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On the number of integers $\leqslant x$ whose prime factors divide n

by

N. G. DE BRUIJN and J. H. VAN LINT (Eindhoven)

1. Introduction. In a paper with the same title as this one, the first author considered the functions

$$F(x) = \sum_{n \leqslant x} f(n, x), \qquad G(x) = \sum_{n \leqslant x} f(n, n), \tag{1}$$

where $f(n, x)$ is the number of integers $\leqslant x$ which are products of powers of prime factors of n. It was shown that, if $x \to \infty$,

$$\log\big(x^{-1}F(x)\big) \sim \log\big(x^{-1}G(x)\big) \sim (8\log x)^{\frac{1}{2}}(\log\log x)^{-\frac{1}{2}}. \tag{2}$$

In this paper we shall prove that $F(x) \sim G(x)$, thus giving the answer to a question raised by P. Erdös (letter of September 15, 1962).

Without using the arithmetical properties of F and G it is possible to derive from (2) that $\limsup G(x)/F(x) = 1$ (see section 2). In order to prove the stronger assertion $F(x) \sim G(x)$ we need more delicate methods (section 3).

In [1] it was shown that

$$\log \sum_{k \leqslant x} \big(a(k)\big)^{-1} \sim (8\log x)^{\frac{1}{2}}(\log\log x)^{-\frac{1}{2}}, \tag{3}$$

$$F(x) = x \sum_{k \leqslant x} \big(a(k)\big)^{-1} + O(x), \tag{4}$$

$$G(x) = \sum_{k \leqslant x} (x-k)\big(a(k)\big)^{-1} + O(x), \tag{5}$$

where $a(k)$ denotes the *kernel* of k, i.e. the product of all different primes dividing k.

The fact that F and G are not too easy to deal with, is connected with the complicated behaviour of the corresponding Dirichlet series

$$\sum_{n=1}^{\infty} a(n) n^{-s} = \prod_p \left(1 + \frac{1}{p(p^s-1)}\right)$$

if $s \to 0$.

We shall not use the rather deep relation (3) explicitly, but only the following weak consequence of (3): for each positive constant M there is a positive constant C such that

$$\sum_{k\leqslant x}\big(a(k)\big)^{-1} > C(\log x)^M \tag{6}$$

for all large values of x.

By (4), (5) and (6), the statement $F(x)\sim G(x)$ is equivalent with

$$\sum_{k\leqslant x} k\big(a(k)\big)^{-1} = o\Big(x\sum_{k\leqslant x}\big(a(k)\big)^{-1}\Big) \qquad (x\to\infty). \tag{7}$$

Our proof of (7) will be based on the following idea: if a positive sequence $a_1, a_2, \ldots$ is such that for every prime number p we have

$$\sum_{n\leqslant x, p\nmid n} a_n = o\Big(\sum_{n\leqslant x} a_n\Big),$$

then also

$$\sum_{n\leqslant x}\big(\varphi(n)/n\big)a_n = o\Big(\sum_{n\leqslant x} a_n\Big).$$

This method can be used for other problems of this type. We shall use it a second time to show (section 4) that (7) remains true if on both sides the summation index k is restricted to 2-full integers, or to 3-full integers, etc. An integer m is called *λ-full* if m is divisible by at least the λth power of each prime which it contains.

Throughout the paper, in the notation $\sum^{(\lambda)}$ the upper suffix (λ) will denote that the summation index is restricted to λ-full integers. And we put

$$S_\lambda(x) = \sum_{k\leqslant x}^{(\lambda)}\big(a(k)\big)^{-1}. \tag{8}$$

Our main interest is, of course, the case $\lambda = 1$, viz.

$$S_1(x) = \sum_{k\leqslant x}\big(a(k)\big)^{-1}.$$

Another way of expressing (7) is (see section 5) that $S_1(x)$ is *slowly oscillating* in the sense of Karamata, and similarly for the analogous sum $S_\lambda(x)$ over λ-full integers. This fact can be used for proving abelian and tauberian theorems involving $S_1(x)$.

It can be proved that the Dirichlet series corresponding to $S_{\lambda+1}$ and S_λ, respectively, satisfy (see [3])

$$\sum_{n=1}^{\infty}{}^{(\lambda)}\frac{\big(a(n)\big)^{-1}}{n^s} \sim \lambda^{-1}e^{-\gamma}\Big(s\log\frac{1}{s}\Big)^{-1}\sum_{n=1}^{\infty}{}^{(\lambda+1)}\frac{\big(a(n)\big)^{-1}}{n^s}$$

if $s>0$, $s\to 0$ (γ is Euler's constant). This leads to the conjecture that

$$S_\lambda(x)\sim\lambda^{-1}e^{-\gamma}(2\log x/\log\log x)^{\frac{1}{2}}S_{\lambda+1}(x)$$

if $x\to\infty$. In order to derive this from the behaviour of the ratio of the corresponding Dirichlet series, it seems that we need to assume that $S_{\lambda+1}/S_\lambda$ is slowly oscillating in a much stronger sense than we actually proved.

