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PRODUCTS OF GENERALIZED ALGEBRAS AND PRODUCTS OF REALIZATIONS

BY

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This paper is a continuation of my paper [2]. The terminology and notation are the same as in [2] and the knowledge of this paper is supposed.

Łoś [1] has recently introduced a very useful notion of products of realizations and models modulo a prime filter. In metamathematical investigations concerning formalized languages of the first order all Boolean homomorphisms and prime filters have usually to satisfy some additional hypothesis, viz. they have to preserve infinite Boolean operations corresponding to logical quantifiers. Therefore it may seem strange that no such hypotheses appear in the construction of the product of realizations or models modulo a prime filter which can be completely arbitrary. This paper contains an explanation of this fact. In § 3,1° and 2° a general definition of product R of realizations of a formalized language of the first order is formulated. In § 5, 1°, 2° the original definition of the product R_0 of semantic realizations R_n modulo a prime filter (in the set N of all indexes n) is quoted. It is shown in § 5 (1) that R_0 is a homomorphic image of R

$$R_0 = hR,$$

in the sense defined in [2], § 8 (11). The homomorphism h in question has to preserve some infinite operations, in general. But in the case of semantic realizations those infinite operations reduce to trivial identities and therefore they are always preserved by h (see theorem 4.2). The notion of the product of generalized abstract algebras (i. e. algebras with infinite operations, which are not always feasible — see [2], § 4) introduced in § 2 is the basis for an examination of products of realizations. Two notions of products of generalized algebras suggest themselves in a natural way: the first is called *product*, the other is called *complete product*. The complete product seems, from a point a view, to be a more natural notion. For instance, the complete product of complete Boolean algebras is a complete algebra in the sense defined in [2], § 4. The product has not this property (see p. 10). However, from the point of view of applications to products of realizations the notion of product is more important than the notion of complete product.

§ 1. Product of abstract algebras. Let

(1) $$\{A_n, \{o_\varphi\}_{\varphi \in \Phi}\} \quad (n \in N)$$

be an indexed set of similar abstract algebras. The Cartesian product

(2) $$A = \mathop{\mathrm{P}}_{n \in N} A_n$$

will be considered as an abstract algebra

(3) $$\{A, \{o_\varphi\}_{\varphi \in \Phi}\}$$

similar to algebras (1), with the following definition of operations o_φ:

(4) $$o_\varphi(\{a_{1,n}\}_{n \in N}, \ldots, \{a_{m,n}\}_{n \in N}) = \{o_\varphi(a_{1,n}, \ldots, a_{m,n})\}_{n \in N},$$

where $\{a_{1,n}\}_{n \in N}, \ldots, \{a_{m,n}\}_{n \in N}$ are any points in A, and m is the number of arguments of o_φ. Roughly speaking, we perform the operation o_φ on elements in A by performing this operation on coordinates of these elements.

The algebra (3) just defined will be called the *product of algebras* (1).

1.1. *If A is the product of similar algebras A_n $(n \in N)$ and, for every $n \in N$, h_n is a homomorphism from a similar algebra A' into A_n, then the equation*

(5) $$h(a) = \{h_n(a)\}_{n \in N} \quad \textit{for} \quad a \in A'$$

defines a homomorphism from A' into A.

The proof is by an easy verification.

§ 2. Product of generalized abstract algebras. Let

(1) $$\{A_n, \{o_\varphi\}_{\varphi \in \Phi}, \{O_\psi\}_{\psi \in \Psi}\} \quad (n \in N)$$

be an indexed set of similar generalized algebras (see [2], p. 11). The Cartesian product

(2) $$A = \mathop{\mathrm{P}}_{n \in N} A_n$$

will be considered as a generalized algebra

(3) $$\{A, \{o_\varphi\}_{\varphi \in \Phi}, \{O_\psi\}_{\psi \in \Psi}\}$$

similar to algebras (1), with the following definition of operations.

The finite operations o_φ are defined as in § 1 (4). In other words, $\{A, \{o_\varphi\}_{\varphi \in \Phi}\}$ is the product of algebras $\{A_n, \{o_\varphi\}_{\varphi \in \Phi}\}$, where $n \in N$, in the sense defined in § 1.

