $c_m \ll 1/m$ and $c_m \ll M^s/m^{s+1}$ and the dash denotes that the term corresponding to $m = 0$ is omitted. Hence

$$(50)\quad \sum_{(n_1,\ldots,n_p)\in D} M^s\int_0^{\pm 1/M}\cdots\int_0^{\pm 1/M} \psi_1\Big(g(n_1, \ldots, n_p) + \sum_{j=1}^{s} y_j\Big)\prod_{1}^{s} dy_j$$
$$\ll \sum_{m=-\infty}^{\infty}{}' |c_m|\Big|\sum_{D} e^{-2\pi i m g(n_1,\ldots,n_p)}\Big| \ll \sum_{m=1}^{\infty}\Big|\sum_{D} e^{-2\pi i m g(n_1,\ldots,n_p)}\Big| \operatorname{Min}\Big(\frac{1}{m}, \frac{M^s}{m^{s+1}}\Big).$$

Again $\psi_1(u)-\psi_1(v) \leqslant u-v$ if $u \geqslant v$. Hence we have

$$\sum_D M^s \int_0^{1/M} \ldots \int_0^{1/M} \psi_1\Big(g(n_1, \ldots, n_p)+\sum_{j=1}^s y_j\Big) \prod_{j=1}^s dy_j$$
$$\leqslant \sum_D \psi_1\big(g(n_1, \ldots, n_p)\big) + \sum_{(n_1,\ldots,n_p)\in D} M^s \int_0^{1/M} \ldots \int_0^{1/M} \Big(\sum_{j=1}^s y_j\Big) \prod_{j=1}^s dy_j$$
$$= \sum_D \psi_1\big(g(n_1, \ldots, n_p)\big) + \frac{s}{2M} \sum_{(n_1,\ldots,n_p)\in D} 1$$
$$\leqslant \sum_D \psi_1\big(g(n_1, \ldots, n_p)\big) + \frac{s}{2M}|D|\,.$$

Similarly

$$\sum_D M^s \int_{-1/M}^0 \ldots \int_{-1/M}^0 \psi_1\Big(g(n_1, \ldots, n_p)+\sum_{j=1}^s y_j\Big) \prod_{j=1}^s dy_j \geqslant \sum_D \psi_1\big(g(n_1, \ldots, n_p)\big) - \frac{s|D|}{2M}\,.$$

So

$$(51)\qquad \sum_{(n_1,\ldots,n_p)\in D} \psi_1\big(g(n_1, \ldots, n_p)\big) \leqslant \Big|\sum_D M^s \int_0^{\pm 1/M} \ldots \int_0^{\pm 1/M} \psi_1\Big(g(n_1, \ldots, n_p)+\sum_1^s y_j\Big) \prod_1^s dy_j\Big| + \frac{s|D|}{2M}\,.$$

(49) now follows from (50) and (51).

Lemma 9. *If $a_1, \ldots, a_p$ are any real numbers, then*

$$\sum\nolimits_r^{a_1,\ldots,a_p} = \sum_{\substack{0\leqslant h_p \leqslant h_{p-1}\leqslant \ldots \leqslant h_1 \\ (r+2)h_1+h_2+\ldots+h_p \leqslant \frac{\log x}{\log 2}}} 2^{h_1a_1+\ldots+h_pa_p} \leqslant \sum_{s=1}^p x^{\frac{a_1+\ldots+a_s}{s+r+1}} (\log x)^{p-s} + (\log x)^p\,.$$

Proof. When $p=1$ we obviously have

$$\sum_{0\leqslant h_1 \leqslant \frac{1}{r+2}\cdot\frac{\log x}{\log 2}} 2^{h_1a_1} \leqslant x^{a_1/(r+2)} + \log x\,.$$