In section 6 we prove that anyway $\log S_\lambda$ has the same asymptotic behaviour as $\log S_1$; in fact we show that $S_1(x)/S_\lambda(x)$ is at least $\big(\log(x+1)\big)^{\lambda-1}$.

2. Proof of $\limsup G(x)/F(x) = 1$. Define $x_n = e^{n^2}$, $y_n = x_{n-1}$. Then by (2) and (8) we have

$$\log S_1(x_n)\sim 2n(\log n)^{-\frac{1}{2}} = o(n). \tag{9}$$

For any ε with $0<\varepsilon<1$ there is an infinite sequence of n's for which $S_1(y_n)\geqslant(1-\varepsilon)S_1(x_n)$, for otherwise we would have

$$S_1(x_n)\geqslant C(1-\varepsilon)^{-n} \quad \text{for all large } n,$$

C not depending on n. This contradicts (9).

If n belongs to the sub-sequence mentioned above, we have

$$\begin{aligned}\sum_{k\leqslant x_n} k\big(a(k)\big)^{-1} &< y_n\sum_{k\leqslant y_n}\big(a(k)\big)^{-1} + x_n\sum_{y_n<k\leqslant x_n}\big(a(k)\big)^{-1}\\ &< y_n\sum_{k\leqslant x_n}\big(a(k)\big)^{-1} + \varepsilon x_n\sum_{k\leqslant x_n}\big(a(k)\big)^{-1}\\ &< 2\varepsilon x_n S_1(x_n),\end{aligned}$$

if n is large enough. Now using (4) and (5) we infer that the ratio $G(x_n)/F(x_n)$ tends to 1 if n runs through the sub-sequence. As obviously $G(x)\leqslant F(x)$ for all x, the proof is complete.

3. Proof of (7). We need two lemmas. The second lemma is stated in a more general form than needed in this section, which only requires the case $\lambda = 1$.

LEMMA 1. *For $k = 1, 2, 3, \ldots$ we have*

$$\frac{k}{a(k)} = \sum_{d\mid k}^{(2)}\varphi\Big(\frac{d}{a(d)}\Big).$$

Proof. By the formula $d/a(d) = t$ the set of 2-full divisors d of k if mapped one-to-one onto the set of all divisors t of $k/a(k)$. So the lemma follows directly from the well-known formula $\sum_{t\mid n}\varphi(t) = n$.

Lemma 2. *Let λ be an integer $\geqslant 1$, let p be any prime, and $B > p^{2\lambda}$. Then we have for all $x > 2$*

$$\sum_{k\leqslant x, p\nmid k}^{(\lambda+1)} \big(\alpha(k)\big)^{-1} \leqslant \frac{2p\log p}{\log B} \sum_{k\leqslant x}^{(\lambda)} \big(\alpha(k)\big)^{-1} + \left(1+\frac{\log x}{\log 2}\right)^{\pi(B)}.$$

(As usual, $\pi(B)$ denotes the number of primes $\leqslant B$.)

Proof. We can split $S_\lambda(x) = \sum_{k\leqslant x}^{(\lambda)}$ into sums extended over all k without factors p, all k containing just λ factors p, all k containing just $\lambda+1$ factors p, etc. We obtain, if ν is chosen such that $\nu \geqslant \lambda$, $p^\nu \leqslant B$,

$$S_\lambda(x) \geqslant \sum_{k\leqslant x, p\nmid k}^{(\lambda)} \big(\alpha(k)\big)^{-1} + \sum_{j=\lambda}^{\nu} \sum_{k\leqslant xp^{-j}, p\nmid k}^{(\lambda)} \big(\alpha(k)\big)^{-1} p^{-1}$$
$$\geqslant \{1+(\nu-\lambda+1)p^{-1}\} \sum_{k\leqslant x/B, p\nmid k}^{(\lambda)} \big(\alpha(k)\big)^{-1}.$$

Since $B > p^{2\lambda}$ we can take $\nu = [\log B/\log p]$ and it follows that

$$S_\lambda(x) \geqslant \frac{\log B}{2p\log p} S_\lambda^*(x/B), \tag{10}$$

where the asterisk indicates that the summation is restricted to k's not divisible by p.

