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TO THE MEMORY OF WITOLD WOLIBNER

ON THE EVALUATION FROM BELOW OF EXTREMAL DETERMINANTS

BY

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1. Let M_n be the maximal value of the determinant of the n-th degree,

$$M_n = \max_{|x_{ik}|\leqslant 1} |x_{ik}|_n,$$

of a matrix composed of elements x_{ik} which are real and absolutely no-greater than 1. Paley [7] has remarked that, since a determinant is a lt near expression in all x_{ik}, the maximum value of a determinant is attained for a matrix

$$(e_{ik}), \quad \text{where} \quad e_{ik} = \pm 1 \quad (1 \leqslant i, k \leqslant n). \tag{1.1}$$

In the sequel we shall generally denote by (e_{ik}) such a matrix and by W_n its determinant, n being its degree.

According to the theorem of Hadamard [5] we have

$$M_n \leqslant n^{n/2}. \tag{1.2}$$

If matrix (1.1) is orthogonal, then

$$M_n = n^{n/2}. \tag{1.3}$$

The inverse theorem is also true.

Matrices (1.1) for which (1.3) holds are called *matrices of Hadamard*. The degress n, for which there exist Hadamard matrices, will be called *Hadamard numbers*. The set of Hadamard numbers will be denoted by H.

Sylvester has proved in [9] that $2^q \epsilon H$ for $q = 1, 2, \ldots$, and that the necessary condition for $n \epsilon H$ is $n = 1, 2$, or $n = 4q$, where $q = 1, 2, \ldots$, making at once a very interesting supposition that $4q \epsilon H$.

The hypotesis of Sylvester raised interest of many authors: Scarpis [8], Gilman [4], Paley [7], Williamson [13], [15] and Brauer [1] have found some classes of Hadamard's numbers. Here we list sufficient conditions for $2 \leqslant n \epsilon H$, which are due to the above mentioned authors; in this list p denotes an odd prime number, while n_1 and n_2 are two arbitrary Hadamard numbers not less that 2:

$$
(1.4)\qquad
\begin{array}{ll}
\text{(a)} & n = 2; \\
\text{(b)} & n = p^h+1 \equiv 0 \pmod 4; \\
\text{(c)} & n = n_1(p^h+1); \\
\text{(d)} & n = n_1 n_2; \\
\text{(e)} & n = (p+1)^2, \text{ where } p+2 \text{ is also a prime number}; \\
\text{(f)} & n = q(q-1), \text{ where } q \text{ is the product of numbers (a) and (b)}; \\
\text{(g)} & n = n_1 n_2 (p^h+1) p^h; \\
\text{(h)} & n = r(r+3), \text{ where } r \text{ and } r+4 \text{ are products of numbers (a) and (b)}; \\
\text{(i)} & n = n_1 n_2 s(s+3), \text{ where } s \text{ and } s+4 \text{ are both of the form } p^h+1; \\
\text{(j)} & n = 172.
\end{array}
$$

Williamson [14] has determined the extremal determinants for some $n \not\equiv 0 \pmod 4$, namely he has shown that

$$(1.5)\qquad M_3 = 2^2,\quad M_5 = 2^4\cdot 3,\quad M_6 = 2^5\cdot 5,\quad M_7 = 2^6\cdot 9.$$

He has also proved that if $4q \epsilon H$, then there exists a matrix (e_{ik}) of degree $4q+1$ with

$$(1.6)\qquad W_{4q+1} = |e_{ik}|_{4q+1} = (5-3/q)(4q)^{2q},$$

and a matrix (e_{ik}) of degree $4q-1$ which is a submatrix of a matrix of Hadamard, with

$$(1.7)\qquad W_{4q-1} = |e_{ik}|_{4q-1} = (4q)^{2q-1}.$$

He also remarked that there exists a matrix (e_{ik}) of degree 21 with $|e_{ik}|_{21} = 2^{20}\cdot 5^{11}$ which is greater than W_{21} in (1.6), and a matrix (e_{ik}) of degree 7 with $|e_{ik}|_7 = 2^6\cdot 9$ which is greater than W_7 in (1.7). In view of (1.6) it is possible to form a matrix (e_{ik}) of degree $2(4q+1)$, namely the direct product [13] of the matrix in (1.6) and the matrix $\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}$, to the effect of

$$(1.8)\qquad |e_{ik}|_{8q+2} = 2(8q)^{4q}(5-3/q)^2.$$

Similarly, forming the direct product of a matrix in (1.7) and matrix $\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}$ we get a matrix (e_{ik}) of degree $8q-2$ with

$$(1.9)\qquad |e_{ik}|_{8q-2} = 2(8q)^{4q-2}.$$

As M_m is the sum of m minors, a minor not less than M_m/m exists among them; hence

$$(1.10)\qquad M_{m-k} \geqslant M_m(m-k)!/m!.$$

The signs of minors and of their elements are, of course, the same, since otherwise we could obtain a greater determinant by changing the signs of elements. This also implies the existence of minors $|e_{ik}|_{m-1}$ of the determinant M_m with

$$(1.11)\qquad |e_{ik}|_{m-1} \leqslant M_m/m.$$

The easy construction of a determinant for proving

$$(1.12)\qquad M_{m+k} \geqslant 2^k M_m$$

is well-known.

Szekeres and Turán [10], and Turán [11], [12], give the evaluation from below of M_n for all n by considering the expression

$$(1.13)\qquad M_n^{2q} = \left\{\frac{1}{2^{n^2}}\cdot\sum_{(e_{ik})}|e_{ik}|_n^{2q}\right\}^{1/2q} < M_n,$$

where the summation is taken upon all different matrices (e_{ik}) of the n-th degree. For (1.13) we have

$$\lim_{q\to\infty} M_n^{(2q)} = M_n.$$

They compute (see [10]) $M_n^{(2q)}$ for $q=1$, and $q=2$ (see [11], [12]), and obtain

$$(1.14)\qquad
\begin{aligned}
M_n &> \sqrt{n!} \approx n^{n/2}(2\pi n)^{1/4}e^{-n/2} \quad \text{(see [10])},\\
M_n &> \sqrt{\frac{n+4}{5}}\sqrt{n!} \approx n^{n/2}\sqrt{\frac{n+4}{5}}(2\pi n)^{1/4}e^{-n/2} \quad \text{(see [11], [12])},
\end{aligned}$$

where $\approx$ denotes that the ratio of the two sides tends to 1 when $n\to\infty$. Turán [12] supposes that it is possible, by making some longer calculations, to obtain $M_n^{(2q)}$ for $q=3$ and better evaluations than (1.14).

Let $\varphi(n)$ be determined for natural n by

$$(1.15)\qquad M_n = n^{n/2}2^{-\varphi(n)/2}.$$

In virtue of (1.2) we have $\varphi(n)\geqslant 0$ and, according to (1.3), $\varphi(n)=0$ if and only if the matrix of the extremal determinant is orthogonal. The aim of the present paper is to estimate from below the determinant M_n by estimating from above the expression $\varphi(n)$.

