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INTERVAL TOPOLOGY OF AN l-GROUP

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1. Introduction. In this section some definitions from the theory of l-groups are given and the main results are formulated. The terminology is that of [1] (¹).

An *l-group* $G = (G; +, \leqslant)$ is a group $(G; +)$ (written additively, but not necessarily commutative, the neutral element of $(G; +)$ being denoted by 0), which is at the same time a lattice $(G; \leqslant)$ such that $x \leqslant y$ implies $a+x+b \leqslant a+y+b$ for all $a, b, x, y \in G$. The l-group G is ordered if any two elements $x, y \in G$ are comparable (i. e., either $x \leqslant y$ or $y \leqslant x$). G^+ is the set of all $x \in G$, $x \geqslant 0$. The elements $x, y \in G^+$ are called *disjoint* if $x \cap y = 0$. If $G \neq \{0\}$, then there is no greatest and no least element in G. A set $A \subset G$ is *convex* in G if from $a_1, a_2 \in A$, $x \in G$, $a_1 \leqslant x \leqslant a_2$ follows $x \in A$. An *l-ideal* of G is a normal subgroup of $(G; +)$ which is a convex sublattice of $(G; \leqslant)$. Every l-ideal H of G determines a congruence relation on G and the factor l-group G/H. The *direct union* $B = \Sigma A_i$ $(i \in I)$ of l-groups A_i is the set of all functions x on I such that $x(i) \in A$ for each $i \in I$ ($x(i)$ is called the *i-th component* of x), the operations $+, \cap, \cup$ in B being performed component-by-component, i. e., $(x \circ y)(i) = x(i) \circ y(i)$, where $\circ \in \{+, \cap, \cup\}$ ([1], p. 222); in [6] the term “direct product” and the notation ΠA_i is used). Let $\{A_\gamma\}$ $(\gamma \in \Gamma)$ be a class of l-ideals of G; the l-group G is a *cardinal sum* of l-groups A_γ (notation $G = \Sigma + A_\gamma$) if the group $(G; +)$ is a direct sum of its subgroups $A_\gamma(+)$ and if from $a_{\gamma_1} + \ldots + a_{\gamma_n} \geqslant 0$ (where $a_{\gamma_i} \in A_{\gamma_i}$ and $\gamma_i \neq \gamma_j$ if $i \neq j$) follows $a_{\gamma_i} \geqslant 0$ for $i = 1, \ldots, n$. In the case $\Gamma = \{1, 2\}$ we put $\Sigma + A_\gamma = A_1 \dot{+} A_2$ (cf. [2], p. 213). The cardinal sum $A_1 \dot{+} A_2$ is non-trivial if $A_1 \neq \{0\} \neq A_2$.

Let G be an l-group, $G \neq \{0\}$, $a \in G$. The sets $I_1(a) = \{x | x \in G, x \leqslant a\}$, $I_2(a) = \{x | x \in G, x \geqslant a\}$, G will be called *infinite intervals* (of G). The interval topology of G is defined by taking as a subbasis for the closed sets the system of all infinite intervals of G (cf. [2], p. 228). Consider the following conditions:

(¹) [1] always means [1], Chapt. XIV.

(t) G is a topological group in its interval topology,

(h) G is a Hausdorff space in its interval topology,

(o) G is an ordered group.

It is well-known that any topological group is a Hausdorff space, i. e., (t) $\Rightarrow$ (h). It is easy to see that (t) is implied by (o). The following theorems are known:

(A) *The additive l-group of all continuous functions defined on a closed unit interval does not satisfy* (t) (E. S. Northam [4]; this is a solution of the Problem 104 of G. Birkhoff [1], p. 233).

(B) *If every non-empty subset of G^+ contains a minimal element, then* (t) $\Rightarrow$ (o) (Choe [3]).

(C) *If G is Archimedean, then* (t) $\Rightarrow$ (o) (cf. [5]).

(D) *If G is a small lexico-extension of ordered groups, then* (t) $\Rightarrow$ (o) (Conrad [2]; the concept of the small lexico-extension will be defined in section 5).

(E) *If G is an l-subgroup of a direct union of ordered Archimedean groups G_i, then* (h) $\Rightarrow$ (o) (Wolk [6]).

P. Conrad and E. S. Wolk have formulated the following problems:

(P 1) Find an example of a non-ordered l-group that is a topological group in its interval topology ([2], p. 230).

(P 2) It remains an open question whether Theorem (E) can be extended (at least for commutative l-groups) to the case where some or all of the factor groups G_i are non-Archimedean.