If $\mathfrak{D}_n$ is the domain of the operation O_ψ in the generalized algebra A_n, then the class of all sets

(4) $$S = \mathop{\mathrm{P}}_{n \in N} S_n, \quad \text{where} \quad S_n \in \mathfrak{D}_n \text{ for } n \in N,$$

is the domain of the operation O_ψ in the generalized algebra A. If S is the set (4), then

(5) $$O_\psi S = \{O_\psi S_n\}_{n \in N}.$$

The generalized algebra (3) just defined will be called the *product of generalized algebras* (1). Note that, by definition, sets (4) only are admissible for the operation O_ψ in the product A.

Since admissible sets in generalized algebras are often given in the form of indexed sets, we shall repeat the definition (4)-(5) in terms of indexed sets.

Let O be one of the generalized operations O_ψ. Suppose that, for every $n \in N$, the indexed set $\{a_{t,n}\}_{t \in T_n}$ is admissible for O in the algebra A_n. Let

(6) $$T = \mathop{\mathrm{P}}_{n \in N} T_n$$

and, for every $t = \{t_n\}_{n \in N} \in T$, let

(7) $$a_t = \{a_{t_n, n}\}_{n \in N} \in A.$$

The indexed set

(8) $$\{a_t\}_{t \in T}$$

just defined is admissible for O in A, and

(9) $$O_{t \in T}\, a_t = \{O_{t_n \in T_n} a_{t_n, n}\}_{n \in N}.$$

2.1. *If A is the product of similar generalized algebras A_n $(n \in N)$ and, for every $n \in N$, h_n is a homomorphism from a similar generalized algebra A' into A_n, then the equation*

(10) $$h(a) = \{h_n(a)\}_{n \in N} \quad \textit{for} \quad a \in A'$$

defines a homomorphism from A' into A.

The proof is obtained by an easy verification.

Since only sets (4) (i. e. sets (8)) are admissible for infinite operations in the product A of the generalized algebras A_n, the algebra A is not complete, in general, even in the case where all factors A_n are complete algebras (in the sense defined in [2], § 4). However, it is possible to introduce another kind of product of generalized algebras, called complete product, which is a complete algebra if all factors are complete.

The *complete product of the generalized algebras* (1) is an algebra (3) with the same set (2) of elements and with the same definition § 1 (4) of finite operations but with a different definition of infinite operations. We give this definition in terms of indexed sets.

Let O be one of the operations O_ψ and let $\{a_t\}_{t\in T}$ be an indexed set of elements $a_t = \{a_{t,n}\}_{n\in N}$ in A. The indexed set $\{a_t\}_{t\in T}$ is said to be *admissible* for the operation O in the complete product, if and only if, for every $n\in N$, the indexed set $\{a_{t,n}\}_{t\in T}$ of elements of A_n is admissible for the operation O in the algebra A_n. Then, by definition,

$$O_{t\in T} a_t = \{O_{t\in T} a_{t,n}\}_{n\in N}. \tag{11}$$

The complete product of generalized algebras A_n is, of course, an extension (in the sense defined in [2], p. 13) of the product of the generalized algebras A_n.

§ 3. Products of realizations. Consider a fixed alphabet of the first order (see [2], p. 13)

$$\mathscr{A} = \{V, \{\Phi_m\}_{m\in M}, \{\Pi_m\}_{m\in M}, \{C_m\}_{m\in M}, Q, \overline{V}\},$$

M denoting the set of all non-negative integers. Similarly as in [2], p. 14, the letter C will denote the set of all connectives, i. e. $C = C_0 \cup C_1 \cup C_2 \cup \ldots$ Connectives will be denoted by o, and quantifiers (i. e. elements of Q) will be denoted by O.

Let

$$\mathscr{L} = \{\mathscr{A}, \mathscr{T}, \mathscr{F}\}$$

be a formalized language of the first order, based on the alphabet $\mathscr{A}$ (see [2], p. 15). We recall that $\mathscr{T}$ is the set of all terms, and $\mathscr{F}$ is the set of all formulas in $\mathscr{L}$.

Suppose that, for every $n\in N$ ($N \neq 0$), R_n is a realization of the language $\mathscr{L}$ in a set $X_n \neq 0$ and in a complete generalized algebra

$$\{A_n, \{o\}_{o\in C}, \{O\}_{O\in Q}\} \tag{1}$$

(see [2], p. 21). Let

$$\{A, \{o\}_{o\in C}, \{O\}_{O\in Q}\} \tag{2}$$

be the product of the generalized algebras (1) (see § 2, (1)-(9)). Let

$$X = \mathop{\mathrm{P}}_{n\in N} X_n \tag{3}$$

and let

$$x_1 = \{x_{1,n}\}_{n\in N}, \quad \ldots, \quad x_m = \{x_{m,n}\}_{n\in N} \tag{4}$$

be any points in X.