We prove that, for a sufficiently great n, we have $\varphi(n)\leqslant bn^a+d$ with a suitable $a<1$ (Corollary 7), and, by assuming Riemanns conjecture, we prove that, for a sufficiently large n, we have $\varphi(n) \leqslant \leqslant bn^{1/2}\log(n/2)+d$ (Corollary 8). (From the result quoted in [3], p. 242, follows only $\varphi(n) = o(n)$).

2. LEMMA 1. *Let (a_{ik}) be a square matrix of degree $r+1$ with $a_{r,r} = a$, $a_{r,r+1} = b$, $a_{r+1,r} = c$, $a_{r+1,r+1} = d$, and such that the remaining parts of its two last columns as well as of its two last rows, are identical.*

If $|a_{ik}|_r$ is the determinant of degree r of a matrix (a_{ik}) deprived of the last column and the last row, and $|a_{ik}|_{r-1}$ is the determinant of degree $r-1$ of a matrix (a_{ik}) without last two columns and rows, then the determinant $|a_{ik}|_{r+1}$ of the degree $r+1$ of a matrix (a_{ik}) is expressed by

$$|a_{ik}|_{r+1} = (a+d-b-c)\,|a_{ik}|_r - (b-a)(c-a)\,|a_{ik}|_{r-1}. \tag{2.1}$$

In order to verify (2.1) we subtract, in the matrix (a_{ik}), the r-th row from the $(r+1)$-th row and then the r-th column from the $(r+1)$-th column and so we get the matrix (b_{ik}), in other words, we have

$$b_{ik} = a_{ik} \quad \text{for} \quad 1 \leqslant i, k \leqslant r,$$

$$b_{i,r+1} = b_{r+1,k} = 0 \quad \text{for} \quad 1 \leqslant i, k \leqslant r-1,$$

$$b_{r,r+1} = b-a, \qquad b_{r+1,r} = c-a, \qquad b_{r+1,r+1} = a+d-b-c.$$

We get (2.1) by developing $|b_{ik}|_{r+1}$ with respect to its last column and row.

THEOREM 1. *If W_m denotes the determinant of a matrix (e_{ik}) with $e_{m,m} = 1$, and W_{m-1} denotes the minor of the element $e_{m,m}$, then there exists a determinant W_{m+1} satisfying the relation*

$$W_{m+1} = 4(W_m - W_{m-1}), \tag{2.2}$$

and thus we have

$$M_{m+1} \geqslant 4(1-1/m)M_m. \tag{2.3}$$

Proof. If we take, in Lemma 1, $r = m$, $a_{ik} = e_{ik}$, $a = d = 1$, $b = c = -1$, and two last columns and rows according to this Lemma, we get $W_{m+1} = |a_{ik}|_{m+1} = 4(W_m - W_{m-1})$, or (2.2).

Consider a matrix (e_{ik}) of the extremal determinant M_m. According to (1.11) there exists a minor $W_{m-1} < M_m/m$. Without loss of generality we can assume that W_{m-1} is a minor corresponding to the element $e_{m,m}$. We get now (2.3) by substituting (e_{ik}) into (2.2).

COROLLARY 1. *If $4q \in H$, then there exists a determinant W_{4q+2} of degree $4q+2$ with*

$$W_{4q+2} = 2(10-7/q)(4q)^{2q}. \tag{2.4}$$

According to (1.6), we have $W_{4q+1} = \left(5-\frac{3}{q}\right)(4q)^{2q}$ and, after Williamson [13], there exists a minor of this determinant, $W_{4q} = 2(4q)^{2q-1}$, to which we apply Theorem 1 and thus get

$$W_{4q+2} = 4(5-3/q-2/4q)(4q)^{2q} = 2(10-7/q)(4q)^{2q},$$

i. e. (2.4).

COROLLARY 2. *If $\varphi(4q-4) \leqslant c+2{,}25$, and $\varphi(4q) \leqslant c$, then, for $4q-4 \leqslant n \leqslant 4q$ and $q \geqslant 8$,*

$$\varphi(n) \leqslant c+1{,}25+2\log_2 q. \tag{2.5}$$

According to (2.3), and to (1.15), we have

$$M_{n+1} = (n+1)^{(n+1)/2}\cdot 2^{-\varphi(n+1)/2} \geqslant 4M_n\left(1-\frac{1}{n}\right) = 4\frac{n-1}{n}\cdot n^{n/2}\cdot 2^{-\varphi(n)/2},$$

which implies, for $n \geqslant 8$,

$$\varphi(n+1) \leqslant \varphi(n)+\log_2\frac{en^2(n+1)}{4^2(n-1)^2} < \varphi(n)+2\log(n+4)-5, \tag{2.6}$$

in view of the inequalities

$$\log_2 e/4^2 = -2{,}55, \qquad \log_2(n/(n-1))^2 \leqslant 0{,}4,$$

$$(n+1)/(n+4) \leqslant 1, \qquad \log_2 1/(n+4) \leqslant -3{,}55.$$

Since

$$2\log_2\frac{n+1}{n-1} \leqslant 0{,}1, \qquad \frac{(n+1)(n+2)}{(n+4)^2} \leqslant 1 \quad \text{for } n \geqslant 60,$$

$$2\log_2\frac{n+1}{n-1} \leqslant 0{,}214, \quad \log_2\frac{(n+1)(n+2)}{(n+4)^2} \leqslant -0{,}114 \quad \text{for } 60 \geqslant n \geqslant 27,$$

we get from (2.6), for $n \geqslant 28$,

$$\varphi(n+2) \leqslant \varphi(n)+\log_2\frac{e^2(n+1)^3(n+2)}{4^4(n-1)^2} \leqslant \varphi(n)+2\log_2(n+4)-5. \tag{2.7}$$

In view of (1.10) we have

$$M_{n-1} \geqslant \frac{M_n}{n} = 2^{-\varphi(n)/2}(n-1)^{(n-1)/2}\cdot\left(1+\frac{1}{n-1}\right)^{n/2}\frac{(n-1)^{1/2}}{n};$$

hence, for $n \geqslant 3$,

$$\varphi(n-1) \leqslant \varphi(n)+2\log_2 n-\log_2 e(n-1) \leqslant \varphi(n)+2\log_2 n-2{,}75. \tag{2.8}$$

From (2.8) we get

$$\varphi(n-2) \leqslant \varphi(n)+2\log_2(n)+\log_2\frac{n-1}{e^2(n-2)} \leqslant \varphi(n)+2\log_2 n-2{,}75. \tag{2.9}$$

Assuming, in (2.6) and (2.7), $n = 4q-4$, and, in (2.8) and (2.9), $n = 4q$, and taking into consideration the assumed inequalities involving c, we get (2.5).