We prove the lemma by induction on p. We have, if $p \geqslant 1$,

$$\sum\nolimits_r^{a_1,\ldots,a_{p+1}} = \sum_{\substack{0\leqslant h_{p+1}\leqslant\ldots\leqslant h_2 \\ (r+3)h_2+h_3+\ldots+h_{p+1}\leqslant\frac{\log x}{\log 2}}} 2^{h_2a_2+\ldots+h_{p+1}a_{p+1}} \sum_{h_2\leqslant h_1 \leqslant \frac{1}{r+2}\left(\frac{\log x}{\log 2}-h_2-\ldots-h_{p+1}\right)} 2^{h_1a_1}$$
$$\leqslant \sum_{\substack{0\leqslant h_{p+1}\leqslant\ldots\leqslant h_2 \\ (r+3)h_2+h_3+\ldots+h_{p+1}\leqslant\frac{\log x}{\log 2}}} 2^{h_2a_2+\ldots+h_{p+1}a_{p+1}}\Big\{2^{h_2a_1} + x^{\frac{a_1}{r+2}} 2^{-\frac{a_1}{r+2}(h_2+\ldots+h_{p+1})}\Big\} \qquad (\text{if } a_1\neq 0)$$

$$= \sum\nolimits_{r+1}^{a_1+a_2,a_3,\ldots,a_{p+1}} + x^{\frac{a_1}{r+2}} \sum\nolimits_{r+1}^{a_2-\frac{a_1}{r+2},a_3-\frac{a}{r+2},\ldots,a_{p+1}-\frac{a_1}{r+2}}$$
$$\leqslant (\log x)^p + \sum_{s=1}^p x^{\frac{a_1+\ldots+a_{s+1}}{s+r+2}} (\log x)^{p-s} +$$
$$+ x^{\frac{a_1}{r+2}}\Big\{(\log x)^p + \sum_{s=1}^p x^{\frac{a_2+\ldots+a_{s+1}-sa_1/(r+2)}{s+r+2}} (\log x)^{p-s}\Big\}$$

(by the induction hypothesis)

$$\leqslant (\log x)^p + \sum_{s=1}^{p+1} x^{\frac{a_1+\ldots+a_s}{s+r+1}} (\log x)^{p+1-s}\,.$$

Hence the lemma follows by induction on p in the case $a_1 \neq 0$.

If

$$a_1 = \ldots = a_s = 0\,, \quad a_{s+1}\neq 0\,, \quad 1 \leqslant s < p\,,$$

then

$$\sum\nolimits_r^{a_1,\ldots,a_p} \leqslant (\log x)^s \sum\nolimits_{r+s}^{a_{s+1},\ldots,a_p}$$

and the lemma again follows in this case. If, however, all the a's are zero, the lemma is trivial. Hence the lemma.

We now consider the exponential sum $S = \sum_D e^{-2\pi i\left(\frac{x}{n_1\ldots n_p}\right)^{1/(r+1)}}$ where D is the region $a_j < n_j \leqslant 2a_j$, $n_1\ldots n_p n_j^{r+1} \leqslant x$, $j = 1, 2, \ldots, p$, and apply Theorem 2 to S. Here

$$f(x_1, \ldots, x_p) = -\Big(\frac{x}{x_1\ldots x_p}\Big)^{1/(r+1)}\,,$$

$$f_{x_j} = \frac{1}{r+1}\cdot\frac{1}{x_j}\Big(\frac{x}{x_1\ldots x_p}\Big)^{1/(r+1)}\,, \qquad j = 1, \ldots, p\,.$$

Δ is a region [2] contained in

$$\frac{1}{r+1}\cdot\frac{1}{2^{(p+r+1)/(r+1)}}\cdot\frac{z}{a_j} < \nu_j \leqslant \frac{1}{r+1}\cdot\frac{z}{a_j}\,, \qquad j = 1, \ldots, p\,,$$

where $z = (x/a_1\ldots a_p)^{1/(r+1)} \geqslant a_j \geqslant 1$ if the sum S is to be non-null. Here we find

$$r = z^{1/2}\,, \qquad R = z^{-1/2}\,, \qquad m = p/2\,,$$

$$f(x_{1\nu}, \ldots, x_{p\nu}) - \sum_{j=1}^p \nu_j x_{j\nu} = -\frac{p+r+1}{(r+1)^{(r+1)/(p+r+1)}}(x\nu_1\ldots\nu_p)^{\frac{1}{p+r+1}}\,,$$

$$\left|\frac{\partial(f_{x_1}, \ldots, f_{x_p})}{\partial(x_1, \ldots, x_p)}\right|_{(x_{1\nu},\ldots,x_{p\nu})} = \frac{p+r+1}{r+1} x^{-\frac{p}{p+r+1}}(r+1)^{\frac{p(r+1)}{p+r+1}}(\nu_1\ldots\nu_p)^{\frac{p+2(r+1)}{p+r+1}}\,.$$

---

[2] Throughout, we are considering only regions satisfying the general conditions we have imposed before.