The realizations R_n determine uniquely a mapping R (defined on the union of the set $\Phi = \Phi_0 \cup \Phi_1 \cup \Phi_2 \cup \ldots$ of all functors and the set $\Pi = \Pi_0 \cup \Pi_1 \cup \Pi_2 \cup \ldots$ of all predicates) such that

1° R assigns to every m-argument functor $\boldsymbol{o} \in \Phi_m$ ($m = 0, 1, 2, \ldots$) the function $\boldsymbol{o}_R\colon X^m \to X$, defined by the equality

$$\boldsymbol{o}_R(x_1, \ldots, x_m) = \{\boldsymbol{o}_{R_n}(x_{1,n}, \ldots, x_{m,n})\}_{n\in N}; \tag{5}$$

2° R assigns, to every m-argument predicate $\pi \in \Pi_m$ ($m = 0, 1, 2, \ldots$) the mapping $\pi_R\colon X^m \to A$, defined by the equality

$$\pi_R(x_1, \ldots, x_m) = \{\pi_{R_n}(x_{1,n}, \ldots, x_{m,n})\}_{n\in N}. \tag{6}$$

The mapping R just defined is said to be the *product of the realizations* R_n ($n\in N$).

We recall that by a valuation in a set X_0 we understand any point $\mathfrak{v} = \{\mathfrak{v}_v\}_{v\in V}$ of the Cartesian product X_0^V, i. e. any mapping from the set V of all free individual variables into X_0.

In particular, valuations $\mathfrak{v} = \{\mathfrak{v}_v\}_{v\in V}$ in the set X defined by (3) are mappings which, to every free individual variable $v\in V$, assign a point $\mathfrak{v}_v = \{\mathfrak{v}_{v,n}\}_{n\in N} \in X$. By definition, $\mathfrak{v}_{v,n} \in X_n$. Thus, for every fixed $n\in N$, $\mathfrak{v}^n = \{\mathfrak{v}_{v,n}\}_{v\in V} \in X_n^V$ is a valuation in X_n. Conversely, if for every $n\in N$, $\mathfrak{v}^n = \{\mathfrak{v}_{v,n}\}_{v\in V}$ is any valuation in X_n, then setting $\mathfrak{v}_v = \{\mathfrak{v}_{v,n}\}_{n\in N}$ we define a valuation $\mathfrak{v} = \{\mathfrak{v}_v\}_{v\in V}$ in X.

Under the above notation, the following theorem holds:

3.1. *The product R of realization R_n of $\mathscr{L}$ in the sets X_n and the complete generalized algebras A_n is a realization of $\mathscr{L}$ in the product X of the sets X_n and in the product A of the generalized algebras A_n. Moreover, for every term τ*

$$\tau_R(\mathfrak{v}) = \{\tau_{R_n}(\mathfrak{v}^n)\}_{n\in N}, \tag{7}$$

and for every formula α

$$\alpha_R(\mathfrak{v}) = \{\alpha_{R_n}(\mathfrak{v}^n)\}_{n\in N}, \tag{8}$$

$\mathfrak{v}$ being any valuation in X.

τ_R, τ_{R_n} and α_R, α_{R_n} denote here mappings defined in [2], p. 22, p. 8-9, and p. 22-23.

According to the remark in [2], p. 26, it is convenient to replace, for a moment, the incomplete algebra A by its complete extension A' defined in [2], p. 13 (7), in order to avoid the difficulties caused by the incompleteness of A. In other words, R will be conceived as a realization of $\mathscr{L}$ in X and A'.

Let $\mathscr{L}_{X_n} = \{\mathscr{A}_{X_n}, \mathscr{T}_{X_n}, \mathscr{F}_{X_n}\}$ be the extended language obtained from the language $\mathscr{L} = \{\mathscr{A}, \mathscr{T}, \mathscr{F}\}$ by the method described in [2], § 7, p. 17-18. We recall that the alphabet $\mathscr{A}_{X_n}$ of $\mathscr{L}_{X_n}$ is obtained from the alphabet A of $\mathscr{L}$ by adding a set X_n' of individual constants (i. e.

zero-argument functors). There is a fixed one-to-one correspondence

$$x_n \leftrightarrow x'_n$$

between elements $x_n \in X_n$ and elements $x'_n \in X'_n$.