3. LEMMA 2. *Let (a_{ik}) be a matrix of degree r and (b_{ik}) a matrix of degree $s \leqslant r$. Define a matrix (c_{ik}) of degree $r+s$ by*

$$c_{ik} = \begin{cases} a_{ik} & \text{for} \quad i \leqslant r,\ k \leqslant r, \\ a_{i,k-r} & \text{for} \quad i \leqslant r,\ k > r, \\ b_{i-r,k-r} & \text{for} \quad i > r,\ k > r, \\ -b_{i-r,k} & \text{for} \quad i > r,\ k \leqslant s, \\ \text{arbitrary} & \text{for} \quad i > r,\ s < k \leqslant r. \end{cases}$$

Then we have

$$|c_{ik}|_{r+s} = |a_{ik}|_r \cdot |b_{ik}|_s \cdot 2^s. \tag{3.1}$$

To verify this, define a matrix (d_{ik}) by

$$d_{ik} = \begin{cases} a_{ik} & \text{for} \quad i \leqslant r,\ k \leqslant r, \\ 0 & \text{for} \quad i \leqslant r,\ k > r, \\ 2b_{i-r,k-r} & \text{for} \quad i > r,\ k > r, \\ c_{ik} & \text{for} \quad i > r,\ k \leqslant r. \end{cases}$$

Thus we see that (d_{ik}) is obtained from (c_{ik}) by subtracting, for $k \leqslant s$, a column with index k from a column with index $r+k$.

Developing the determinant $|d_{ik}|_{r+s}$ after the first r rows and taking into consideration the zero-elements in (d_{ik}) we get

$$|c_{ik}|_{r+s} = |d_{ik}|_{r+s} = |a_{ik}|_r \cdot |2b_{ik}|_s = |a_{ik}|_r |b_{ik}|_s 2^s,$$

and thus (3.1).

THEOREM 2. *If $\varphi(n)$ is determined by* (1.15), *then for every integer m and n*

$$\varphi(m+n) \leqslant \varphi(m)+\varphi(n)+|m-n|; \tag{3.2}$$

if $m \neq n$ in (3.2) *we have the sharp inequality.*

Proof. Let (a_{ik}) be a matrix of the determinant M_m, (b_{ik}) a matrix of the determinant M_n, (c_{ik}) a matrix formed from (a_{ik}) and (b_{ik}) in a way described in Lemma 2, and $M'_{m+n} = |c_{ik}|_{m+n}$. Owing to (1.15) and (3.1) we get

$$M_{m+n} \geqslant M'_{m+n} = M_m \cdot M_n 2^{\min(m,n)}$$

and, since $2\min(m,n) = m+n-|m-n|$, we have

$$(m+n)^{(m+n)/2} \cdot 2^{-\varphi(m+n)/2} \geqslant M'_{m+n} = m^{m/2} \cdot n^{n/2} \cdot 2^{-[\varphi(m)+\varphi(n)-m-n+|m-n|]/2} \tag{3.3}$$

$$= (m+n)^{(m+n)/2}\left(1+\frac{m-n}{m+n}\right)^{m/2}\left(1+\frac{n-m}{m+n}\right)^{n/2} \cdot 2^{-[\varphi(m)+\varphi(n)+|m-n|]/2}.$$

Consider the inequality

$$e^{-x/r} \geqslant 1-\frac{x}{r} > 0 \quad \text{for} \quad x < r,\ 0 < r, \tag{3.4}$$

which is sharp for $x \neq 0$. If we raise it to power $-r < 0$,

$$e^x \leqslant \left(\frac{r}{r-x}\right)^r,$$

and substitute $r = s+x > 0$, $s > 0$, we get

$$e^x \leqslant \left(1+\frac{x}{s}\right)^{s+x}. \tag{3.5}$$

After substituting $s = (m+n)/4 > 0$ and $x = (m-n)/4$, and once more $x = (n-m)/4$, we obtain from (3.5) the following inequalities:

$$\left(1+\frac{m-n}{m+n}\right)^{m/2} \geqslant e^{(m-n)/4}; \qquad \left(1+\frac{n-m}{m+n}\right)^{n/2} \geqslant e^{(n-m)/4}.$$

In view of this, inequality (3.3) implies

$$(m+n)^{(m+n)/2} \cdot 2^{-\varphi(m+n)/2} \geqslant (m+n)^{(m+n)/2} \cdot 2^{-[\varphi(m)+\varphi(n)+|m-n|]/2}. \tag{3.6}$$

Hence (3.2) follows. In virtue of the sharpness of (3.4) for $x \neq 0$ we see that inequality (3.6), and therefore also inequality (3.2), is sharp for $m-n \neq 0$.

COROLLARY 3. *We have*

$$\varphi(4q) \leqslant \frac{4q}{11} - 4 \quad \textit{for} \quad q = 11, 12, 13, \ldots \tag{3.7}$$

According to Table I of this paper, for $q = 1, 2, \ldots, 22$ we have $\varphi(4q) = 0$; therefore, for $s = 0$, we have (3.7) in the interval

$$11 \cdot 2^s \leqslant q \leqslant 11 \cdot 2^{s+1}. \tag{3.8}$$

Assuming that (3.7) holds for q in (3.8) with $s = k$, we see, according to (3.2), for $m = 4[q/2]$, $n = 4(q-[q/2])$ that (3.7) holds for q in (3.8) with $s = k+1$; namely,

$$\varphi(4q) \leqslant \frac{4[q/2]}{11} + \frac{4(q-[q/2])}{11} - 8 + 4(q-2[q/2]) \leqslant \frac{4q}{11} - 4,$$

for $[q/2]$ and $q-[q/2]$ belong to the interval (3.8) if $s = k$. Thus we see, by induction, that (3.7) holds for $q \geqslant 11$.

4. LEMMA 3. *If*

$$\varphi(p) = \varphi(q) = 0, \quad q > p \geqslant 4, \tag{4.1}$$

$$\varphi(n) \leqslant c, \quad 1 \leqslant c \quad \textit{for} \quad p \leqslant n \leqslant q, \tag{4.2}$$

then, for $k = 0, 1, 2, \ldots$, *we have*

$$\varphi(n) < c + \frac{2}{3}(q-p)2^k - \frac{1}{3}(q-p) - 1 \quad \textit{for} \quad 2^k p \leqslant n \leqslant 2^k q. \tag{4.3}$$

Proof. Condition (4.1), according to (1.4) (a) and (d), implies

$$\varphi(2^k \cdot p) = \varphi(2^k \cdot q) = 0 \quad \text{for} \quad k = 0, 1, 2, \ldots \tag{4.4}$$

Instead of (4.3) we prove by induction (4.5) and (4.6):

$$\left.\begin{aligned} &(4.5)\quad \varphi(n) \leqslant c(k), \\ &(4.6)\quad \varphi(n) \leqslant c(k) + d(k) - |n-(p+q)2^{k-1}| \end{aligned}\right\} \quad \text{for} \quad 2^k \cdot p \leqslant n \leqslant 2^k \cdot q,$$

where

$$c(k) = c + \frac{2}{3}(q-p)(2^k-1) + \frac{1-(-1)^k}{6}(q-p-3), \tag{4.7}$$

$$d(k) = \frac{q-p}{3}2^{k-1} + \frac{(-1)^k}{3}(q-p-3). \tag{4.8}$$

Let us remark that

$$c(k) = c(k-1) - d(k) + (q-p)2^{k-1}, \tag{4.9}$$

$$c(k) = c(k-1) + d(k-1) + (q-p)2^{k-2}. \tag{4.10}$$

We verify equalities (4.9) and (4.10) substituting (4.7) and (4.8) into (4.9) and (4.10). Inequalities (4.5) and (4.6) hold for $k = 0$ in virtue of (4.1) and (4.2).