Hence by Theorem 2, we have

$$S - e^{-\frac{i\pi}{4}p}\left(\frac{r+1}{p+r+1}\right)^{\frac{1}{2}}\left(\frac{x^{1/(r+1)}}{r+1}\right)^{\frac{p(r+1)}{2(p+r+1)}} \times \tag{52}$$

$$\times \sum_{(\nu_1,\dots,\nu_p)\in\Delta} (\nu_1\nu_2\dots\nu_p)^{-\frac{1}{2}\cdot\frac{p+2(r+1)}{p+r+1}} e^{-2\pi i\frac{p+r+1}{(r+1)^{(r+1)/(p+r+1)}}(x\nu_1\dots\nu_p)^{1/(p+r+1)}}$$

$$\ll z^{\frac{p(p+4)-1}{2(p+4)}} + \sum_{i=1}^{p}\left\{a_i z^{-\frac{1}{2}} + \log\left(\frac{z}{a_i}+2\right)\right\}\prod_{s\neq i}\left(z^{\frac{1}{2}} + z^{-\frac{1}{2}}a_s\right)$$

$$\ll z^{\frac{p(p+4)-1}{2(p+4)}} + z^{\frac{p}{2}-1}\left(\sum_{i=1}^{p} a_i\right),$$

since

$$\log\left(\frac{z}{a_i}+2\right)\prod_{s\neq i}\left(z^{\frac{1}{2}} + z^{-\frac{1}{2}}a_s\right) \ll z^{\frac{1}{2}\cdot\frac{p+3}{p+4}}\prod_{s\neq i} z^{\frac{1}{2}} \ll z^{\frac{p(p+4)-1}{2(p+4)}},$$

because $z \geqslant a_j \geqslant 1$.

Next we consider the sum

$$S' = \sum_{(\nu_1,\dots,\nu_p)\in\Delta'} e^{2\pi i f(\nu_1,\dots,\nu_p)}$$

where now

$$f(x_1,\dots,x_p) = \frac{p+r+1}{(r+1)^{(r+1)/(p+r+1)}}(x x_1\dots x_p)^{1/(p+r+1)}$$

and Δ' is a region contained in

$$\frac{1}{(r+1)\,2^{(p+r+1)/(r+1)}}\cdot\frac{z}{a_j} < \nu_j \leqslant \frac{1}{r+1}\cdot\frac{z}{a_j}, \qquad j = 1,\dots,p,$$

and apply Theorem 3 to S'. Here we easily find that

$$\omega = z^k/\omega', \qquad \omega' = \prod_{j=1}^{k} a_j^{k_j},$$

$$a_s = z^{-p/2}(z^{-k}\omega')^{p/2-s}\sigma_{p-s}$$

where

$$\sigma_s = \begin{cases} s\text{-th elementary symmetric function of } a_1,\dots,a_p \text{ if } s \neq \frac{p}{2}-1, \\ s\text{-th elementary symmetric function of } a_1,\dots,a_p \prod_{j=1}^{p}\left(\log\frac{z}{a_j}\right)^{k_j} \text{ if } s = \frac{p}{2}-1 \end{cases}$$

the convention being $\sigma_s = 0$ if s is not an integer.

Hence, by Theorem 3,

$$\frac{S'}{z^p/a_1\dots a_p} \ll {\sum_{\mu=1}^{p}}'\left(\frac{z}{a_\mu}\right)^{\frac{-1}{K_\mu}} + (z^{-k}\omega')^{\frac{1}{2K-2}} + (z^{-\frac{p}{2}}\sigma_{(p-1)/2})^{\frac{1}{K}} + \tag{53}$$

$$+ z^{-\frac{p}{2K}}\sum_{\frac{p}{2}+1\leqslant s\leqslant p}\{(z^{-k}\omega')^{\frac{p}{2}-s+1}\sigma_{p-s}\}^{\frac{1}{K}} + \sum_{0\leqslant s\leqslant p/2}\{z^{-\frac{p}{2}}(z^{-k}\omega')^{\frac{p}{2}-s}\sigma_{p-s}\}^{\frac{1}{(K-1)(p-2s)+K}} +$$

