Hence

$$\hat{u}(t) = \exp\left\{\int_0^t \left[i \sum_{j|1}^{n} \left(-\sigma_j a_j(\tau)\right) + b(\tau)\right] d\tau\right\} \hat{u}(0) + \int_0^t \hat{R}(\eta) \exp\left\{\int_\eta^t \left[i \sum_{j|1}^{n} \left(-\sigma_j a_j(\tau)\right) + b(\tau)\right] d\tau\right\} d\eta. \tag{8}$$

On the other hand, $\hat{u}(0) \leqslant 0$ and $\hat{R}(\eta) \leqslant 0$, and the function $\xi(t, \eta; \sigma) = \exp\{\int_\eta^t [i \sum_{j|1}^{n} (-\sigma_j a_j(\tau)) + b(\tau)] d\tau\}$ is a multiplier in Z for fixed t and η. Hence, by our lemma, $\xi(t, \eta; \sigma)\hat{u}(0) \leqslant 0$ and $\xi(t, \eta; \sigma)\hat{R}(\eta) \leqslant 0$ for $\eta \leqslant t$. Obviously $\int_0^t \xi(t, \eta; \sigma)\hat{R}(\eta)d\eta \leqslant 0$. Hence both parts of the right-hand member of (8) are negative, q. e. d.

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SOME REMARKS ON A CERTAIN METHOD OF SUCCESSIVE APPROXIMATIONS IN DIFFERENTIAL EQUATIONS

BY

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In papers [1] and [2] a method of successive approximations in differential equations was discussed. Some sufficient conditions for the convergence of iteration process were given. These conditions were obtained by reducing the problem to the solving of a system of Volterra's equations by successive approximations method. In the present paper we shall give some remarks which allow to weaken the assumption of theorems formulated in [1] and [2].

1. Let us consider Volterra's integral equation of the form

$$x(t) = A(t)x(t) + \int_0^t B(t, \xi)x(\xi)d\xi + f(t), \tag{1}$$

where matrices $A(t)$, $B(t, \xi)$ are continuous for $t \geqslant 0$, and $t \geqslant 0$, $0 \leqslant \xi < t$ respectively; vector function $f(t)$ is continuous for $t \geqslant 0$.

Definition 1. Let $\|x\|$ be an arbitrary norm of the vector

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

i. e. a non-negative number satisfying conditions:

a) $\|x\| > 0$, for $x \neq 0$ and $\|0\| = 0$,

b) $\|cx\| = |c| \cdot \|x\|$, c — an arbitrary real number,

c) $\|x+y\| \leqslant \|x\| + \|y\|$.

Definition 2. Let $\|A\| = \max\limits_{\|x\|=1} \|Ax\|$ be the norm of matrix A (see [3], p. 124-127).

This norm is consistent with the given norm of vectors x and it fulfils the following conditions:

a) $\|A\| > 0$ for $A \neq 0$, $\|0\| = 0$,

b) $\|cA\| = |c|\cdot\|A\|$ (c — real number),

c) $\|A+B\| \leqslant \|A\|+\|B\|$,

d) $\|Ax\| \leqslant \|A\|\,\|x\|$,

e) $\|\int_0^t A(\xi)\,d\xi\| \leqslant \int_0^t \|A(\xi)\|\,d\xi$.

Let us construct a sequence $\{x_m\}$, $m = 0, 1, 2, \ldots$, assuming

$$x_0(t) = f(t),$$

(2) $$x_{m+1}(t) = A(t)x_m(t) + \int_0^t B(t, \xi)x_m(\xi)\,d\xi, \quad m = 0, 1, 2, \ldots,$$

and let

(3) $$x(t) = \sum_{m=0}^{+\infty} x_m(t).$$

THEOREM 1. *If there exists a non-singular diagonal matrix $P(t)$, continuous for $t \geqslant 0$, and such that for each real number $a \geqslant 0$ we have*

$$\max_{0\leqslant t\leqslant a} \|P^{-1}(t)A(t)P(t)\| < 1,$$

then the series (3) *is quasi-uniformly convergent to the unique solution of equation* (1), *continuous for $t \geqslant 0$.*

Proof. Let $P(t)$ be a non-singular diagonal matrix, and let us define a sequence of vector functions $z_m(t)$ by