Supposing that (4.5) holds for $k = l$ we shall verify that (4.6) will be valid for $k = l+1$.

In order to do this, divide the interval $2^{l+1}p \leqslant n \leqslant 2^{l+1}q$ into two subintervals:

(i) $$2^{l+1}p \leqslant n = 2^l p + h \leqslant 2^l(p+q)$$

and

(ii) $$2^l(p+q) \leqslant n = 2^l q + h \leqslant 2^{l+1}\cdot q.$$

(Notice that in both cases $2^l p \leqslant h \leqslant 2^l q$).

Now, applying formula (3.2) to the numbers $2^l p$ and h in the case of the first subinterval, and then to the numbers $2^l q$ and h, and since $\varphi(2^l p) = \varphi(2^l q) = 0$ in virtue of (4.4), we get

$$\varphi(n) = \varphi(2^l \cdot p + h) \leqslant c(l) + h - 2^l \cdot p = c(l) + n' - 2^{l+1}\cdot p$$

in subinterval (i) and

$$\varphi(n) = \varphi(2^l \cdot q + h) \leqslant c(l) + 2^l \cdot q - h = c(l) + 2^{l+1}\cdot q - n$$

in subinterval (ii). Further, according to (4.9), where $k = l+1$, we have

$$\begin{aligned} \varphi(n) &\leqslant c(l+1) + d(l+1) + n - (q-p)\cdot 2^l - 2^{l+1}\cdot p \\ &= c(l+1) + d(l+1) + n - (p+q)2^l, \end{aligned}$$

for $2^{l+1}p \leqslant n \leqslant 2^l(p+q)$ and

$$\begin{aligned} \varphi(n) &\leqslant c(l+1) + d(l+1) - n - (q-p)\cdot 2^l + 2^{l+1}\cdot q \\ &= c(l+1) + d(l+1) - n + (q+p)\cdot 2^l \end{aligned}$$

for $2^l(p+q) \leqslant n \leqslant 2^{l+1}q$; hence we obtain (4.6) for $k = l+1$.

Let us remark that in (4.6) the inequality continues to hold if we substitute a plus sign for the minus sign before the absolute value. Therefore we may write

$$\varphi(n) \leqslant c(k) + d(k) \mp (n-(p+q)2^{k-1}) \quad \text{for} \quad 2^k p \leqslant n \leqslant 2^k q. \tag{4.11}$$

Assuming that (4.6) holds for $k = l$ (which means that (4.11) holds too), we shall see that (4.5) is valid for $k = l+1$.

Consider again the subintervals (i) and (ii). Applying formula (3.2) in the same way as before we get

$$\begin{aligned} \varphi(n) = \varphi(2^l p + h) &\leqslant c(l) + d(l) - (h-(p+q)2^{l-1}) - 2^l p + h \\ &= c(l) + d(l) + (q-p)2^{l-1}, \end{aligned}$$

for n in (i), and

$$\begin{aligned} \varphi(n) = \varphi(2^l q + h) &\leqslant c(l) + d(l) + (h-(p+q)2^{l+1}) + 2^l q - h \\ &= c(l) + d(l) + (q-p)2^{l-1}, \end{aligned}$$

for n in (ii). Moreover, according to (4.10), where $k = l+1$, we write

$$\varphi(n) \leqslant c(l+1) \quad \text{for} \quad 2^{l+1}p \leqslant n \leqslant 2^{l+1}q,$$

which implies (4.5) for $k = l+1$. Since (4.5) and (4.6) hold for $k = 0$, and it was shown that if they hold for $k = l$, then they hold for $k = l+1$, so they hold for $k = 0, 1, 2, \ldots$

According to theorem 2, in all cases where $2^l p \neq h$ or $2^l q \neq h$ the inequality (4.5) is sharp. But if $2^l p = h$ or $2^l q = h$, then, according to (4.4), the inequality (4.5) is sharp as well. Hence the inequality (4.3) is sharp in all cases for $k = 1, 2, \ldots$ Therefore, according to (4.5) and (4.7), we have

$$\varphi(n) < c + \frac{2}{3}(q-p)(2^k-1) + \frac{1-(-1)^k}{6}(q-p-3)$$

for $k = 1, 2, \ldots$ and $2^k p \leqslant n \leqslant 2^k q$.

Since, according to the theorem of Sylvester for $q > p \geqslant 4$, we have $q-p \geqslant 4$ or $q-p-3 \geqslant 1$, we see that for $k = 1, 2, \ldots$

$$\varphi(n) < c + \frac{2}{3}(q-p)2^k - \frac{q-p}{3} - 1 \quad \text{for} \quad 2^k p \leqslant n \leqslant 2^k q,$$

or (4.3) for $k = 1, 2, \ldots$; but (4.3) holds also for $k = 0$ which is immediately verified in view of (4.2) and of the fact that $(q-p)/3 \geqslant 4/3$.

THEOREM 3. *If*

$$\varphi(n_i) = 0 \quad \textit{for} \quad i = 0, 1, 2, \ldots, m, \tag{4.12}$$

where $n_{i+1} > n_i$ *and* $8 \leqslant 2n_0 = n_m$,

$$\alpha \geqslant \max_{i=0,1,2\ldots m-1}\left(\frac{n_{i+1}-n_i}{n_i}\right), \tag{4.13}$$

$$\beta \geqslant \max_{i=0,1,2,\ldots,m-1}\ \max_{n_i \leqslant n \leqslant n_{i+1}}\left(\varphi(n) - \frac{n_{i+1}-n_i}{3},\ 1 - \frac{n_{i+1}-n_i}{3}\right), \tag{4.14}$$

then for $n \geqslant n_0$ *we have*

$$\varphi(n) < \frac{2}{3}\alpha n + \beta - 1. \tag{4.15}$$

Proof. Substitute $p = n_i$, $q = n_{i+1}$, in Lemma 3. Then, in virtue of (4.12), we have (4.1). Because of (4.14) we have (4.2) with $c_i = \beta + (n_{i+1}-n_i)/3$. Therefore, according to (4.3) we have, for every $k = 0, 1, 2, \ldots$, and $i = 0, 1, 2, \ldots, m-1$

$$\varphi(n) < \beta + \frac{n_{i+1}-n_i}{3} + \frac{2}{3}(n_{i+1}-n_i)2^k - \frac{1}{3}(n_{i+1}-n_i) - 1$$

$$= \beta - 1 + \frac{2}{3}(n_{i+1}-n_i)2^k \leqslant \beta - 1 + \frac{2}{3}\,\frac{n_{i+1}-n_i}{n_i}\cdot 2^k \cdot n_i$$

for $2^k \cdot n_i \leqslant n \leqslant 2^k n_{i+1}$.

Replacing, in this inequality, $(n_{i+1}-n_i)/n_i$ by α and $2^k n_i$ by n we get, according to (4.13), the inequality (4.15) for $2^k n_i \leqslant n \leqslant 2^k n_{i+1}$, $k = 0, 1, 2, \ldots$, $i = 0, 1, 2, \ldots, m-1$.

Since, by assumption, $2n_0 = n_m$, we see that for $n \geqslant n_0$ (4.15) holds.