$$+ z^{-\frac{p}{2K}}\Bigg[\sum_{0\leqslant s\leqslant p/2}\{\sigma_{p-s}\sigma_{(p-1)/2}^{p-2s}\}^{\frac{1}{K(p+1-2s)}} +$$

$$+ \sum_{\substack{0\leqslant s\leqslant p/2\\ p/2+1\leqslant s'\leqslant p}}\{(z^{-k}\omega')^{\left(\frac{p}{2}-s\right)\left(\frac{p}{2}-s'+1\right)}\sigma_{p-s}\sigma_{p-s'}^{\frac{p}{2}-s}\}^{\frac{1}{K(p/2-s+1)}}\Bigg].$$

We can now assume without loss of generality that

$$z \geqslant a_1 \geqslant a_2 \geqslant \dots \geqslant a_p \geqslant 1.$$

Then, in the above, we can choose $\sigma_s = a_1\dots a_s$ if $s \neq p/2-1$. Hence, if $0 \leqslant s \leqslant p/2$,

$$\{\sigma_{p-s}\sigma_{(p-1)/2}^{p-2s}\}^2(\sigma_{(p+1)/2}\sigma_{(p-1)/2})^{-(p-2s+1)} = \left(\frac{\sigma_{p-s}}{\sigma_{(p-1)/2}}\right)^2\left(\frac{\sigma_{(p+1)/2}}{\sigma_{(p-1)/2}}\right)^{-(p-2s+1)}$$

$$= (a_{(p+1)/2}\dots a_{p-s})^2 a_{(p+1)/2}^{-(p-2s+1)} \leqslant 1.$$

So the sixth term on the right-hand side of (53)

$$\ll z^{-p/2K}(\sigma_{(p+1)/2}\sigma_{(p-1)/2})^{1/2K},$$

and the third term

$$\leqslant z^{-p/2K}(\sigma_{(p+1)/2}\sigma_{(p-1)/2})^{1/2K}.$$

Also from (52) and Lemma 1 we have

$$Sz^{-p/2} \ll \frac{S'}{z^p/a_1\dots a_p} + z^{-1/2(p+4)} + z^{-1}\sigma_1. \tag{54}$$

From (53) and (54) we find

$$S \ll \sum_{a} z^{\alpha_0}a_1^{\alpha_1}\dots a_p^{\alpha_p}, \text{ say}, \quad \text{where} \quad \alpha_0 > 0\ [3].$$

---

[3] When $p = 1$, the term $z^{-1}\sigma_1$ in (54) gives rise to an $\alpha_0 < 0$; but in this case $z^{-1}\sigma_1 = z^{-1}a_1 \leqslant (z^{-1}a_1)^{1/K}$ and so this term can be omitted.

Hence by Lemma 8, for arbitraray $M>0$ and $s>\operatorname{Max}\alpha_0$,

$$S_1=\sum_{(n_1,\dots,n_p)\in D}\left(\psi_1\left(\frac{x}{n_1\dots n_p}\right)^{1/(r+1)}\right)$$

$$\ll\sum_{m=1}^{\infty}\sum_{a}(mz)^{a_0}a_1^{a_1}\dots a_p^{a_p}\operatorname{Min}\left(\frac{1}{m},\frac{M^s}{m^{s+1}}\right)+\frac{\sigma_p}{M}$$

$$=\frac{\sigma_p}{M}+\sum_a z^{a_0}a_1^{a_1}\dots a_p^{a_p}\Big\{\sum_{1\leqslant m\leqslant M}m^{a_0-1}+M^s\sum_{m>M}m^{a_0-s-1}\Big\}$$

$$\ll\frac{\sigma_p}{M}+\sum_a(Mz)^{a_0}a_1^{a_1}\dots a_p^{a_p}.$$

Now by Lemma 3, there is $M>0$ so that

$$S_1\ll\sum_a\{(z\sigma_p)^{a_0}a_1^{a_1}\dots a_p^{a_p}\}^{1/(1+a_0)}.$$

Hence, by Lemma 1,

$$(55)\quad \sum_D\left(\frac{x}{n_1\dots n_p}\right)^{\frac{r(1+\varrho)}{r+1}}\psi_1\left(\left(\frac{x}{n_1\dots n_p}\right)^{\frac{1}{r+1}}\right)$$

