But $p > 2$, and consequently $q < 2 < p$. Hence $L^q(G) \subset L^p(G)$, and consequently $L^p(G) = L^2(G) = L^q(G)$. Therefore $L^2(G)$ is an algebra under convolution.

Hence G is finite.

Note. The author has proved after submitting this paper that for any locally compact group G the space $L^p(G)$ is closed for convolution for some $p > 2$ if G is compact.

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A NOTE ON L_p-ALGEBRAS

BY

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In [3] it was shown that if G is a locally compact Abelian group, then $L_p(G)$ for $p > 1$ is a Banach algebra under convolution if and only if G is compact. Further, Rajagopalan [2] extended this result to the case when G is discrete but not Abelian and $p \geqslant 2$. In this paper we prove this result for an arbitrary locally compact group under the assumption that $p > 2$.

Let G be a locally compact group. Its elements will be denoted by t, τ; group operation will be written multiplicatively. Unit element will be de noted by e. If A, B are subsets of G, then AB is a set of all elements of G written in the form $t\cdot\tau$, where $t \in A$, $\tau \in B$, and A^{-1} is defined as the set of all t^{-1}, such that $t \in A$. U, V will stand for compact neighbourhoods of the unit e. It is known that for every neighbourhood U, there exists a symmetric neighbourhood $V \subset U$ (i. e. such that $V = V^{-1}$) for which $V^2 \subset U$. μ will denote the left invariant Haar measure on G. We recall that if A is open and B compact, then $\mu(A) > 0$, and $\mu(B) < \infty$. Generally speaking the left invariant measure is not the right invariant one, but there exists such a continuous function $\Delta(t)$, called *modular function*, that $\mu(At) = \mu(A)\Delta(t)$ for every measurable A, and $t \in G$. We have $\Delta(t) > 0$ for every $t \in G$, $\Delta(e) = 1$, and

$$\Delta(t\tau) = \Delta(t)\Delta(\tau). \tag{1}$$

In the case when $\Delta(t) \equiv 1$ the group G is called *unimodular*. In this case we have

$$\int f(t\tau)\mu(d\tau) = \int f(\tau t)\mu(d\tau) = \int f(\tau^{-1})\mu(d\tau) = \int f(\tau)\mu(d\tau) \tag{2}$$

for every integrable function f defined on G and $t \in G$. $L_p(G)$ will denote the space of all complex functions (or more exactly of equivalence classes) such that

$$\|x\|_p = \left(\int |x(t)|^p \mu(dt)\right)^{1/p} < \infty.$$

The convolution is defined as

$$x * y(t) = \int x(t\tau^{-1}) y(\tau) \mu(d\tau).$$

The following lemma reduces our problem to the case when G is a unimodular group.

LEMMA 1. *If G is not unimodular locally compact group, then $L_p(G)$ is not an algebra under the convolution for any $p > 1$.*

Proof. By our assumption there exists in G such a t_0 that $\Delta(t_0) \neq 1$. Let $G(t_0)$ be a subgroup of G generated by t_0. It is the intersection of all closed subgroups of G containing t_0. We have either $\Delta(t_0) > 1$, or $\Delta(t_0^{-1}) > 1$, so $G(t_0)$ is not compact, since the continuous modular function is unbounded on the sequence (t_0^n), $n = 0, \pm 1, \pm 2, \ldots$, contained in $G(t_0)$. Consequently $G(t_0)$ is discrete and consists exactly of all positive and negative powers of t_0 (cf. [4], lemma 3, or [1]). But in this case the proof of our conclusion given in [3] holds; so does the proof of the theorem 2, section 2°, pp. 117 and 118, q. e. d.