Consequently we have a one-to-one correspondence

$$x = \{x_n\}_{n \in N} \leftrightarrow x' = \{x'_n\}_{n \in N}$$

between elements $x \in X = \mathop{\mathrm{P}}_{n \in N} X_n$ and elements $x' \in X' = \mathop{\mathrm{P}}_{n \in N} X'_n$. Adding the set X' just defined to the set of all individual constants in $\mathscr{L}$, we have obtained the extended language $\mathscr{L}_X = \{\mathscr{A}_X, \mathscr{T}_X, \mathscr{F}_X\}$, according to [2], § 7, p. 17-18.

Let

$$f_n: X \to X_n$$

be the projection of X onto the n^{th} axis X_n, i. e.

$$f_n(\{x_n\}_{n \in N}) = x_n.$$

By [2], § 7 (9), (11) and 7.3, the mapping f_n induces a homomorphism

$$f'_n: \mathscr{T}_X \to \mathscr{T}_{X_n}$$

from the algebra

(9) $$\{\mathscr{T}_X, \{\boldsymbol{o}\}_{\boldsymbol{o} \in \Phi'}\}$$

of all terms in $\mathscr{L}_X$ (where $\Phi' = \Phi \cup X'$ is the set of all functors in $\mathscr{L}_X$) onto the algebra

(10) $$\{\mathscr{T}_{X_n}, \{\boldsymbol{o}\}_{\boldsymbol{o} \in \Phi'_n}\}$$

of all terms in $\mathscr{L}_{X_n}$ (where $\Phi'_n = \Phi \cup X'_n$ is the set of all functors in $\mathscr{L}_{X_n}$), such that

(11) $$f'\tau = \tau \quad \text{for all terms} \quad \tau \text{ in } \mathscr{L}.$$

By [2], § 7, (10)-(12), (14), and 7.4, the mapping f_n induces a mapping

$$f_n^*: \mathscr{F}_X \to \mathscr{F}_{X_n}$$

such that the equation

$$f^{**}(|\alpha|) = |f^*\alpha| \quad \text{for} \quad \alpha \in \mathscr{F}_X$$

defines a homomorphism

$$f^{**}: F_X \to F_{X_n}$$

of the X-algebra

(12) $$\{F_X, \{o\}_{o \in C}, \{O\}_{O \in Q}\}$$

of the language $\mathscr{L}$ (see [2], p. 18-19) onto the X_n-algebra

(13) $$\{F_{X_n}, \{o\}_{o \in C}, \{O\}_{O \in Q}\}$$

of the language $\mathscr{L}$. Moreover

(14) $$f^*\alpha = \alpha \quad \text{and} \quad f^{**}(|\alpha|) = |\alpha| \text{ for all formulas } \alpha \text{ in } \mathscr{L}.$$

According to [2], § 8, p. 22-23, we extend the realizations R and R_n of $\mathscr{L}$ to realizations R' and R'_n (in X and A', or in X_n and A_n respectively) of $\mathscr{L}_X$ or $\mathscr{L}_{X_n}$ by assuming

$$x'_{R'} = x \quad \text{for all} \quad x' \in X',$$
$$\boldsymbol{o}_{R'} = \boldsymbol{o}_R \quad \text{for all} \quad \boldsymbol{o} \in \Phi,$$
$$\pi_{R'} = \pi_R \quad \text{for all} \quad \pi \in \Pi,$$

and similarly

$$x'_{nR'_n} = x_n \quad \text{for all} \quad x'_n \in X'_n,$$
$$\boldsymbol{o}_{R'_n} = \boldsymbol{o}_{R_n} \quad \text{for all} \quad \boldsymbol{o} \in \Phi,$$
$$\pi_{R'_n} = \pi_{R_n} \quad \text{for all} \quad \pi \in \Pi.$$

The realization R' is the product of the realization R'_n.

P r o o f of 3.1. First we shall prove the following generalizations of (7) and (8):

(15) $$\tau_{R'}(\mathfrak{v}) = \{f'\tau_{R_n}(\mathfrak{v}^n)\}_{n \in N} \quad \text{for every term } \tau \text{ in } \mathscr{L}_X,$$

(16) $$\alpha_{R'}(\mathfrak{v}) = \{f^*\alpha_{R_n}(\mathfrak{v}^n)\}_{n \in N} \quad \text{for every formula } \alpha \text{ in } \mathscr{L}_X.$$