(4) $$z_m(t) = P^{-1}(t)x_m(t), \quad m = 0, 1, \ldots$$

Then the relation (2) will take the form

(5) $$z_{m+1}(t) = [P^{-1}(t)A(t)P(t)]z_m(t) + \int_0^t [P^{-1}(t)B(t, \xi)P(\xi)]z_m(\xi)\,d\xi.$$

Owing to the given properties of vector and matrix norm we obtain the estimation

(6) $$\|z_m(t)\| \leqslant F\left[\sum_{s=0}^{m} \binom{m}{s} K^{m-s}\frac{(Mt)^s}{s!}\right], \quad m = 0, 1, \ldots,$$

for $0 \leqslant t \leqslant a$, where

$$F = \max_{0\leqslant t\leqslant a} \|P^{-1}(t)f(t)\|, \quad K = \max_{0\leqslant t\leqslant a} \|P^{-1}(t)A(t)P(t)\|,$$

$$M = \max_{\substack{0\leqslant t\leqslant a\\ 0\leqslant \xi\leqslant t}} \|P^{-1}(t)B(t, \xi)P(\xi)\|.$$

If $K < 1$, then the series

(7) $$\sum_{m=0}^{\infty} z_m(t)$$

is uniformly convergent in the interval $\langle 0, a\rangle$, since under this assumption a series with terms equal to the right hand side of the inequality (6) is convergent (see [1]). In view of the relation (4) a uniform convergence of series (7) implies uniform convergence of series (3) in the interval $\langle 0, a\rangle$.

In view of the theorem on integration of uniformly convergent series, we can state that the sum of series (3) is the solution of equation (1). In the same way we can easily state (proceeding as usually in the case of successive approximations) that this solution is unique (see [1]). Thus theorem 1 is proved.

Theorem 1 is more general than theorem 3 in [1], which is due to the general definition of the vector and matrix norm, and to the weaker condition.

Let us now consider three most frequently encountered definitions of the norm of vector

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

as well as the corresponding definitions of the norm of matrix.

Definition 3.

a) $\|x\|_1 = \max_i |x_i|$, $\quad \|A\|_1 = \max_i \sum_{k=1}^{n} |a_{ik}|$;

b) $\|x\|_2 = \sum_{k=1}^{n} |x_k|$, $\quad \|A\|_2 = \max_k \sum_{i=1}^{n} |a_{ik}|$;

c) $\|x\|_3 = \left(\sum_{k=1}^{n} |x_k|^2\right)^{1/2}$, $\quad \|A\|_3 = \sqrt{\lambda_1}$,

where λ_1 is the greatest eigenvalue of the matrix A^TA (A^T — matrix transposed to the matrix A).

We know that $\|A\|_3 \leqslant \left(\sum_{i,k=1}^{n} |a_{ik}|^2\right)^{1/2}$. Let us consider matrices of the form

(7^a) $$P(t) = \{p_1(t), p_2(t), \ldots, p_n(t)\}.$$

Then, from theorem 1, we get the following

Corollary 1. *If there exists a non-singular diagonal matrix of the form* (7^a), *continuous for* $t \geqslant 0$, *and such that at least one of the conditions given below is satisfied*

a) $\max\limits_{0\leqslant t\leqslant a} \max\limits_{i} \sum_{k=1}^{n} |a_{ik}(t)| \cdot \dfrac{|p_k(t)|}{|p_i(t)|} < 1$,

b) $\max\limits_{0\leqslant t\leqslant a} \max\limits_{k} \sum_{i=1}^{n} |a_{ik}(t)| \cdot \dfrac{|p_k(t)|}{|p_i(t)|} < 1$,

c) $\max\limits_{0\leqslant t\leqslant a} \Big(\sum_{i,k=1}^{n} |a_{ik}(t)|^2 \cdot \dfrac{|p_k(t)|^2}{|p_i(t)|^2}\Big)^{1/2} < 1$,

d) $\max\limits_{0\leqslant t\leqslant a} \sqrt{\lambda_1(t)} < 1$,

where $\lambda_1(t)$ *is the greatest eigenvalue of the matrix* $P(t)A^T(t)(P^{-1}(t))^2 \times \times A(t)P(t)$, *and* a *is an arbitrary real number, then series* (3) *is quasi-uniformly convergent for* $t \geqslant 0$.

Remark 1. Theorem 3 in [1] is equivalent to the mentioned corollary when condition a) is satisfied, and $P(t)$ is unit matrix.