Next we consider the sum $S_{\lambda+1}^*(x)$ occurring in the lemma on the left-hand side. We split this sum into two parts according to

$$\alpha(k) > B \quad \text{or} \quad \alpha(k) \leqslant B.$$

First consider the terms with $\alpha(k) > B$. The formula $k/\alpha(k) = h$ provides a one-to-one mapping of all $(\lambda+1)$-full numbers onto all λ-full numbers, and with this mapping the k's with $k \leqslant x$, $\alpha(k) > B$ are mapped onto h's not exceeding x/B. It follows that the first part of our sum is at most $S_\lambda^*(x/B)$, to which we can apply (10).

The second part of the sum does not exceed the number of integers k with $\alpha(k) \leqslant B$, $k \leqslant x$. These are products of powers of the first $\pi(B)$ primes $p_1, \ldots, p_{\pi(B)}$. For each exponent there are at most $1+(\log x)/(\log 2)$ possibilities. Hence the second part of the sum is at most equal to the $\pi(B)$-th power of that expression, and the proof of the lemma is complete.

We now prove (7), i.e.

Theorem 1. *If $\alpha(k)$ denotes the product of the different prime divisors of k then*

$$\sum_{k\leqslant x} \frac{k}{\alpha(k)} = o\left(x\sum_{k\leqslant x}\frac{1}{\alpha(k)}\right) \quad (x\to\infty).$$

Proof. By Lemma 1 we have

$$\sum_{k\leqslant x} k/\big(\alpha(k)\big) = \sum_{k\leqslant x}\sum_{d|k}^{(2)} \varphi\big(d/\alpha(d)\big)$$
$$= \sum_{d\leqslant x}^{(2)} \varphi\big(d/\alpha(d)\big)[x/d] \leqslant x\sum_{d\leqslant x}^{(2)} \varphi\big(d/\alpha(d)\big)/d.$$

We shall split the latter sum into two parts. We choose a number $A > 2$, and we consider the $\pi(A)$ primes which do not exceed A, viz. $p_1, \ldots, p_{\pi(A)}$. Now the first part of the sum is extended over all d which are 2-full, $\leqslant x$ and divisible by $(p_1, \ldots, p_{\pi(A)})^2$.

For these d we have

$$\varphi\big(d/\alpha(d)\big) \leqslant \big(d/\alpha(d)\big)W, \quad \text{where} \quad W = \prod_{i=1}^{\pi(A)} (1-p^{-1}),$$

whence this first part is at most

$$xW\sum_{d\leqslant x}^{(2)} \big(\alpha(d)\big)^{-1} \leqslant xWS_1(x).$$

The second part of our sum runs over d's which are, for at least one i (with an $i \leqslant \pi(A)$) not divisible by p_i. Therefore, this part of the sum is at most

$$\sum_{i\leqslant\pi(A)} \sum_{d\leqslant x, p_i\nmid d}^{(2)} x/\alpha(d).$$

This can be estimated by Lemma 2, where we take $\lambda = 1$, and $B > A^2$. It results that

$$x^{-1}\sum_{k\leqslant x} k/\alpha(k) \leqslant \left(W + \frac{2A\log A}{\log B}\right) S_1(x) + \left(1+\frac{\log x}{\log 2}\right)^{\pi(B)}.$$

If $\varepsilon > 0$ is given, we can choose A such that $W < \frac{1}{3}\varepsilon$; next we can choose B such that $2\log A < \frac{1}{3}\varepsilon\log B$. Finally, by (6) we know that

$$\big(1+(\log x)/(\log 2)\big)^{\pi(B)} < \tfrac{1}{3}\varepsilon S_1(x)$$

for all large x. It follows that, for all large x,

$$x^{-1}\sum_{k\leqslant x} k/\alpha(k) < \varepsilon S_1(x).$$

As this holds for every $\varepsilon > 0$, this proves the lemma.

4. Restriction to λ-full integers. The proof of Theorem 2 (p. 355) requires, instead of Lemma 1, a slightly more complicated convolution formula.