We shall give a partial solution of (P 1): it is not possible to construct a commutative non-ordered l-group satisfying (t) (Theorem 4.3). Since each l-group G fulfilling the assumptions of the theorems (A), (B), (C) or (E) is commutative (cf. [1], § 7, § 13), it follows that these theorems are corollaries of the theorem 4.3. Further we prove that the answer to the problem (P 2) is affirmative (Theorem 6. 2) and we give some sufficient conditions for (t) $\Rightarrow$ (o) to be fulfilled in a general (non-commutative) l-group. Theorem 5.3 is a generalization of (D).

2. Lexico-extensions. Let A be an l-ideal of G. The l-group G is called a *lexico-extension* of A (notation $G = \langle A\rangle$; cf. [2], p. 214), if for each $x \in G^+ \setminus A$ and each $a \in A$ the relation $x > a$ holds. The lexico-extension in non-trivial if $\{0\} \neq A \neq G$. In such case, if $x \in G \setminus A$, then either $x > a$ or $x < a$ for all elements $a \in A$ (cf. [2], lemma 9. 1). This implies that G/A is an ordered group.

Let $N(G)$ be the set of all $x \in G$, $x > 0$, such that there exists $y(x) \in G$ satisfying $y(x) > 0$, $y(x) \frown x = 0$. (In such a case clearly $y(x) \in N(G)$ also holds.) If n is a positive integer and $x \in N(G)$, then $nx \frown y = 0$ (cf. [1], Theorem 6); hence $nx \in N(G)$. Let $B(G)$ be the subgroup of $(G; +)$ generated by $N(G)$. Then $B(G)$ is an l-ideal of G and $G = \langle B(G)\rangle$ (cf. [2], § 9).

2.1. Let $a_i, b_j \in G$, $i = 1, \ldots, n$; $j = 1, \ldots, m$,

$$M = \{I_1(a_1), \ldots, I_1(a_n); I_2(b_1), \ldots, I_2(b_m)\}.$$

Let $a, b \in G$. The system M will be called an *s-system* for a, b in G if M covers G (i. e., $(\bigcup I_1(a_i)) \smile (\bigcup I_2(b_i)) = G$) and if none of the intervals of M contains both a and b. M is *minimal* if no proper subsystem of M is an s-system for a, b in G. From the definition of the Hausdorff space it follows immediately (cf. also [2], lemma 6.4) that G satisfies (h) if and only if for any two distinct elements $a, b \in G$ there exists an s-system in G; from each such system M a minimal system $M' \subset M$ can be constructed.

Remark. If there exist finite sets $\{a_i\}$, $\{b_j\}$ such that $G = (\bigcup I_1(a_i)) \smile (\bigcup I_2(b_j))$, then both $\{a_i\}$ and $\{b_j\}$ are non-empty. Indeed, suppose $G = \bigcup I_s(a_i)$ and write $\bigcup a_i = a$; since $G \neq \{0\}$, we can choose $a' \in G$ such that $a' > a$; hence $a' \notin \bigcup I_1(a_i)$, which is a contradiction. In the case $G = \bigcup I_2(b_j)$ the proof is analogous.

2.2. *Let $G = \langle S\rangle$, $a, b \in S$, let M be a minimal s-system for a, b in S. Then all a_i, b_j belong to S.*

Proof. If $i_1, i_2 \in \{1, \ldots, n\}$, $i_1 \neq i_2$, $a_{i_1} - a_{i_2} \notin S$, then the elements a_{i_1}, a_{i_2} are comparable; hence one of the sets $I_1(a_{i_1})$, $I_1(a_{i_2})$ is a subset of the second, which contradicts the minimality of M; this shows that $a_{i_1} - a_{i_2} \in S$. Analogously, $b_{j_1} - b_{j_2} \in S$ if $j_1 \neq j_2$. If $a_i \notin S$, $a_i > 0$, then $a_i > a$, $a_i > b$; hence $a, b \in I_1(a_i)$ which is impossible; therefore $a_i \notin S$ implies $a_i < 0$. Analogously, $b_j \in S$ or $b_j > 0$.

Suppose $a_i \notin S$; write $\bigcap b_j = u$. By the preceding result either u belongs to S or u is greater than any element of S. In either case there exists a $u_1 \in S$, $u_1 < u$. Then $u_1 \notin \bigcup I_1(a_i)$, $u_1 \notin I_2(u) \supset \bigcup I_2(b_j)$; hence M is not an s-system for a, b in G, which is a contradiction. Consequently $a_i \in S$; applying an analogous argument we get $b_j \in S$.