Remark 2. If matrix $A(t)$ is symmetric, then the condition d) is equivalent to the condition

$$\max_{0\leqslant t\leqslant a} |\lambda_1(t)| < 1,$$

where $\lambda_1(t)$ is eigenvalue of the matrix $A(t)$ with the greatest absolute value.

To justify corollary 1 it suffices to see that conditions a) and b) follow immediately from theorem 1 and from definitions of the norm (Df. 3, a), b)); that condition c) results from theorem 1 and from the well-known estimation of the norm of matrix (see Df. 3, c)) by the left hand side of the condition c); finally, that condition d) results from theorem 1, from definition of the norm of matrix (Df. 3, c)), and from the equality

$$[P^{-1}(t)A(t)P(t)]^T \cdot [P^{-1}(t)A(t)P(t)] = P(t)A^T(t)(P^{-1}(t))^2 A(t)P(t).$$

Remark 2 results from the fact that for symmetric matrices the equality

$$[P^{-1}(t)A(t)P(t)]^T [P^{-1}(t)A(t)P(t)] = A^2(t)$$

holds.

Since for $t = 0$ integral equation (1) is reduced to the algebraic system of equations the following remark holds.

Remark 3. In order that series (3) be convergent it is necessary that eigenvalues of the matrix $A(0)$ fulfil inequality $|\lambda_k(0)| < 1$ (see [3], p. 208).

2. The preceding considerations enable us to formulate sufficient condition for the applicability of an iteration method of solving differential equations.

Let us consider equation

$$\sum_{i=0}^{n} a_i(t) x^{(n-i)}(t) = f(t), \tag{8}$$

where functions $a_i(t)$, $i = 0, 1, \ldots, n$, $f(t)$, are continuous for $t \geqslant 0$, and $a_0(t) \neq 0$.

Assuming arbitrary functions $\bar{a}_i(t)$, $i = 0, 1, \ldots, n$ (in particular they may be constant), $\bar{a}_0(t) \neq 0$, $\bar{a}_0(t) \not\equiv a_0(t)$, let $\bar{\bar{a}}_i(t) = \bar{a}_i(t) - a_i(t)$.

Let us construct a sequence of functions $x_m(t)$, $m = 0, 1, \ldots$, where $x_0(t)$ is the solution of the equation

$$\sum_{i=0}^{n} \bar{a}_i(t) x^{(n-i)}(t) = f(t) \tag{9}$$

fulfilling initial conditions $x_0^{(k)}(0) = c_k$, $k = 0, 1, \ldots, n-1$, and $x_{m+1}(t)$ is the solution of the equation

$$\sum_{i=0}^{n} \bar{a}_i(t) x^{(n-i)}(t) = \sum_{i=0}^{n} \bar{\bar{a}}_i(t) x_m^{(n-i)}(t), \tag{10}$$

satisfying initial conditions $x_{m+1}^{(k)}(0) = 0$, $k = 0, 1, \ldots, n-1$.

Put

$$x(t) = \sum_{m=0}^{\infty} x_m(t). \tag{11}$$

In the sequel we shall denote by L, $\bar{L}$ and $\bar{\bar{L}}$, respectively, differential operators occuring on the left hand side of equations (8), (9) and on the right hand side of equation (10).

Theorem 2. *In order that series* (11) *be quasi-uniformly convergent in the interval* $(0, +\infty)$ *to the solution of equation* (8) *fulfilling initial conditions* $x^{(k)}(0) = c_k$, $k = 0, 1, \ldots, n-1$, *it is sufficient that for each real number* a *the inequality*

$$\max_{0\leqslant t\leqslant a} \frac{|a_0(t) - \bar{a}_0(t)|}{|\bar{a}_0(t)|} < 1$$

holds.

Proof. Let us denote by $\bar{K}(t, \xi)$ the solution of the equation $\bar{L}(t)x(t) = 0$ depending on parameter ξ and satisfying initial conditions

$$\bar{K}_t^{(k)}(\xi, \xi) = \begin{cases} 0 & \text{for} \quad k = 0, 1, \ldots, n-2, \\ 1 & \text{for} \quad k = n-1. \end{cases}$$

We find the following relation between the function $x_{m+1}(t)$ and the function $x_m(t)$:

$$x_{m+1}(t) = \int_0^t \bar{K}(t,\xi)\frac{\bar{\bar{L}}(\xi)x_m(\xi)}{\bar{a}_0(\xi)}d\xi. \tag{12}$$