Consider the sets X and X_n ($n \in N$) as abstract algebras

(17) $$\{X, \{\boldsymbol{o}_{R'}\}_{\boldsymbol{o} \in \Phi}\}$$

and

(18) $$\{X_n, \{\boldsymbol{o}_{R'_n}\}_{\boldsymbol{o} \in \Phi}\}.$$

respectively (see [2], p. 21). It follows from 1° that the algebra (17) is the product of the algebras (18), $n \in N$, in the sense defined in § 1. For every fixed $\mathfrak{v}$, the left side of (16) considered as a function of τ is a homomorphism from the algebra (9) of terms (see [2], p. 15 and p. 9) into the algebra (17). Similarly, $f'\tau_{R'_n}(\mathfrak{v}^n)$ interpreted as a function of τ is a homomorphism from the algebra (9) into the algebra (18). By 1.1 the right side of (15) is a homomorphism from the algebra (9) of terms into the algebra (17). The both homomorphisms (on the left and right side of (15)) coincide on the set V which generates the algebra (9). Thus they are equal, i. e. (15) holds for all terms τ.

For every fixed $\mathfrak{v}$ the left side of (16), considered as a function of $|\alpha| \in F_X$ is a homomorphism from the algebra (12) into A' (see [2], p. 23). Similarly, $f^*\alpha_{R'_n}(\mathfrak{v}^n)$ considered as a function of $|\alpha| \in F_X$ is a homomorphism from the algebra (12) into A_n. Hence it follows that the right side of (16), considered as a function of $|\alpha| \in F_X$, is a homomorphism from the algebra (12) into A', by 2.1 and in virtue of the fact that the identity mapping from the generalized algebra A into A' is a homomorphism. It follows from 2° and (15) that the homomorphisms on the left and right side of (16) assume the same values if α is an elementary formula. Since elementary formulas generate F_X, the both homomorphisms are equal, i. e. (16) holds for all formulas α in $\mathscr{L}_X$.

If τ is a term in $\mathscr{L}$, then (15) coincides with (7) on account of (11). If α is a formula in $\mathscr{L}$, then (16) coincides with (8), by (14). This proves (7) and (8).

Since the right side of (8) always belongs to the product A, so does the left side. This proves that R is a realization of R in X and A (since A is incomplete, in general, we understand the word "realization in A" in the extended sense defined in [2], p. 26).

Let now

$$\{B, \{o\}_{o \in C}, \{O\}_{O \in Q}\} \tag{19}$$

be another complete generalized algebra similar to the algebras (1), and let h be a homomorphism from the product A (see (2)) of all A_n into the algebra (19). Let R denote, as previously, the product of realizations R_n of $\mathscr{L}$ in sets X_n and algebras A_n respectively. By 3.1 R is a realization of $\mathscr{L}$ in X and A.

According to [2], p. 24, the symbol hR will denote the following realization in J and B:

$$\boldsymbol{o}_{hR} = \boldsymbol{o}_R \quad \text{for every functor } \boldsymbol{o} \in \Phi,$$

$$\pi_{hR} = h\pi_R \quad \text{for every predicate } \pi \in \Pi.$$

Under the above hypotheses, the following theorem holds:

3.2. *For every formula α in $\mathscr{L}$, and every valuation $\mathfrak{v}$ in X,*

$$\alpha_{hR}(\mathfrak{v}) = h(\alpha_R(\mathfrak{v})).$$

This is an immediate consequence of [2], 8.1, p. 24. The only difference is that [2], 8.1 was proved under the hypothesis that the domain A of h is complete. However in [2], p. 26, we have observed that all theorems proved in [2], § 8, in particular theorem 8.1, remain true for realizations in incomplete algebras provided that all infinite operations appearing in the inductive definition of $\alpha_R(\mathfrak{v})$ are feasible. On account of 3.1, this hypothesis is satisfied in the case examined in 3.2.

All remarks and theorems proved in § 3 remain true if the product of all algebras A_n is replaced ewerywhere by the complete product of all A_n. The proof of 3.1 and 3.2 is then simpler because the complete product of complete algebras is a complete algebra (it is not necessary to

introduce the extension A'). However, the case of the product of all generalized algebras A_n is more important in view of its applications than the case of the complete product of all A_n (see 4.2 and 5.1).

§ 4. Products of Boolean algebras. If an abstract algebra A or a generalized abstract algebra A has only a finite number of operations, then instead of

$$\{A, \{o_\varphi\}_{\varphi \in \Phi}\} \quad \text{or} \quad \{A, \{o_\varphi\}_{\varphi \in \Phi}, \{O_\psi\}_{\psi \in \Psi}\}$$

we shall write

$$\{A, o_1, \ldots, o_r\} \quad \text{or} \quad \{A, o_1, \ldots, o_r, O_1, \ldots, O_s\}$$

respectively. The last notation, in particular will be applied in the case of Boolean algebras to be examined below.