LEMMA 2. *If the group G is not compact then for every compact subset $A \subset G$ there exists a sequence of elements $t_n \in G$, $n = 1, 2, \ldots$, such that*

$$t_n A \cap t_k A = \emptyset \quad \text{for} \quad n \neq k. \tag{3}$$

Proof. Take an arbitrary element as t_1, choose t_2 in such a way that $t_1 A \cap t_2 A = \emptyset$, t_3 in such a way that $t_3 A \cap t_2 A = \emptyset$ and $t_3 A \cap t_1 A = \emptyset$, and so on. If the n-th step is impossible, then for any $t \in G$ there exists a t_k, $k < n$, such that $t_k A \cap tA \neq \emptyset$, which is equivalent to $t \in t_k A A^{-1}$. But in this case G would be covered by the finite family of compact sets, which is impossible, q. e. d.

LEMMA 3. *If $L_p(G)$ is a Banach algebra under the convolution, and if $p > 2$, then G is compact.*

Proof. By lemma 1 we may assume that G is a unimodular group. It is to be proved that $\mu(G) < \infty$ which is equivalent to its compactness. Suppose then that $\mu(G) = \infty$ and the proof will be given if we get a contradiction. Let $x(t) \in L_p(G)$; so, by (2), $\tilde{x}(t) = x(t^{-1}) \in L_p(G)$. Let $z(t) \in L_q(G)$, where $1/p + 1/q = 1$, and consider the functional $(z, \tilde{x} * y)$ generated on $L_p(G)$ by the function z, taken at the point $\tilde{x} * y$, $y \in L_p(G)$. For fixed $z \in L_q$, and $x \in L_p$ it is a continuous linear functional defined on L_p. So there exists a $w \in L_q$ such that $(z, \tilde{x} * y) = (w, y)$ for every $y \in L_p$. In the same way as in [2] we shall show that $w = x * z$. In fact,

$$(z, \tilde{x} * y) = \int z(t) \int \tilde{x}(t\tau^{-1}) y(\tau) \mu(d\tau) \mu(dt)$$
$$= \int y(\tau) \; x(\tau t^{-1}) z(t) \mu(dt) \mu(d\tau) = (x * z, y).$$

So for every $x \in L_p(G)$, $z \in L_q(G)$ we have $x * z \in L_q(G)$. To prove our lemma it is sufficient to construct such an $x \in L_p$, and $z \in L_q$, that $x * z$ is not in L_q. Let U be a compact symmetric neighbourhood of the unit $e \in G$. U^2 is also compact, so by our assumption and by lemma 2 we can choose such sequence $\{t_n\}$ of elements of G that (3) holds with U^2 instead of A. It is clear that for this sequence (3) also holds if we take U instead of A. We put now

$$x(t) = \sum_{n=1}^{\infty} a_n \chi_{t_n U^2}(t),$$

$$y(t) = \sum_n a_n \chi_{t_n U}(t),$$

$$z(t) = \chi_U(t),$$

where a_n is a sequence of positive reals such that

$$\sum_{n=1}^{\infty} a_n^p = a^p < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} a_n^q = \infty.$$

This is possible because $q < p$ for $p > 2$; $\chi_A(t)$ denotes the characteristic function of the set A, i. e.

$$\chi_A(t) = \begin{cases} 1 & \text{if} \quad t \in A, \\ 0 & \text{if} \quad t \notin A. \end{cases}$$

It is clear that $x, y \in L_p$, $z \in L_q$, and x and y are not members of L_q. We shall see that $x * z \notin L_q$. In fact, consider the convolution

$$\chi_{t_n U^2} * \chi_U(t) = \int_U \chi_{t_n u^2}(t\tau^{-1}) d\tau.$$

Let $t \in t_n U$. If $\tau \in U$, then $t\tau^{-1} \in t_n U^2$ and

$$\chi_{t_n U^2} * \chi_U(t) = \mu(U^2) \quad \text{if} \quad t \in t_n U.$$

Consequently

$$\chi_{t_n U^2} * \chi_U(t) \geqslant \mu(U^2) \chi_{t_n U}(t)$$

for every $t \in G$, and so $x * z(t) \geqslant \mu(U^2) y(t) \geqslant 0$. But $\|y\|_q = \infty$, and $\mu(U^2) > 0$, so $\|x * z\|_q = \infty$ and $x * z$ is not in L_q, q. e. d.