Differentiating the relation (12) n times we get the following system of relations

$$x_{m+1}^{(k)}(t) = \begin{cases} \displaystyle\int_0^t \bar{K}_t^{(k)}(t,\xi)\frac{\bar{\bar{L}}(\xi)x_m(\xi)}{\bar{a}_0(\xi)}d\xi, & k = 0,1,\ldots,n-1, \\ \displaystyle\frac{\bar{\bar{L}}(t)x_m(t)}{\bar{a}_0(t)} + \int_0^t \bar{K}_t^{(k)}(t,\xi)\frac{\bar{\bar{L}}(\xi)x_m(\xi)}{\bar{a}_0(\xi)}d\xi, & k = n. \end{cases} \tag{13}$$

Using notations

$$y_m(t) = \begin{bmatrix} x_m(t) \\ x_m'(t) \\ \cdots \\ x_m^{(n)}(t) \end{bmatrix}, \quad a_{ik}(t) = \begin{cases} 0 & \text{for } \begin{matrix} k = 0,1,\ldots,n \\ i = 0,1,\ldots,n-1 \end{matrix} \\ \dfrac{\bar{\bar{a}}_{n-k}(t)}{\bar{a}_0(t)} & \text{for } \begin{matrix} k = 0,1,\ldots,n \\ i = n \end{matrix} \end{cases} \tag{14}$$

$$b_{ik}(t,\xi) = \bar{K}_t^{(i)}(t,\xi)\frac{\bar{\bar{a}}_{n-k}(\xi)}{\bar{a}_0(\xi)}, \quad i,k = 0,1,\ldots,n,$$

we state that relations (13) are of form (2).

Thus in order to prove the quasi-uniform convergence of the series

$$\sum_{m=0}^{\infty} y_m(t) \quad \text{for} \quad t \geqslant 0$$

and thus the quasi-uniform convergence of series (11), it is sufficient to show that there exists a non-singular diagonal matrix $P(t)$ such that

$$\max_{0\leqslant t\leqslant a}\max_{k=0,1,\ldots,n}\left\{\left|\frac{\bar{\bar{a}}_{n-k}(t)}{\bar{a}_0(t)}\right|\cdot\left|\frac{p_k(t)}{p_n(t)}\right|\right\} < 1. \tag{15}$$

To prove this it is sufficient to take a matrix

$$P(t) = \{p_0(t), p_1(t), \ldots, p_n(t)\}$$

with $p_i(t) = 1$ for $i = 0,1,\ldots,n-1$, and

$$p_n(t) = \left\{\left[\max_{0\leqslant t\leqslant a}\max_{k=1,2,\ldots,n}\left|\frac{\bar{\bar{a}}_k(t)}{\bar{a}_0(t)}\right|\right] : \left[1-\max_{0\leqslant t\leqslant a}\left|\frac{\bar{\bar{a}}_0(t)}{\bar{a}_0(t)}\right|\right]\right\}+1.$$

Hence we get inequality

$$\max_{0\leqslant t\leqslant a}\max_{k=1,2,\ldots,n}\left|\frac{\bar{\bar{a}}_k(t)}{\bar{a}_0(t)}\right|\left|\frac{1}{|p_n(t)|}\right| + \max_{0\leqslant t\leqslant a}\left|\frac{\bar{\bar{a}}_0(t)}{\bar{a}_0(t)}\right| < 1,$$

and hence inequality

$$\max_{0\leqslant t\leqslant a}\max_{k=0,1,\ldots,n}\left\{\left|\frac{\bar{\bar{a}}_{n-k}(t)}{\bar{a}_0(t)}\right|\cdot\left|\frac{p_k(t)}{p_n(t)}\right|\right\} < 1,$$

which was to be shown.

According to corollary 1, series (11) is quasi-uniformly convergent. By the theorem on differentiation of series, we can easily state that series (11) is quasi-uniformly convergent, for $t \geqslant 0$, to the solution of equation (8), satisfying initial conditions $x^{(k)}(0) = c_k$, $k = 0,1,\ldots,n-1$. Theorem 2 is completely proved.

Theorem 2 is a generalization of theorems 1 and 2 in [1], this being possible owing to the presence of the matrix $P(t)$ in the formulation of theorem 1.