2.3. Theorem. *Let $G = \langle S\rangle$, $S \neq \{0\}$. Then G satisfies* (h) *if and only if S satisfies* (h).

Proof. (a) Suppose S satisfies (h). If $c \in S$, we denote by $J_i(c)$ the set $I_i(c) \frown S$ $(i = 1, 2)$. Let $a, b \in S$, $a \neq b$. Then by 2.1 there exist $a_1, \ldots, a_n$; $b_1, \ldots, b_m$, $n \geqslant 1$, $m \geqslant 1$, such that

$$M_1 = \{J_1(a_1), \ldots, J_1(a_n); J_2(b_1), \ldots, J_2(b_m)\}$$

is an s-system for a, b in S. The system M is now an s-system for a, b in G. (If $x \in G \setminus S$, then either $x > 0$ or $x < 0$; in the former case we have $x \in I_1(a_1)$, in the latter $x \in I_2(b_1)$.) If $u, v \in G$, $u \neq v$, $u - v \in S$, put $a = 0$, $b = u - v$ and let a_i, b_j have the same meaning as above. Write $K_i =$

$= I_1(a_i)+v$, $K'_j = I_2(b_j)+v$. The set of intervals K_i, K'_j $(i = 1, \ldots, n;$ $j = 1, \ldots, m)$ is an s-system for u, v in G. If $u, v \in G$, $u-v \notin S$, then the elements u, v are comparable; we may assume $u < v$. Since there does not exist the greatest and the least element in S (and hence there does not exist the greatest element in $S+u$ and the least element in $S+v$), we can choose elements $u_1, v_1 \in G$ such that $u-u_1 \in S$, $v-v_1 \in S$, $u < u_1$, $v_1 < v$. The set $\{I_2(u_1), I_1(v_1)\}$ is an s-system for u, v in G.

(b) Suppose now that G satisfies (h) and let $a, b \in S$, $a \neq b$. There exists an s-system M for a, b in G; we may suppose that M is minimal. By 2.2, a_i, $b_j \in S$; hence M_1 is an s-system for a, b in S.

3. Condition (P). Let $x, y \in G$; by $x \ll y$ we mean that $0 \leqslant y$ and that $nx < y$ for every positive integer n (cf. [1], p. 225). Consider the following conditions:

(P) If $n > 1$ is a positive integer, $b_1, \ldots, b_n$, $c \in N(G)$, $b_1 \ll b_2+\ldots+$ $+b_n$, $c \ll b_1+b_2+\ldots+b_n$, then $c \ll b_2+\ldots+b_n$.

(v_n) There are elements $b_1, b_2, \ldots, b_n \in N(G)$ such that for any $c \in N(G)$ the relation $c \leqslant b_1+b_2+\ldots+b_n = b$ holds.

3.1. *If* (v_n) *holds, then* $n \geqslant 2$.

Proof. Let $b_1 \in N(G)$; by 1, $y(b_1) \in N(G)$, $y(b_1) \frown b_1 = 0$; then $y(b_1)$ non$\leqslant b_1$.

3.2. *Let* $G \neq \{0\}$ *be covered by the system* M. *Then there exists an element* $v' \in G$, $v' \geqslant 0$, *such that* $G = I_1(v') \smile I_2(0)$.

Proof. Write $u = \bigcap b_j$, $v = (\bigcup a_i) \smile u$; from $I_1(a_i) \subset I_1(v)$, $I_2(b_j) \subset$ $\subset I_2(u)$ follows $G = I_1(v) \smile I_2(u)$. The mapping $x \to x-u$ $(x \in G)$ is an automorphism of the lattice $(G; \leqslant)$, therefore the relation $G = I_1(v) \smile$ $I_2(u)$ implies $G = I_1(v') \smile I_2(0)$, where $v' = v-u \geqslant 0$.