LEMMA 4. *Let f be a positive function defined on the group G such that $f * x \in L_r(G)$ for every $x \in L_p(G)$, $r, p \geqslant 1$; then the mapping $x \to f * x$ is a continuous linear mapping of L_p into L_r.*

Proof. Suppose that the mapping $x \to f * x$ is not continuous. Then there exists a sequence $\{x_n\}$ of elements of L_p, and a positive constant C

such that $\lim\|x_n\|_p = 0$, and $\|f*x_n\|_r \geqslant C$, $n = 1, 2, \ldots$ Taking $|x_n(t)|$ instead of $x_n(t)$ we have also $\lim\|x_n(t)|\|_p = 0$ and $\|f*|x_n|\|_r \geqslant C$, so we may assume that $x_n(t) \geqslant 0$. Taking a suitable sequence of positive scalars a_n we obtain $\lim\|a_n x_n\|_p = 0$, and $\lim\|f*a_n x_n\|_r = \infty$, so by passing, if necessary, to a subsequence we may assume that

$$\|x_n\|_p \leqslant 1/2^n \quad \text{and} \quad \|f*x_n\|_r \geqslant n$$

for $n = 1, 2, \ldots$ Now let $y = \sum_{n=1}^{\infty} x_n$; we have $y \in L_p$, so $\|f*y\|_r < \infty$. On the other hand, $y \geqslant x_n$, and so $f*y \geqslant f*x_n \geqslant 0$. Consequently $\|f*y\|_r \geqslant \|f*x_n\|_r \geqslant n$ which is the contradiction mentioned above, q. e. d.

COROLLARY. *If $L_p(G)$, $p \geqslant 1$, is an algebra under the convolution, then it is a Banach algebra* (*i. e. there exists a submultiplicative norm equivalent to the norm $\|x\|_p$*).

We may formulate now our main result

THEOREM 1. *Let G be a locally compact group and $p > 2$; then the space $L_p(G)$ is an algebra under the convolution if and only if the group G is compact.*

We may rewrite also the main result of [3] in the following form:

THEOREM 2. *Let G be a locally compact Abelian group and $p > 1$; then the space $L_p(G)$ is an algebra under the convolution if and only if the group G is compact.*

The following problem is open:

P 392. Is the conclusion of theorem 1 true for $1 < p \leqslant 2$?

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ON DECOMPOSITION OF A COMMUTATIVE p-NORMED ALGEBRA INTO A DIRECT SUM OF IDEALS

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1. In the theory of commutative complex Banach algebras it is known that a Banach algebra A is decomposable into a direct sum of its two non-trivial ideals

$$A = I_1 \oplus I_2, \tag{1.1}$$

if and only if the compact space $\mathfrak{M}$ of all multiplicative linear functionals of A may be written in the form

$$\mathfrak{M} = \mathfrak{M}_1 \cup \mathfrak{M}_2, \tag{1.2}$$

where $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are disjoint closed subsets of $\mathfrak{M}$.

The decompositions (1.1) and (1.2) are equivalent to the decomposition of the unit $e \in A$ into a sum of two non-zero idempotents

$$e = e_1 + e_2, \tag{1.3}$$

where

$$e_1^2 = e_1, \quad e_2^2 = e_2, \quad e_1 e_2 = 0. \tag{1.4}$$

When we have the decomposition (1.3) with (1.4) the decompositions (1.1) and (1.2) may be written by means of the formulas

$$I_1 = e_1 A, \quad I_2 = e_2 A, \tag{1.5}$$

and

$$\mathfrak{M}_1 = \{f \in \mathfrak{M} \colon f(e_1) = 1\}, \quad \mathfrak{M}_2 = \{f \in \mathfrak{M} \colon f(e_2) = 1\}. \tag{1.6}$$

This result was obtained by Šilov [4], who used analytic functions of several variables of elements of A. Here is presented a similar result for the class of p-normed algebras.

2. A *p-normed algebra* A is a metric algebra complete in the norm $\|x\|$ satisfying

$$\|xy\| \leqslant \|x\|\|y\|, \quad \|\alpha x\| = |\alpha|^p\|x\|,$$