COROLLARY 4. *For* $n \geqslant 4$, *we have*

$$\varphi(n) < \frac{2}{3}n - 0{,}996. \tag{4.16}$$

In view of Table I we have $\varphi(4) = \varphi(8) = 0$, and, according to (1.5), we have

$$\varphi(5) = 5\log_2 5 - 8 - 2\log_2 3 \leqslant 11{,}7 - 8 - 3{,}16 = 0{,}54,$$

$$\varphi(6) = 6 + 6\log_2 3 - 2\log_2 5 - 10 \leqslant 6 + 9{,}5 - 4{,}63 - 10 = 0{,}87,$$

$$\varphi(7) = 7\log_2 7 - 4\log_2 3 - 12 \leqslant 19{,}7 - 6{,}33 - 12 = 1{,}37,$$

i. e. we have $\varphi(5) \leqslant \varphi(6) \leqslant \varphi(7) \leqslant 1{,}37 \leqslant 0{,}04 + \frac{4}{3}$.

Hence, if we put $n_0 = 4$, $n_1 = 8$, $m = 1$, in theorem 3, then $\alpha = 1$ fulfils (4.13) and, according to the above estimation, $\beta = 0.04$ fulfils (4.14), which proves (4.16).

COROLLARY 5. *For* $n \geqslant 48$ *we have*

$$\varphi(n) < \frac{2}{33}n + 10{,}65. \tag{4.17}$$

In view of (4.13) and of Table I we have $\alpha = \frac{2}{22} = \frac{1}{11}$ for $48 \leqslant n \leqslant 96$, and according to (2.5), we have

$$\varphi(n) \leqslant 2\log_2 22 + 1{,}25 \leqslant 2\cdot 4{,}47 + 1{,}25 \leqslant 10{,}2 \quad \text{for} \quad 48 \leqslant n \leqslant 88;$$

according to (2.5) and in view of Table II we have

$$\varphi(n) \leqslant 2\log_2 23 + 5{,}25 \leqslant 2{,}453 + 5{,}25 \leqslant 14{,}31 \quad \text{for} \quad 88 \leqslant n \leqslant 96.$$

Both cases imply, in view of (4.14),

$$14{,}31 - \frac{8}{3} = \beta = 11{,}65,$$

whence follows (4.17) by Theorem 3.

COROLLARY 6. *For* $n \geqslant 1080$, *we have*

$$\varphi(n) < \frac{n}{192} + 37{,}7. \tag{4.18}$$

In view of (4.13) and of Table I we find that $\alpha = \frac{4}{512} = \frac{1}{128}$ for $270 \leqslant n/4 \leqslant 540$. Looking through Table II, according to (2.5) and in view of (4.14) we find that

$$2\log_2(535)+23-\frac{8}{3} \leqslant \beta = 38,7.$$

Having α and β we obtain (4.18) by (4.15).

5. THEOREM 4. *If*

$$\varphi(n_i) = 0, \quad n_{i+1} > n_i \quad \textit{for} \quad i = 0, 1, 2, \ldots, \tag{5.1}$$

and if $g(n)$ is a function defined for integers $n \geqslant n_0$ in such a manner that

$$g(n) \geqslant \min_{n_i \leqslant n}(n-n_i), \tag{5.2}$$

and $\psi(x)$ is a function defined for real values $x \geqslant n_0$, non-decreasing and such that

$$1 \leqslant \psi(x), \tag{5.3}$$

$$\psi\big(2n-g(n)\big) \geqslant \psi(n)+g(n), \tag{5.4}$$

then from the assumption

$$\varphi(n) \leqslant \psi(n) \quad \textit{for} \quad n_0 \leqslant n \leqslant 2n_0 \tag{5.5}$$

it follows that inequality (5.5) *holds for every* $n \geqslant n_0$.

Proof. Obviously, according to (1.4) (a) and (d), $\varphi(2n_i) = 0$; hence we may complete the sequence (5.1) so that $n_{i+1} \leqslant 2n_i$. If for $n_0 \leqslant n < 2n_i$ formula (5.5) holds, which is verified according to the assumption for $i = 0$, we shall see, by Theorem 2, that (5.5) holds for $n_0 \leqslant n \leqslant 2n_{i+1}$.

For $2n_i \leqslant n = n_i+h \leqslant n_i+n_{i+1}-1$ we have $n_i \leqslant h \leqslant n_{i+1}-1 < 2n_i$. In view of (5.1), (5.2), (5.3), and (5.4), where $n = h$ and applying formula (3.2) to the numbers n_i and h, we get

$$\begin{aligned}\varphi(n_i+h) &\leqslant \psi(h)+h-n_i \leqslant \psi(h)+g(h) \leqslant \psi\big(2h-g(h)\big)\\ &\leqslant \psi(2h-h+n_i) = \psi(n_i+h).\end{aligned} \tag{5.6}$$

For $n = n_i+n_{i+1}$, according to (5.2), (5.3), (5.4), (5.1), and to Theorem 2, we have

$$\begin{aligned}\varphi(n_i+n_{i+1}) &\leqslant n_{i+1}-n_i = 1+(n_{i+1}-1-n_i) \leqslant 1+g(n_{i+1}-1)\\ &\leqslant \psi(n_{i+1}-1)+g(n_{i+1}-1) \leqslant \psi\big(2(n_{i+1}-1)-g(n_{i+1}-1)\big)\\ &\leqslant \psi(2n_{i+1}-2+n_i-n_{i+1}+1) \leqslant \psi(n_{i+1}-n_i).\end{aligned} \tag{5.7}$$

For $n_i+n_{i+1}+1 \leqslant n = n_{i+1}+h \leqslant 2n_{i+1}-1$ we also have $n_i+1 \leqslant h \leqslant n_{i+1}-1 < 2n_i$; therefore, for $n = h$, according to the induction hypothesis, (5.5) holds, and in view of (5.2), (5.3), (5.4), (5.1) and applying formula (3.2) to the numbers n_{i+1} and h we have

$$\begin{aligned}\varphi(n_{i+1}+h) &\leqslant \psi(h)+n_{i+1}-h \leqslant \psi(h)+g(n_{i+1}-1)\\ &\leqslant \psi(n_{i+1}-1)+g(n_{i+1}-1) \leqslant \psi\big(2(n_{i+1}-1)-g(n_{i+1}-1)\big)\\ &\leqslant \psi(n_{i+1}+n_i) \leqslant \psi(n_{i+1}+h).\end{aligned} \tag{5.8}$$

Since (5.6), (5.7), and (5.8), imply (5.5) for $2n_i \leqslant n < 2n_{i+1}$, we get (5.5) for $n_0 \leqslant n \leqslant 2n_{i+1}$; hence (5.5) holds for $n \geqslant n_0$.

COROLLARY 7. *There exist an $a < 1$ and an N such that for $n > N$ we have*

$$\varphi(n) \leqslant bn^a+d. \tag{5.9}$$

Let $n_i = 2(p_i+1)$, where p_i is the i-th prime number. Then, according (1.4), (a), (c), and (d), we have $\varphi(n_i) = 0$, and thus (5.1).