Remark 4. Without using matrix $P(t)$ in theorem 1 we could only deduce a theorem weaker than theorem 2, where sufficient condition (depending on the assumed definition of the norm) should have been replaced by one of the following stronger conditions:

a) $$\max_{0\leqslant t\leqslant a}\left\{\sum_{i=0}^{n}\left|\frac{\bar{\bar{a}}_i(t)}{\bar{a}_0(t)}\right|\right\} < 1,$$

b) $$\max_{0\leqslant t\leqslant a}\left\{\max_i\left|\frac{\bar{\bar{a}}_i(t)}{\bar{a}_0(t)}\right|\right\} < 1,$$

c) $$\max_{0\leqslant t\leqslant a}\left\{\sum_{i=0}^{n}\left|\frac{\bar{\bar{a}}_i(t)}{\bar{a}_0(t)}\right|^2\right\} < 1.$$

Let us consider equation

$$x'(t) = A(t)x(t)+B(t)x'(t)+f(t), \tag{16}$$

where matrices $A(t)$, $B(t)$ and vector function $f(t)$ are continuous for $t \geqslant 0$.

Assuming an arbitrary matrix $\bar{A}(t)$ (in particular the matrix $\bar{A}(t)$ may be constant), let

$$\bar{\bar{A}}(t) = \bar{A}(t)-A(t),$$

and let us construct a sequence $\{x_m(t)\}$, $m = 0,1,\ldots$, where $x_0(t)$ is the solution of the equation

$$x'(t) = \bar{A}(t)x(t)+f(t), \tag{17}$$

satisfying the initial condition $x_0(0) = c$, and $x_{m+1}(t)$, $m = 0, 1, \ldots$, is the solution of the equation

$$x'(t) = \bar{A}(t)x(t) + \bar{\bar{A}}(t)x_m(t) + B(t)x'_m(t), \tag{18}$$

satisfying the initial condition $x_{m+1}(0) = 0$.

Let

$$x(t) = \sum_{m=0}^{+\infty} x_m(t). \tag{19}$$

THEOREM 3. *If for any real number $a \geqslant 0$*

1° *matrix $B(t)$ has, in the interval $\langle 0, a\rangle$, a bounded derivative,*

2° *there exists a non-singular diagonal matrix $P(t)$, continuous for $t \geqslant 0$, and such that*

$$\max_{0 \leqslant t \leqslant a} \|P^{-1}(t)B(t)P(t)\| < 1,$$

then series (19) *is quasi-uniformly convergent, in the interval $\langle 0, +\infty)$, to the solution of equation* (16), *satisfying initial condition $x(0) = c$.*

Proof. We denote by $\bar{X}(t, \xi)$ a matrix being the solution of the differential equation

$$\bar{X}'(t) = \bar{A}(t)X(t),$$

depending on parameter ξ and satisfying the initial condition

$$\bar{X}(t, t) = I, \quad I \text{ — unit matrix.}$$

We find the following relation between the functions $x_m(t)$ and $x_{m+1}(t)$

$$x_{m+1}(t) = \int_0^t \bar{X}(t, \xi)\bar{\bar{A}}(\xi)x_m(\xi)\,d\xi + \int_0^t \bar{X}(t, \xi)B(\xi)x'_m(\xi)\,d\xi, \quad m = 0, 1, \ldots \tag{20}$$

Integrating by parts we obtain

$$\int_0^t \bar{X}(t, \xi)B(\xi)x'_m(\xi)\,d\xi = B(t)x_m(t) - \int_0^t \frac{\partial}{\partial\xi}[\bar{X}(t, \xi)B(\xi)]x_m(\xi)\,d\xi \quad \text{for} \quad m = 1, 2, \ldots$$

Relation (20) will take the form

$$x_{m+1}(t) = B(t)x_m(t) + \int_0^t D(t, \xi)x_m(\xi)\,d\xi, \quad m = 1, 2, \ldots, \tag{21}$$

$$\text{where} \quad D(t, \xi) = \bar{X}(t, \xi)\bar{\bar{A}}(\xi) - \frac{\partial}{\partial\xi}[\bar{X}(t, \xi)B(\xi)].$$

In view of theorem 1 we get the conclusion of the present theorem; thus the proof is complete.

Theorem 3 is a generalization of theorem 2 in [2].

Remark 5. It is evident that by theorem 3 a corollary analogous to corollary 1 can be obtained.