Every multiplicative function f is completely defined when for each prime p the numbers $f(p), f(p^2), \ldots$ are given, in other words if its p-component

$$1+f(p)z+f(p^2)z^2+\ldots$$

is given. If we have, between f, g and h, the following relation for each p

$$\sum_{j=0}^{\infty} f(p^j)z^j \sum_{j=0}^{\infty} g(p^j)z^j = \sum_{j=0}^{\infty} h(p^j)z^j$$

then h is the convolution of f and g, i.e.

$$\sum_{d|k} f(d)g(k/d) = h(k)$$

for all k, whence, for all $x>0$

$$\sum_{k\leqslant x} h(k) = \sum_{d\leqslant x} f(d) \sum_{t\leqslant [x/d]} g(t)\,.$$

We take for h the function defined by the series

$$1+p^{\lambda-1}z^{\lambda}+p^{\lambda}z^{\lambda+1}+p^{\lambda+1}z^{\lambda+2}+\ldots,$$

for g the function defined by

$$1+p^{\lambda-1}z^{\lambda}+p^{2(\lambda-1)}z^{2\lambda}+p^{3(\lambda-1)}z^{3\lambda}+\ldots,$$

whence we have to take for the series which defines f:

$$(1-p^{\lambda-1}z^{\lambda})(1+p^{\lambda-1}z^{\lambda}+p^{\lambda}z^{\lambda+1}+\ldots) = 1+\sum_{j=\lambda+1}^{2\lambda-1} p^{j-1}z^j + \sum_{j=2\lambda}^{\infty} \frac{p-1}{p} p^{j-1}z^j\,.$$

It follows that $f(k)=0$ if k is not $(\lambda+1)$-full, and, otherwise $f(k) = k(\alpha(k))^{-1}\prod(1-p^{-1})$, where the product is extended over all primes p for which $p^{2\lambda}$ divides k.

Notice that $h(k)=k/\alpha(k)$ if k is λ-full, and $h(k)=0$ otherwise. Hence

$$\sum_{k\leqslant x}{}^{(\lambda)} k/\alpha(k) = \sum_{k\leqslant x} h(k)\,.$$

We have, for all $y>0$,

$$\sum_{t\leqslant y} g(t) = 1^{\lambda-1}+2^{\lambda-1}+\ldots+[y^{1/\lambda}]^{\lambda-1} \leqslant y$$

and it follows that

(11) $$\sum_{k\leqslant x}{}^{(\lambda)} k/\alpha(k) \leqslant x\sum_{d\leqslant x} f(d)/d\,.$$

We can now prove

THEOREM 2. *For every λ $(\lambda=1,2,3,\ldots)$ we have*

$$\sum_{k\leqslant x}{}^{(\lambda)} \frac{k}{\alpha(k)} = o\Big(x\sum_{k\leqslant x}{}^{(\lambda)} \frac{1}{\alpha(k)}\Big)\,.$$

Proof. As in the proof of Theorem 1, we take a number $A>2$, and we split the sum in the right-hand side of (11) into two parts. The first part consists of all d which are $(\lambda+1)$-full and divisible by $(p_1\ldots p_{\pi(A)})^{2\lambda}$. For such d we have $f(d)/d\leqslant W/\alpha(d)$, whence this first part is at most $xWS_\lambda(x)$.

The second part consists of all $(\lambda+1)$-full d's which are, for at least one i $(1\leqslant i\leqslant \pi(A))$ not divisible by $p_i^{2\lambda}$. By Lemma 2, the contribution of these terms is at most

$$\frac{2\lambda A\log A}{\log B} S_\lambda(x) + \Big(1+\frac{\log x}{\log 2}\Big)^{\pi(B)}\,.$$

The extra factor λ arises from the fact that we do not only consider d's which are not divisible by p_i, but also d's with exactly $\lambda+1$, or $\lambda+2,\ldots$, or $2\lambda-1$ factors p_i, and we use the fact that $\sum^{(\lambda)}(\alpha(k))^{-1}$, extended over all λ-full $k\leqslant x$ with exactly j factors p_i dividing k, equals $p_i^{-1}\sum^{(\lambda)}(\alpha(k))^{-1}$, extended over all λ-full $k\leqslant x/p_i^j$ with $p_i\nmid k$. The remaining part of the proof can be copied from the proof of Theorem 1.

5. The $S_\lambda(x)$ are slowly oscillating. For each λ, the function $S_\lambda(x)$ is slowly oscillating (in the sense of Karamata), i.e. for every fixed $c>0$ we have $S_\lambda(cx)/S_\lambda(x)\to 1$ as $x\to\infty$. For, if $0<c<1$ (a restriction we may obviously make), we have

$$\sum_{k\leqslant x}{}^{(\lambda)} (\alpha(k))^{-1} - \sum_{k\leqslant cx}{}^{(\lambda)} (\alpha(k))^{-1} = \sum_{cx<k\leqslant x}{}^{(\lambda)} \leqslant (cx)^{-1}\sum_{cx<k\leqslant x}{}^{(\lambda)} k/\alpha(k)\,,$$

and the latter expression is $o(S_\lambda(x))$, by Theorem 2.