$$\ll\left(\frac{x}{\sigma_p}\right)^{\frac{r(1+\varrho)}{r+1}}\sum_a\{(x\sigma_p^r)^{\frac{a_0}{r+1}}a_1^{a_1}\dots a_p^{a_p}\}^{\frac{1}{1+a_0}}$$

$$\ll\sum_a x^{1+\frac{1}{r+1}\left(r\varrho-\frac{1}{1+a_0}\right)}\prod_{j=1}^{p}a_j^{\frac{a_j}{1+a_0}-\frac{r}{r+1}\left(\varrho+\frac{1}{1+a_0}\right)}.$$

Hence

$$S_2=\sum_{\substack{n_1\dots n_pn_j^{r+1}\leqslant x\\ j=1,\dots,p}}\left(\frac{x}{n_1\dots n_p}\right)^{\frac{r(1+\varrho)}{r+1}}\psi_1\left(\left(\frac{x}{n_1\dots n_p}\right)^{\frac{1}{r+1}}\right)$$

$$=\sum_{h_1,\dots,h_p}\sum_{\substack{2^{h_j}\leqslant n_j<2^{h_j+1}\\ n_1\dots n_pn_j^{p+1}\leqslant x,\ j=1,\dots,p}}\left(\frac{x}{n_1\dots n_p}\right)^{\frac{r(1+\varrho)}{r+1}}\psi_1\left(\left(\frac{x}{n_1\dots n_p}\right)^{\frac{1}{r+1}}\right)$$

$$=\sum_{h_1,\dots,h_p}S_{h_1,\dots,h_p}\quad(\text{say}).$$

Since $S_{h_1,\dots,h_p}$ is obviously invariant for interchange of any two h's, we have

$$(56)\quad |S_2|\ll\sum_{\substack{0\leqslant h_p\leqslant\dots\leqslant h_1\\ (r+2)h_1+h_2+\dots+h_p\leqslant\frac{\log x}{\log 2}}}|S_{h_1,\dots,h_p}|.$$

From (55), (56) and Lemma 9, we deduce

$$(57)\quad S_2\ll\sum_{\lambda=0}^{p}(\log x)^{p-\lambda}\sum_a x^{i_\lambda^{(a)}}$$

where

$$(58)\quad i_\lambda^{(a)}=1+\frac{r\varrho-j_\lambda^{(a)}}{\lambda+r+1},\qquad j_\lambda^{(a)}=\frac{\lambda+1-\sum_{s=1}^{\lambda}a_s}{1+a_0},$$

$\sum_{s=1}^{\lambda}a_s$ denoting zero if $\lambda=0$.

We now choose $k_1=\dots=k_{p-1}=0$, $k_p=k$ in (53). Writing down the various values of $j_\lambda^{(a)}$, we find, for fixed λ, that $j_\lambda^{(a)}$ is a monotonic function of the variables of summation s, s' in (53), and so $j_\lambda^{(a)}$ corresponding to the end-values of s, s' alone matter. Substituting the resulting values of $j_\lambda^{(a)}$ in the expression for $i_\lambda^{(a)}$, we find that the values for $i_\lambda^{(a)}$ are again monotonic functions of λ, and so here again $i_0^{(a)}$ and $i_p^{(a)}$ alone matter. We find the best possible choice of k to be $k=1$, if $p\geqslant4$, $k=2$ if $p=2$ or 3 and $k=3$, if $p=1$. Choosing these values of k, we find

$$(59)\quad j_0=\operatorname*{Min}_a j_0^{(a)}=\frac{2(p+4)}{(p+2)(p+4)-1},\ \frac{16}{23}$$

if $p\geqslant2$, $p=1$ respectively;

$$(60)\quad j_p=\operatorname*{Min}_a j_p^{(a)}=\frac{2(p+1)(p+4)}{(p+2)(p+4)-1},\frac{23}{14},\frac{26}{17},\frac{55}{41}$$

if $p\geqslant4$, $p=3,2,1$ respectively.