According to the theorem of Ingham [6], we have

$$p_{i+1}-p_i \leqslant p^{38/61+\eta},$$

where p_i and p_{i+1} are two consecutive prime numbers, c is a constant, and η is an arbitrary positive number. If we substitute $n_i = 2(p_i+1)$ in this inequality, we get

$$n_{i+1}-n_i \leqslant 2(n_i/2-1)^{38/61+\eta} \quad \text{for} \quad n_i \geqslant 2(c^{1/\eta}+1) = N.$$

Let $g(n) = \alpha n^a$, where $a = 38/61+\eta$ and $\alpha = 2^{1-a}$. Then, for each $n > N$ there exists such an i that

$$0 \leqslant n-n_i \leqslant 2(n/2-1)^{38/61+\eta}-1 < \alpha n^a = g(n),$$

and thus (5.2) holds. The function $\psi(n) = bn^a+d$ satisfies (5.3) and (5.4) if

$$b(2n-\alpha n^a)^a \geqslant bn^a+\alpha n^a,$$

i. e. for $b = \alpha\colon\{(2-\alpha N^{a-1})^a-1\} \geqslant \alpha\colon\{(2-\alpha n^{a-1})^a-1\}$. Choosing a d such that (5.9) holds for $N \leqslant n \leqslant 2N+2g(N)$, we have, in view of Theorem 4, formula (5.9) for $n \geqslant N$.

COROLLARY 8. *If we assume the hypothesis of Riemann about the distances of consecutive prime numbers, we have*

$$\varphi(n) \leqslant bn^{1/2}\log\left(\frac{n}{2}\right)+d \quad \textit{for} \quad n \geqslant N. \tag{5.10}$$

According to Riemann's hypothesis [2] we have, for the i-th prime number $p_i > \frac{1}{2}N_1-1$,

$$p_{i+1}-p_i < cp_i^{1/2}\log p_i.$$

Taking $n_i = 2(p_i+1)$ we have (5.1) and

$$n_{i+1}-n_i < 2c\left(\frac{n_i}{2}-1\right)^{1/2}\log\left(\frac{n_i}{2}-1\right);$$

therefore for $n \geqslant N_1$ we have (5.2) with

$$g(n) = \sqrt{2}\cdot c\cdot\sqrt{n}\cdot\log\frac{n}{2}.$$

The function $\psi(n) = bn^{1/2}\log(n/2)+d$ satisfies (5.3) and (5.4) if

$$b \geqslant \frac{\sqrt{2}\cdot c}{\sqrt{2-\sqrt{2}\cdot c\log(N_2/2):\sqrt{N_2}-1}}, \qquad n \geqslant N = \max(N_1, N_2),$$

which we verify by substituting $\psi(n)$ in (5.4). Choosing a d such that (5.10) holds for $N \leqslant n \leqslant 2N+2g(N)$ we obtain (5.10) for $n \geqslant N$.

Corollary 9. *If we assume the hypothesis of H. Cramér, then for $n \geqslant N$ we get*

$$\varphi(n) \leqslant b\log^3\left(\frac{n}{2}\right)+d. \tag{5.11}$$

According to Cramér's hypothesis [2], we have

$$p_{i+1}-p_i \leqslant c\log^2 p_i \quad \text{for} \quad p_i \geqslant \frac{N_1}{2}-1,$$

which gives (5.2), for $n_i = 2(p_i+1)$ and

$$g(n) = 2c\log^2\left(\frac{n}{2}\right) \quad \text{for} \quad n \geqslant N_1.$$

The function $\psi(n) = b\log^3(n/2)+d$ satisfies (5.3) and (5.4) if

$$b \geqslant \frac{2c}{3\log\left(2-\dfrac{2c}{N_2}\log^2\dfrac{N_2}{2}\right)}, \qquad n \geqslant N = \max(N_1, N_2)$$

which is to be verified by substituting $\psi(n)$ in (5.4). Choosing a d such that (5.11) holds for $N \leqslant n \leqslant 2N+2g(N)$ we obtain (5.11) for $n \geqslant N$.

It is interesting to compare Corollary 9 with Corollary 2, which, if Sylvester's hypothesis is assumed, gives the following evaluation:

$$\varphi(n) \leqslant d+2\log_2 n/\log 2,$$

where d is a positive constant.

6. The evaluation of the maximal value of the determinant of matrix (1.1) may be used for the evaluation of the maximal value of the determinant of a matrix of a more general form owing to the following

Theorem 5. *If the matrix (x_{ik}) is of the n-th degree, such that*

$$a_ib_k(c-1) \leqslant x_{ik} \leqslant a_ib_k(c+1), \quad 1 \leqslant i, k \leqslant n \tag{6.1}$$
$$a_i \geqslant 0, \quad b_i \geqslant 0, \quad c \geqslant 0,$$

and X_n is the maximal value of the determinant of matrix (x_{ik}), then

$$X_n \geqslant \prod_{i=1}^{n} a_i \prod_{k=1}^{n} b_k M_n\left(\frac{3n-4}{n}c+1\right), \tag{6.2}$$

$$X_n \geqslant \prod_{i=1}^{n} a_i \prod_{k=1}^{n} b_k M_{n+1}\frac{cn+1}{n+1}. \tag{6.3}$$

Proof. By interchanging columns and rows and multiplying by ± 1 matrix (1.1) of the determinant M_n can be reduced to a matrix (e_{ik}) such that $|e_{ik}|_n = M_n$, and the minor W_{n-1} of the element e_{nn} is the absolutely least one and that

$$e_{nk} = 1 \text{ for } 1 \leqslant k \leqslant n, \qquad e_{in} = -1 \quad \text{for } 1 \leqslant i \leqslant n-1, \tag{6.4}$$

which is possible for $n \geqslant 3$, for we can change the sign of the determinant by permuting the first two columns; for $n = 2$ we have (6.4) since

$$M_2 = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}. \tag{6.5}$$

In view of (1.11) we have $0 \leqslant |e_{ik}|_{n-1} = W_{n-1} \leqslant M_n/n$ and, since $c \geqslant 0$, we get

$$\left(3c+1-\frac{4c}{n}\right)M_n \leqslant c(3M_n-4W_{n-1})+M_n$$
$$= c(3|e_{ik}|_n-4|e_{ik}|_{n-1})+|e_{ik}|_n,$$

Further, according to Lemma 1 we get (2.1), where $n = r$, $a_{ik} = e_{ik}$ for $1 \leqslant i, k \leqslant n$ and $a = 1$, $d = 0$, $b = c = -1$, and the corresponding a_{ik} for the last column and for the last row, we have

$$\left(3c+1-\frac{4c}{n}\right)M_n \leqslant c\begin{vmatrix} & & -1 \\ & (e_{ik}) & \vdots \\ & & -1 \\ 1 & \ldots 1 & 0 \end{vmatrix}_{n+1} + |e_{ik}|_n.$$

Multiplying the n-th column by -1 and permuting the first two rows (which is possible for $n \geqslant 2$) we obtain the matrix (e'_{ik}) and

$$(6.6)\qquad \left(3c+1-\frac{4c}{n}\right) M_n \leqslant c \begin{vmatrix} (e'_{ik}) & \begin{matrix} -1 \\ \vdots \\ -1 \end{matrix} \\ 1 \ldots 1 & 0 \end{vmatrix}_{n+1} + |e'_{ik}|_n$$

$$= \begin{vmatrix} (e'_{ik}) & \begin{matrix} -1 \\ \vdots \\ -1 \end{matrix} \\ c \ldots c & 0 \end{vmatrix}_{n+1} + \begin{vmatrix} (e'_{ik}) & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ c \ldots c & 1 \end{vmatrix}_{n+1} = \begin{vmatrix} (e'_{ik}) & \begin{matrix} -1 \\ \vdots \\ -1 \end{matrix} \\ c \ldots c & 1 \end{vmatrix}_{n+1} = \begin{vmatrix} (d_{ik}) & \begin{matrix} 1 \\ \vdots \\ 1 \end{matrix} \\ c \ldots c & 1 \end{vmatrix}_{n+1} = |d_{ik}|_n,$$

where $d_{ik} = e'_{ik}+c$.