The fact that $S_\lambda(x)$ is slowly oscillating can be used for proving abelian and tauberian theorems involving that notion. For example, it can be used to show that

$$\sum_{1\leqslant k<\infty}{}^{(\lambda)} e^{-ky}/\alpha(k) \sim S_\lambda(y)^{-1} \quad \text{if} \quad y>0,\ y\to 0\,.$$

This follows by application of an abelian lemma (see [2], Lemma 1) which is the abelian counterpart of a tauberian theorem of Karamata.

6. Behaviour of $S_\lambda(x)$. It is not difficult to show that $\log S_\lambda(x)$ has the same asymptotic behaviour as $\log S_1(x)$. We can prove a stronger result, viz. $\log S_\lambda(x) = \log S_1(x) + O(\log\log x)$. This follows from

THEOREM 3. *For $\lambda=1,2,\ldots$; $x>0$ we have*

$$S_\lambda(x)/\log(x+1) \leqslant S_{\lambda+1}(x) \leqslant S_\lambda(x)\,.$$

Proof. The upper estimate is trivial. In order to prove the lower one we notice that every λ-full integer k can be uniquely represented in the form $k = nr^{\lambda}$, with the restrictions that n is $(\lambda+1)$-full, r is squarefree, and $(n, r) = 1$. As $a(k) = a(n)r$, we obtain

$$S_{\lambda}(x) = \sum_{n\leqslant x}^{(\lambda+1)} \big(a(n)\big)^{-1} \sum_{r\leqslant x/n}' r^{-1},$$

where the dash indicates restriction to squarefree r with $(r, n) = 1$. Ignoring these restrictions, we obtain

$$S_{\lambda}(x) \leqslant S_{\lambda+1}(x) \sum_{r\leqslant x/n} r^{-1} \leqslant S_{\lambda+1}(x)\log(x+1) .$$

This proves the theorem.

References

[1] N. G. de Bruijn, *On the number of integers $\leqslant x$ whose prime factors divide n*, Illinois J. Math. 6 (1962), pp. 137-141.

[2] — and P. Erdös, *On a recursion formula and on some tauberian theorems*, J. Res. Nat. Bur. Standards 50 (1953), pp. 161-164.

[3] — and J. H. van Lint, *On the asymptotic behaviour of some Dirichlet series with a complicated singularity*, Nieuw Archief v. Wiskunde (3), 11 (1963), pp. 68-75.

TECHNOLOGICAL UNIVERSITY, EINDHOVEN, NETHERLANDS

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Verschärfung der Abschätzung von $\zeta(\frac{1}{2}+it)$

von

W. Haneke (Marburg/Lahn)

Einleitung. In dieser Arbeit soll eine neue obere Abschätzung für die Riemannsche Zetafunktion $\zeta(s)$ auf der „kritischen Geraden“ $s = \frac{1}{2}+it$ hergeleitet werden.

Grundlegend ist dabei die Idee von Weyl und van der Corput, die in der approximativen Funktionalgleichung der Zetafunktion auftretenden trigonometrischen Summen in Exponentialsummen überzuführen, die unmittelbar eine nichttriviale Abschätzung gestatten.

Bei Weyl und van der Corput waren diese Exponentialsummen stets eindimensional (vgl. [7], Chap. 5, S. 81-101) (1).

Titchmarsh erzielte durch Einschaltung zweidimensionaler Exponentialsummen bessere Abschätzungen (vgl. [5] und [6]).

Min [3] verfeinerte die zweidimensionale Methode und fand damit für jedes beliebige aber feste positive ε

$$\zeta(\tfrac{1}{2}+it) = O(t^{15/92+\varepsilon}) \qquad (t\to\infty) .$$

In der Literatur wurde bisher noch keine Verschärfung dieser Abschätzung veröffentlicht.

Im Beweis von Min tritt das Problem auf, für die Nullstellenmannigfaltigkeit einer Hesseschen Determinante eine Umgebung mit bestimmten Eigenschaften zu konstruieren. Diese Umgebung ist nun im wesentlichen durch die Anzahl der frei wählbaren Parameter in drei vorbereitenden „Weylschen Schritten“ bestimmt. Min beschränkte sich auf vier Parameter (vgl. [3], S. 459, (4.7)) und konnte daher als Umgebung einen Parallelstreifen wählen, was bei mehr als vier Parametern nicht mehr möglich gewesen wäre. Infolgedessen konnte die dreimalige Anwendung der „Weylschen Idee“ nicht voll mit insgesamt sechs Parametern ausgenutzt werden.

(1) Die Zahlen in den eckigen Klammern beziehen sich auf das Literaturverzeichnis am Schluß der Arbeit.