From the point of view of applications to Mathematical Logic it is convenient to conceive any Boolean algebra A as an abstract algebra

$$\{A, \cup, \cap, \Rightarrow, -\}, \tag{1}$$

where $\cup$ is the join, $\cap$ is the meet, $-$ is the complementation, and $\Rightarrow$ is defined as follows:

$$a \Rightarrow b = (-a) \cup b \quad \text{for} \quad a, b \in A.$$

The operations $\cup$, $\cap$, $\Rightarrow$ are binary, the operation $-$ is unary. Let

$$\{A_n, \cup, \cap, \Rightarrow, -\} \quad (n \in N) \tag{2}$$

be an indexed set of Boolean algebras. By the general definition from § 1, the product of all Boolean algebras (2) is an abstract algebra (1), where $A = \mathop{\mathrm{P}}_{n \in N} A_n$ and the operations in (1) are defined as follows:

$$\begin{gathered}\{a_n\}_{n \in N} \cup \{b_n\}_{n \in N} = \{a_n \cup b_n\}_{n \in N}, \qquad \{a_n\}_{n \in N} \cap \{b_n\}_{n \in N} = \{a_n \cap b_n\}_{n \in N}, \\ \{a_n\}_{n \in N} \Rightarrow \{b_n\}_{n \in N} = \{a_n \Rightarrow b_n\}_{n \in N}, \qquad -\{a_n\}_{n \in N} = \{-a_n\}_{n \in N},\end{gathered} \tag{3}$$

where a_n, b_n are any elements in A_n. It is easy to verify that product (1) of Boolean algebras (2) is also a Boolean algebra.

Often it is necessary to consider Boolean algebras as generalized algebras

$$\{A, \cup, \cap, \Rightarrow, -, \bigcup, \bigcap\} \tag{4}$$

with two infinite operations: the infinite join $\bigcup$ and the infinite meet $\bigcap$. If we write (4), we assume by definition that the domain of the operation $\bigcup$ is the class of all non-empty sets $S \subset A$ such that the infinite join $\bigcup S$ exists in A, and the domain of the operation $\bigcap$ is the class of all non-empty sets $S \subset A$ such that the infinite meet $\bigcap S$ exists in A. Thus

(4) is a complete algebra in the sense defined in [2], § 4 if and only if the Boolean algebra A is complete in the ordinary sense from Lattice Theory.

Sometimes it is useful to restrict the domains of $\bigcup$ and $\bigcap$ to smaller classes $\mathfrak{D}_1$ and $\mathfrak{D}_2$ of sets $S \subset A$, respectively. Then we write

$$\{A, \cup, \cap, \Rightarrow, -, \bigcup|\mathfrak{D}_1, \bigcap|\mathfrak{D}_2\} \tag{5}$$

instead of (3).

Let

$$\{A_n, \cup, \cap, \Rightarrow, -, \bigcup|\mathfrak{D}_{1,n}, \bigcap|\mathfrak{D}_{2,n}\} \quad (n \in N) \tag{6}$$

be an indexed set of Boolean algebras conceived as generalized algebras. It is not difficult to verify that

4.1. *The product of all the Boolean algebras* (6) *is the Boolean algebra* (5), *where* $A = \mathop{\mathrm{P}}_{n \in N} A_n$, $\mathfrak{D}_1$ *is the class of all sets* $S = \mathop{\mathrm{P}}_{n \in N} S_n$, *where* $S_n \in \mathfrak{D}_{1,n}$, *and* $\mathfrak{D}_2$ *is the class of all the sets* $S = \mathop{\mathrm{P}}_{n \in N} S_n$, *where* $S_n \in \mathfrak{D}_{2,n}$. *Moreover, if* $S = \mathop{\mathrm{P}}_{n \in N} S_n$, *then*

$$\bigcup S = \{\bigcup S_n\}_{n \in N}, \qquad \bigcap S = \{\bigcap S_n\}_{n \in N}. \tag{7}$$

Observe that the Boolean algebra

$$\{\mathop{\mathrm{P}}_{n \in N} A_n, \cup, \cap, \Rightarrow, -, \bigcup, \bigcap\} \tag{8}$$

is not the product of Boolean algebras

$$\{A_n, \cup, \cap, \Rightarrow, -, \bigcup, \bigcap\} \quad (n \in N). \tag{9}$$