Choosing $x_{ik} = a_i b_k d_{ik}$ we see that (6.1) holds and get

$$(6.7)\qquad X_n \geqslant |x_{ik}|_n = \prod_{i=1}^{n} a_i \prod_{k=1}^{n} b_k \, |d_{ik}|_n .$$

After substituting $|d_{ik}|$ in (6.7) we obtain, for $n \geqslant 2$, formula (6.2), and for $n=1$ we have $X_1 = a_1 b_1 (c+1)$; hence (6.2) holds for all integers n.

Permuting columns and rows and multiplying by -1, respectively, we reduce matrix (1.1) of the determinant M_{n+1} to a form in which

$$(6.8)\qquad e_{i,n+1} = -1 \text{ for } 1 \leqslant i \leqslant n, \qquad e_{n+1,k} = 1 \text{ for } 1 \leqslant k \leqslant n+1,$$

and the minor $|e_{ik}|_n \geqslant 0$ of the element $e_{n+1,n+1}$ is the greatest one if $1-c \geqslant 0$ and absolutely the least one if $1-c < 0$. This is possible for $n \geqslant 2$ since the sign of the determinant M_{n+1} may remain unchanged when the first two columns are permuted respectively. Formula (6.8) can be obtained also for $n-1$ in view of (6.5). Applying (1.10) to $|e_{ik}|_n$, if $1-c \geqslant 0$, and (1.11), if $1-c < 0$, we get the following relation:

(6.9)

$$\left(c+\frac{1-c}{n+1}\right) M_{n+1} \leqslant c M_{n+1} + (1-c)|e_{ik}|_n = c \begin{vmatrix} (e_{in}) & \begin{matrix} -1 \\ \vdots \\ -1 \end{matrix} \\ 1 \ldots 1 & \end{vmatrix}_{n+1} + (1-c)|e_{ik}|_n$$

$$= \begin{vmatrix} (e_{ik}) & \begin{matrix} -1 \\ \vdots \\ -1 \end{matrix} \\ c \ldots c & \end{vmatrix}_{n+1} + \begin{vmatrix} (e_{ik}) & \begin{matrix} -1 \\ \vdots \\ -1 \end{matrix} \\ 0 \ldots 0 & (1-c) \end{vmatrix}_{n+1} = \begin{vmatrix} (e_{ik}) & \begin{matrix} -1 \\ \vdots \\ -1 \end{matrix} \\ c \ldots c & 1 \end{vmatrix}_{n+1} = \begin{vmatrix} (d_{ik}) & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ c \ldots c & 1 \end{vmatrix}_{n+1} = |d_{ik}|_n,$$

where $d_{ik} = e_{ik}+c$. Choosing $x_{ik} = a_i b_k d_{ik}$ we see that (6.1) holds and we get (6.7); then, substituting (6.9) into (6.7) we obtain (6.3) for integers n.

Table I. For $1 < q \leqslant 568$ the numbers $4q$, contained in column H, either have one of the forms (1.4) or are in column S. If q_1 is expressed in one of the forms (1.4) and $q_2 = 2^k q_1$ $(k = 1, 2, \ldots)$, then in Table I there are no numbers q_2, as they obviously are integers of the form (1.4) according to (1.4), (a) and (d).

Table II. Table II contains numbers $1 \leqslant q \leqslant 544$ for which it is not known whether the matrices H of degree $4q$ exist.

In column $\Phi(q)$ are presented the values of the function $\Phi(q)$ defined in the following manner:

(a) $\Phi(q) = 0$ if $4q$ satisfies one of the conditions (1.4);

(b) If the value of $\Phi(q)$ is known for $q \leqslant q_0-1$, then

$$\Phi(q_0) = \min_{q_0=q_1+q_2} \{\Phi(q_1)+\Phi(q_2)+|q_2-q_1|\}.$$

In column Z are given the components $q_0 = q_1+q_2$ for which the minimum value is reached.

The function $\Phi(q)$ can be used for determining β appearing in theorem 3.

TABLE I

q	H	S	q	H	S	q	H	S
1	$3+1$		47		188	91	$2(181+1)$	
3	$2(5+1)$		49	$2(97+1)$		92	$367+1$	
5	$19+1$		51	$2(101+1)$		93		372
7	$2(13+1)$		53	$211+1$		94		376
9	$2(17+1)$		55	$2(109+1)$		95	$379+1$	
11	$43+1$		57	$2(113+1)$		97	$2(193+1)$	
13	$2(5^2+1)$		58		232	99	$2(197+1)$	
15	$2(29+1)$		59		236	101		404
17	$67+1$		61	$2(11^2+1)$		103		412
19	$2(37+1)$		63	$251+1$		105	$419+1$	
21	$2(41+1)$		65		260	107		428
23		92	67		268	109		436
25	$2(7^2+1)$		69	$2(137+1)$		111	$443+1$	
27	$2(53+1)$		71	$283+1$		113		452
29		116	73		292	115	$2(229+1)$	
31	$2(61+1)$		75	$2(149+1)$		116	$463+1$	
33	$131+1$		77	$307+1$		117	$2(233+1)$	
35	$139+1$		78	$311+1$		118		472
37	$2(73+1)$		79	$2(157+1)$		119		476
39		156	81	$(17+1)^2$		121	$2(241+1)$	
41	$163+1$		83	$331+1$		122	$487+1$	
43	172		85	$2(13^2+1)$		123	$491+1$	
45	$2(89+1)$		87	$2(173+1)$		125	$499+1$	
46		184	89		356	127		508