4. Finally let us consider the differential equation

$$L(t)i'(t) + R(t)i(t) + S(t)\int_0^t i(\tau)\,d\tau = f(t) \tag{22}$$

occurring in studies on parametric linear electric systems, where matrices $L(t)$, $R(t)$, $S(t)$ are continuous for $t \geqslant 0$. Assuming arbitrary matrices $\bar{L}(t)$, $\bar{R}(t)$, $\bar{S}(t)$ (in particular these matrices may be constant), $\det \bar{L}(t) \neq 0$, given $\bar{\bar{L}}(t) = \bar{L}(t) - L(t)$, $\bar{\bar{R}}(t) = \bar{R}(t) - R(t)$, $\bar{\bar{S}}(t) = \bar{S}(t) - S(t)$, let us construct a sequence $\{i_m(t)\}$, $m = 0, 1, \ldots$, where $i_0(t)$ is the solution of the equation

$$\bar{L}(t)i'(t) + \bar{R}(t)i(t) + \bar{S}(t)\int_0^t i(\tau)\,d\tau = f(t),$$

satisfying the initial condition $i(0) = c$; and $i_{m+1}(t)$ is the solution of the equation

$$\bar{S}(t)i'(t) + \bar{R}(t)i(t) + \bar{L}(t)\int_0^t i(\tau)\,d\tau = \bar{\bar{L}}(t)i'_m(t) + \bar{\bar{R}}(t)i_m(t) + \bar{\bar{S}}(t)\int_0^t i_m(\tau)\,d\tau,$$

satisfying the initial condition $i_{m+1}(0) = 0$.

Let

$$i(t) = \sum_{m=0}^{\infty} i_m(t). \tag{23}$$

In view of theorem 3 we can easily get the following theorem

THEOREM 4. *If for any real number $a \geqslant 0$*

1° *matrices $L(t)$, $\bar{L}(t)$ have bounded derivatives in the interval $\langle 0, a\rangle$,*

2° $\det \bar{L}(t) \neq 0$ *for $t \in \langle 0, a\rangle$,*

3° *there exists a non-singular diagonal matrix $P(t)$ continuous for $t \geqslant 0$, and such that*

$$\max_{0 \leqslant t \leqslant a} \|P^{-1}(t)\bar{L}^{-1}(t)[\bar{L}(t) - L(t)]P(t)\| < 1,$$

then series (23) *is quasi-uniformly convergent in the interval $\langle 0, +\infty)$, to the solution of equation* (22) *which satisfies initial condition $i(0) = c$.*

Theorem 4 is a generalization of theorem 1 in [2].

Remark 6. Corollary, analogous to Corollary 1, can be obtained.

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A REMARK ON A PROBLEM OF M. KRZYŻAŃSKI CONCERNING SECOND ORDER PARABOLIC EQUATIONS

BY

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Consider the linear second order equation of parabolic type

$$Fu \equiv \sum_{i,j=1}^{m} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j=1}^{m} b_j(x,t)\frac{\partial u}{\partial x_j} + c(x,t)u - \frac{\partial u}{\partial t} = 0, \tag{1}$$

where $x = (x_1, \ldots, x_m)$ varies in the whole m-dimensional Euclidean space E^m and $0 < t < T$. Denote by D^T the topological product of E^m with the interval $(0, T)$.

By a solution of (1) is meant a function $u(x,t)$, which is continuous in the closure $\overline{D^T}$ of D^T and which has continuous partial derivatives $\partial u/\partial x_i$, $\partial^2 u/\partial x_i \partial x_j$, $\partial u/\partial t$ in D^T satisfying (1).

Let $u(x,t)$ be a solution of (1) satisfying the initial condition

$$u(x,0) = \varphi(x) \quad \text{for} \quad x \in E^m, \tag{2}$$

$\varphi(x)$ being a given continuous function. Assume that there exist positive constants M, K such that the solution fulfils the inequality

$$u(x,t) \geqslant -M\exp(K|x|^2) \quad \text{for} \quad (x,t) \in D^T, \tag{3}$$

where $|x| = (\sum_{i=1}^{m} x_i^2)^{1/2}$. Krzyżański's problem is whether the condition (3) is sufficient for equation (1) to have at most one solution satisfying (2). It is known that, for instance, the condition

$$|u(x,t)| \leqslant M\exp(K|x|^2), \quad (x,t) \in D^T, \tag{4}$$

is sufficient if certain growth conditions concerning the coefficients are fulfilled (see [2]).

In the case when $\varphi(x) \equiv 0$ a positive answer can be obtained from the below mentioned theorems. Using the fundamental solution constructed by Dressel, Friedman [1] has proved the following