The algebra (8) is the complete product of all algebras (9), in the sense defined in § 2. If all Boolean algebras (9) are complete, then their complete product (8) is also complete, but their product is, in general, not any complete algebra. In fact, the product of all the complete Boolean A_n is the generalized algebra (with operations defined by (3) and (7))

$$\{\mathop{\mathrm{P}}_{n \in N} A_n, \cup, \cap, -, \bigcup|\mathfrak{D}, \bigcap|\mathfrak{D}\}, \tag{10}$$

where the common domain $\mathfrak{D}$ for the both infinite operations is the class of all sets $S = \mathop{\mathrm{P}}_{n \in N} S_n$, where $0 \neq S_n \subset A_n$ for every $n \in N$. The Boolean algebra $A = \mathop{\mathrm{P}}_{n \in N} A_n$ is then a complete Boolean algebra in the sense of the Lattice Theory, but (10) is not any complete generalized algebra in the sense of § 2 since the domains of infinite operations are artificially restricted to a proper sub-class of the class of all non-open subsets of A.

4.2. *If* A_n *is a two-element Boolean algebra for every* $n \in N$, $A = \mathop{\mathrm{P}}_{n \in N} A_n$, *and* B *is a complete Boolean algebra, then every homomorphism* h *from the product* (1) *of all the algebras* (2) *into the algebra* $\{B, \cup, \cap, \Rightarrow, -\}$ *is a homomorphism from the product* (10) *of all the generalized algebras* (9) *into* $\{B, \cup, \cap, \Rightarrow, -, \bigcup, \bigcap\}$.

In other words, *if a mapping* $h: A \to B$ *preserves all finite Boolean operations, then it preserves also infinite joins and meets* $\bigcup S$ *and* $\bigcap S$ *where* $S \in \mathfrak{D}$, *i. e.*

$$h(\bigcup S) = \bigcup h(S) \quad \text{and} \quad h(\bigcap S) = \bigcap h(S), \tag{11}$$

provided

$$S = \mathop{\mathrm{P}}_{n \in N} S_n, \quad 0 \neq S_n \subset A_n \quad \text{for every } n \in N.$$

Let $\bigcup S = a = \{a_n\}_{n \in N}$. By (7), $\bigcup S_n = a_n$. Since A_n has only two elements, we have $a_n \in S_n$. Thus $a = \{a_n\}_{n \in N} \in S$. Consequently $h(a) \leqslant \bigcup h(S)$. The converse inequality is true for every mapping h preserving $\cup$ and $\cap$. This proves that $h(a) = \bigcup h(S)$. Similarly we prove the second of the equalities (11).

Let $\mathfrak{B}(N)$ denote the Boolean algebra of all subsets of the set N, the Boolean operations being set-theoretical.

4.3. *Suppose that all* A_n *are two-element Boolean algebras. For every element* $a = \{a_n\}_{n \in N}$ *in the product* $A = \mathop{\mathrm{P}}_{n \in N} A_n$ *let* $h_0(a)$ *be the set of all* $n \in N$ *such that* a_n *is the unit element of* A_n. *The mapping* h_0 *is an isomorphism from the Boolean algebra* $\{A, \cup, \cap, \Rightarrow, -\}$ *onto the Boolean algebra* $\{\mathfrak{B}(N), \cup, \cap, \Rightarrow, -\}$.

The proof is by an easy verification.

§ 5. Product of semantic realizations modulo a prime filter.

In this section $\mathscr{L}$ is a language (of the first order) described in [2], § 5, Example 4. We recall that $\mathscr{L}$ is a language such that: the sets V and $\overline{V}$ of free and bound individual variables are infinite, there is only one unary connective $\mathbf{N}$ in C_1 and three connectives $\mathbf{D}$, $\mathbf{C}$, $\mathbf{I}$ in C_2, all the sets C_0, C_3, C_4, ... being empty. The set Q of quantifiers contains only two signs $\mathbf{E}$ and $\mathbf{U}$. Thus Boolean algebras $\{A, \cup, \cap, \Rightarrow, -, \bigcup, \bigcap\}$ are similar to the Q-algebra $\{F, \mathbf{D}, \mathbf{C}, \mathbf{I}, \mathbf{N}, \mathbf{E}, \mathbf{U}\}$ of $\mathscr{L}$, and we can examine realizations of $\mathscr{L}$ in any set $X \neq 0$ and any Boolean algebra A.