TABLE I (continued)

q	H	S	q	H	S	q	H	S
129	2(237+1)		206	823+1		281	1123+1	
130	$20(5^2+1)$		207	827+1		283		1132
131	523+1		209		836	285	2(569+1)	
133		532	211	2(421+1)		287		1148
134		536	213		852	289	2(577+1)	
135	2(269+1)		214		856	291	1163+1	
137	547+1		215	859+1		292		1167
139	2(277+1)		217	2(433+1)		293	1171+1	
141	2(281+1)		218		872	295		1180
143	571+1		219		876	297	2(593+1)	
145	$2(17^2+1)$		221	883+1		298		1192
146		584	223		892	299		1196
147	2(293+1)		225	2(449+1)		301	2(601+1)	
149		596	226		904	302		1208
151		604	227	907+1		303		1212
153		612	229	2(457+1)		305		1220
155	619+1		231	2(461+1)		306	1223+1	
157	2(313+1)		233		932	307	2(613+1)	
159	2(317+1)		235		940	309	2(617+1)	
161	643+1		236		944	311		1244
163		652	237	947+1		313	$2(5^4+1)$	
165	659+1		238	(13+1)17·4		315	1259+1	
167		668	239		956	317		1268
169	2(337+1)		241		964	319		1276
170	(9+1)4·17		243	971+1		321	2(641+1)	
171	863+1		245		980	323	1291+1	
173	691+1		247		988	325		1300
175	2(349+1)		249		996	326	1303+1	
177	2(353+1)		251		1004	327	2(653+1)	
178		712	253		1012	329		1316
179		716	254		1016	331	2(661+1)	
181	$2(19^2+1)$		255	2(509+1)		333	11^3+1	
183		732	257		1028	334		1336
185	739+1		259		1036	335		1340
186	743+1		260	1039+1		337	2(673+1)	
187	2(373+1)		261	2(521+1)		339	2(677+1)	
188	751+1		263	1051+1		341		1364
189	$3^3(3^3+1)$		265	$2(23^2+1)$		343		1372
191		764	266	1063+1		345		1380
193		772	267		1068	347		1388
195	2(389+1)		268		1072	349		1396
197	787+1		269		1076	351	2(701+1)	
199	2(397+1)		271	2(541+1)		353		1412
201	2(401+1)		273	1091+1		355	2(709+1)	
202		808	275		1100	356	1423+1	
203	811+1		277		1108	357	1427+1	
205	2(409+1)		279	2(557+1)		358		1432

TABLE I (continued)

q	H	S	q	H	S	q	H	S
359		1436	433		1732	501	2003+1	
361		1444	435		1740	502		2008
362	1447+1		436		1744	503	2011+1	
363	1451+1		437	1747+1		505	2(1009+1)	
365	1459+1		438		1752	506		2024
366		1464	439	2(877+1)		507	2(1013+1)	
367	2(733+1)		441	2(881+1)		508		2032
369		1476	443		1772	509		2036
371	1483+1		445		1780	511	2(1021+1)	
373		1492	446	1783+1		513		2052
375	1499+1		447	1787+1		514		2056
377		1508	449		1796	515		2060
378	1511+1		451		1804	517	2(1033+1)	
379	2(757+1)		452		1808	518	(13+1)37·4	
381	2(761+1)		453	1811+1		519		2076
382		1528	455		1820	521	2083+1	
383	1531+1		457		1828	523		2092
385	2(769+1)		459		1836	525	2(1049+1)	
386	1543+1		461		1844	527		2108
387	2(773+1)		463		1852	529		2116
389		1556	465	2(929+1)		531	2(1061+1)	
391		1564	466		1864	533	2131+1	
393	1571+1		467	1867+1		534		2136
395	1579+1		469	2(937+1)		535	2(1069+1)	
397		1588	470	1879+1		536	2143+1	
399	2(791+1)		471	2(941+1)		537		2148
401		1604	472		1888	538		2152
403		1612	473	44(44−1)		539		2156
404		1616	475		1900	541		2164
405	2(809+1)		477	2(953+1)		543		2171
407	1627+1		478		1912	545		2180
409		1636	479		1916	547	2(1093+1)	
411	2(821+1)		481	$2(31^2+1)$		549	2(1097+1)	
413		1652	482		1928	550	(49+1)11·4	
415	2(829+1)		483	1931+1		551	2203+1	
417	1667+1		485		1940	553		2212
418	(37+1)44		487		1948	554		2216
419		1676	489	2(977+1)		555	2(1109+1)	
421		1684	490	(9+1)49·4		557		2228
423		1692	491		1964	559	2(1117+1)	
425	1699+1		493		1972	561	2243+1	
426	(5+1)4·71		494	(25+1)4·19		563	2251+1	
427	2(853+1)		495	1979+1		565	2(1129+1)	
428		1721	497	1987+1		566		2264
429	2(857+1)		498	(5+1)83·4		567	2267+1	
431	1723+1		499	2(997+1)				

TABLE II

q	Z	$\Phi(q)$	q	Z	$\Phi(q)$	q	Z	$\Phi(q)$	q	Z	$\Phi(q)$
23	11+12	1	213	106+107	2	325	162+163	2	449	224+225	1
29	14+15	1	214	107+107	2	329	164+165	1	451	224+227	3
39	19+20	1	218	109+109	2	334	167+167	2	452	227+225	2
46	23+23	2	219	109+110	2	335	167+168	2	455	228+227	1
47	23+24	2	223	111+112	1	341	170+171	1	457	229+228	1
58	29+29	2	226	113+113	2	343	171+172	1	459	229+230	1
59	30+29	2	233	116+117	1	345	172+173	1	461	230+231	1
65	32+33	1	235	117+118	3	347	173+174	1	463	231+232	1
67	33+34	1	236	116+120	4	349	174+175	1	466	233+233	2
73	36+37	1	239	119+120	4	353	176+177	1	472	234+238	4
89	44+45	1	241	120+121	1	358	179+179	4	475	237+238	1
93	45+48	3	245	122+123	1	359	179+180	3	478	238+240	2
94	47+47	4	247	123+124	1	361	180+181	1	479	238+241	4
101	50+52	1	249	124+125	1	366	183+183	2	482	241+241	2
103	51+58	1	251	125+126	1	369	184+185	1	485	242+243	1
107	53+54	1	253	126+127	2	373	186+187	1	487	243+244	1
109	54+55	1	254	127+127	2	377	188+189	1	491	245+246	2
113	57+56	1	257	128+129	1	382	191+191	2	493	246+247	2
118	59+59	4	259	129+130	1	389	194+195	1	502	251+251	2
119	59+60	3	267	132+135	3	391	195+196	1	506	253+253	4
127	63+64	1	268	134+134	4	397	198+199	1	508	254+254	4
133	66+67	2	269	134+135	3	401	200+201	1	509	254+255	3
134	67+67	2	275	137+138	1	403	201+202	3	513	256+257	2
146	73+73	2	277	138+139	1	404	201+203	2	514	257+257	2
149	74+75	1	283	141+142	1	409	204+205	1	515	257+258	2
151	75+76	1	287	143+144	1	413	206+207	1	519	259+260	2
153	76+77	1	292	145+147	2	419	209+210	2	523	261+262	1
163	81+82	1	295	147+148	1	421	210+211	1	527	263+264	1
167	83+84	1	298	149+149	2	423	211+212	1	529	264+265	1
178	89+89	2	299	149+150	2	428	214+214	4	534	267+267	6
179	89+90	2	302	151+151	2	433	216+217	1	537	266+271	5
183	91+92	1	303	151+152	2	435	217+218	3	538	269+269	6
191	95+96	1	305	152+153	2	436	218+218	4	539	269+270	4
193	96+97	1	311	155+156	1	438	219+219	4	541	270+271	1
202	101+101	2	317	158+159	1	443	221+222	1	543	271+272	1
209	104+105	1	319	159+160	1	445	222+223	2	545	272+273	1

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