So finally we find that (57) reduces to

$$S_2\ll x^{1+\frac{r\varrho-j_p}{p+r+1}}+x^{1+\frac{r\varrho-j_0}{r+1}}(\log x)^p.$$

Noting that $j_p<2$, we have from (7)

$$(61)\quad \Delta_{p+r+1}^{(r,\varrho)}(x)\ll x^{1+\frac{r\varrho-j_p}{p+r+1}}+x^{1+\frac{r\varrho-j_0}{r+1}}(\log x)^p.$$

(12) is now an immediate consequence of (59), (60) and (61).

In conclusion, I wish to record with great pleasure my sincere thanks to Professors V. S. Krishnan and C. T. Rajagopal for the keen interest they evinced in the preparation of this paper and last but not least, to Professor V. Granapathy for his valuable suggestions and criticism.

References

[1] Van der Corput, Math. Annalen 89 (1922), pp. 39-65.
[2] — ibidem, 98 (1928), pp. 697-716.
[3] — Math. Zeitschrift 29 (1929), pp. 397-426.
[4] L. K. Hua, Quart. Jour. Math (Oxford Series) 13 (1942), pp. 18-29.
[5] E. Landau, *Über die Anzahl der Gitterpunkte is gewissen Bereichen*, Nachrichten der K. Gesell. der Wissenschaften zu Göttingen (1912), pp. 687-771.
[6] H. E. Richert, Math. Zeitschrift 58 (1953), pp. 204-218.
[7] B. R. Srinivasan, Proc. of the National Inst. of Sciences of India 28, A (1962), pp. 732-742.
[8] — *Lattice points in a circle*, to appear in the Proc. of the National Inst. of Sciences of India.
[9] — *The Lattice point problem in many-dimensional hyperboloids I*, Acta Arith. this volume, pp. 153-172.
[10] E. C. Titchmarsh, Proc. of London Math. Soc. (2) 36 (1934), pp. 485-500.
[11] — ibidem (2) 38 (1935), pp, 96-115.
[12] — *The theory of the Riemann zeta function*, Oxford 1951.

MADRAS UNIVERSITY

Reçu par la Rédaction le 6. 6. 1962

ACTA ARITHMETICA
VIII (1963)

Another note on Hardy-Littlewood's theorem

by

S. Knapowski and W. Staś (Poznań)

1. In this paper we return to the subject of [3], i.e. to the investigation of the behaviour of

$$F(y) = \sum_{n=1}^{\infty} \{\Lambda(n)-1\} e^{-ny}, \quad y > 0, \tag{1.1}$$

as $y \to 0+$. Unlike in [3] we shall be interested here in oscillatory properties of the function (1.1). Hardy and Littlewood showed [1] that on the Riemann hypothesis there is a constant K such that each of the inequalities

$$F(y) < -\frac{K}{y^{1/2}}, \quad F(y) > \frac{K}{y^{1/2}} \tag{1.2}$$

is satisfied for an infinity of values of y tending to zero. In connection with this result we shall supply here inequalities similar, though somewhat weaker, to (1.2) holding however in an explicit form and without any hypothesis. In the proof we shall use the method of Turán (see [5]), particularly its development to the study of oscillatory questions in prime number theory (see [4]). Our result reads as follows:

Theorem. *For* $0 < \delta < c_1$ (¹) *we have*

$$\max_{\delta \leqslant y \leqslant \delta^{1/3}} F(y) > \delta^{-1/2} \exp\left(-14 \frac{\log(1/\delta)\log\log\log(1/\delta)}{\log\log(1/\delta)}\right) \tag{1.3}$$

and

$$\min_{\delta \leqslant y \leqslant \delta^{1/3}} F(y) < -\delta^{-1/2} \exp\left(-14 \frac{\log(1/\delta)\log\log\log(1/\delta)}{\log\log(1/\delta)}\right). \tag{1.4}$$

Corollary. *Replacing the exponent* $\frac{1}{2}$ *in* (1.2) *by* $\frac{1}{2}-\varepsilon$, $\varepsilon > 0$ *and arbitrary, the inequalities are satisfied* (*without any hypothesis* !) *for an infinity of values of* y *tending to zero.*

(¹) c_1 and further $c_2, c_3, \ldots$ denote positive, numerically calculable constants.