By a *semantic realization* of $\mathscr{L}$ we shall understand any realization of $\mathscr{L}$ in a set $X \neq 0$ and in a two-element Boolean algebra.

The zero element and the unit element of a two-element Boolean algebra will be always denoted by $\wedge$ and $\vee$ respectively.

For every $n \in N$, let R_n be a semantic realization of $\mathscr{L}$ in a set $X_n \neq 0$ and a two-element Boolean algebra A_n. Let ∇ be a prime filter in the

Boolean algebra $\mathfrak{B}(N)$. By the *product of realizations* R_n $(n \in N)$ *modulo the prime filter* ∇ we shall understand the semantic realization R_0 of $\mathscr{L}$ in the set $X = \mathop{\mathrm{P}}_{n \in N} X_n$ and a two-element Boolean algebra A_0 defined as follows: for any points $x_1 = \{x_{1,n}\}_{n \in N}, \ldots, x_m = \{x_{m,n}\}_{n \in N}$ in X,

1° if $\boldsymbol{o}$ is an m-argument functor $(m = 0, 1, 2, \ldots)$, then

$$\boldsymbol{o}_{R_0}(x_1, \ldots, x_m) = \{\boldsymbol{o}_{R_n}(x_{1,n}, \ldots, x_{m,n})\}_{n \in N};$$

2° if π is an m-argument predicate $(m = 0, 1, 2, \ldots)$, then

$$\pi_{R_0}(x_1, \ldots, x_m) = \begin{cases} \bigvee & \text{if} \quad N_{x_1, \ldots, x_m} \in \nabla, \\ \bigwedge & \text{if} \quad N_{x_1, \ldots, x_m} \notin \nabla, \end{cases}$$

where $N_{x_1, \ldots, x_m}$ is the set of all $n \in N$ such that

$$\pi_{R_n}(x_{1,n}, \ldots, x_{m,n}) = \bigvee .$$

Since all two-element Boolean algebras are isomorphic, we may assume that $A_0 = \mathfrak{B}(N)/\nabla$. Denote by h_1 the natural homomorphism from $\{\mathfrak{B}(N), \cup, \cap, \Rightarrow, -\}$ onto $\{A_0, \cup, \cap, \Rightarrow, -\}$.

Let R be the product of the realizations R_n. It follows directly from 1°, 2°, and from § 3 1°, 2°, that

$$R_0 = hR, \tag{1}$$

where h is the superposition $h = h_1 h_0$, h_0 being the isomorphism defined in 4.2. By definition, h is a homomorphism from the product $\{A, \cup, \cap, \Rightarrow, -\}$ of the two-element Boolean algebras § 4 (2) into the Boolean algebra $\{A_0, \cup, \cap, \Rightarrow, -\}$. By 4.2, h is a homomorphism from the product § 4 (10) of generalized algebras § 4 (9) into $\{A_0, \cup, \cap, \Rightarrow, -, \bigcup, \bigcap\}$. Thus we can apply theorem 3.2. Thus, in view of 3.1, we have using the notation from § 3, p. 5,

$$a_{R_0}(\mathfrak{v}) = h\big(\{a_{R_n}(\mathfrak{v}^n)\}_{n \in N}\big). \tag{2}$$

Hence it follows that

5.1. *For every formula a in $\mathscr{L}$ and for every valuation $\mathfrak{v}$ in $X = \mathop{\mathrm{P}}_{n \in N} X_n$,*

$$a_{R_0}(\mathfrak{v}) = \bigvee \quad \textit{if and only if} \quad N(a, \mathfrak{v}) \in \nabla,$$

where $N(a, \mathfrak{v})$ is the set of all $n \in N$ such that $a_{R_n}(\mathfrak{v}^n) = \bigvee$.

A semantic realization R' is said to be a *model* for a formula a provided $a_{R'}(\mathfrak{v}) = \bigvee$ for every valuation $\mathfrak{v}$.

The following theorem follows immediately from 5.1:

5.2. *The product R_0 of semantic realizations R_n $(n \in N)$ modulo a prime filter ∇ is a model for a formula a if and only if for every valuation $\mathfrak{v}$ the set of all $n \in N$ such that $a_{R_n}(\mathfrak{v}^n) = \bigvee$ belongs to ∇.*

In particular, if the set of all $n \in N$ such that R_n is a model for a belongs to ∇, then R_0 is also a model for a.

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