3.3. *If* G *is non-ordered and satisfies* (h), *then there exists a positive integer* n *such that the condition* (v_n) *is fulfilled.*

Proof. Since G is non-ordered, there exist incomparable elements $f, q \in G$. Write $p_1 = p-(p \frown q)$, $q_1 = q-(p \frown q)$; then the elements p_1, q_1 also are incomparable, $p_1 > 0$, $q_1 > 0$ and $p_1 \frown q_1 = 0$; hence $p_1, q_1 \in N(G)$, $N(G) \neq \emptyset$, $B(G) \neq \{0\}$. By 2.3, $B(G)$ satisfies (h) and by 3.2 there exists $v' \in B(G)$ such that $0 \leqslant v'$, $B(G) = \big(I_2(v') \frown B(G)\big) \smile \big(I_1(0) \frown B(G)\big)$. From the definition of $B(G)$ it follows that there are elements $b_1, \ldots, b_n \in N(G)$ such that $v' \leqslant b_1+\ldots+b_n = b$. Let $x \in N(G)$; there exists $x' \in N(G)$ such that $x \frown x' = 0$. Write $z = x-x'$. Since x and x' are disjoint, the elements $0, z$ are incomparable (cf. [1], p. 220, lemma 2); hence $z \in I_1(v')$, $z \leqslant v'$. Consequently $z \leqslant b$, $0 < b$, $x = z \smile 0 \leqslant b$.

3.4. *If* G *fulfills* (P), *then there is no positive integer* n *satisfying* (v_n).

Proof. Let $n \geqslant 2$ (cf. 3.1) be the least positive integer such that (v_n) is safisfied; suppose G fulfills (P). If $c \in N(G)$, then $mc \in N(G)$, $(m+1)b_1 \in$

$\in N(G)$ for any positive integer m; hence $c \ll b$, $b_1 \ll b_2+\ldots+b_n$. From (P) follows $c \leqslant b_2+\ldots+b_n$; therefore (v_{n-1}) holds, which is a contradiction.

3.5. Theorem. *If* G *satisfies* (P), *then* (h) $\Rightarrow$ (o).

The proof follows from 3.3 and 3.4.

Remark. I do not know whether there exists any non-ordered l-group which does not satisfy (P).

4. Commutative l**-groups.** We have

4.1. *Let* $B(G)$ *be commutative. Then* (v_n) *is false for any positive integer* n.

Proof. Suppose that n is the least positive integer such that (v_n) is satisfied. By 3.1, $n > 1$. From $b_1 \in N(G)$ follows $2b_1 \in N(G)$, and hence $b_1 \leqslant b_2+\ldots+b_n$, $b \leqslant 2(b_2+\ldots+b_n) = 2b_2+\ldots+2b_n$. Therefore (v_{n-1}) holds, which contradicts the minimality of n.

4.2. Theorem. *Let* $B(G)$ *be commutative. If* $B(G)$ *satisfies* (h), *then* $B(G) = \{0\}$.

Proof. From 4.1 and 3.3 it follows that G is ordered; hence $N(G) = \emptyset$, $B(G) = \{0\}$.

4.3. Theorem. *If* G *is commutative, then* (h) $\Rightarrow$ (0).

Proof. If G is a commutative l-group satisfying (h), then, by 4.2, $B(G) = \{0\}$; hence $G = G/B(G)$, and by section 2 the factor l-group $G/B(G)$ is ordered.

5. Small lexico-sums. Conrad [2] investigated the small lexico-sums of ordered groups C^1_γ. Under the more general assumption that C^1_γ are l-groups, the definition of this concept is as follows (cf. [2], p. 224):

Let N be the set of all positive integers or $N = \{1, 2, \ldots, k\}$. Let C^1_γ $(\gamma \in \Gamma_1 \neq \emptyset)$ be l-ideals of an l-group G. For every $n \in N$ let C^n be an l-ideal of G such that:

(a) $C^1 = \Sigma + C^1_\gamma$ $(\gamma \in \Gamma_1)$;

(b) if $n+1 \in N$, then $C^n \subset C^{n+1}$, and $\bigcup_{n \in N} C^n = G$;

(c) for $n > 1$, $C^n = \Sigma + C^n_\gamma$ $(\gamma \in \Gamma_n \neq \emptyset)$, where each C^n_γ is a convex subgroup of G and either a non-trivial lexico-extension of a finite cardinal sum of two or more of the components C^{n-1}_β or C^n_γ is equal to one of the C^{n-1}_β.

Under these suppositions G is called a *small lexico-sum* of l-groups C^1_γ.

Remark. The l-groups C^1_γ in the preceding definition are not uniquely determined.

Example. Let G be the additive l-group of all integers with the natural ordering. Put

(a) $\Gamma_1 = \Gamma_2 = \{1\}$, $C^1_1 = C^2_1 = C^1 = C^2 = G$;

(b) $\Gamma_1 = \{1, 2\}$, $\Gamma_2 = \{1\}$, $C^1_1 = C^1_2 = C^1 = \{0\}$, $C^2_1 = C^2 = G$.

5.1. *Let $G \neq \{0\}$ be a small lexico-sum of ordered groups. Then we can choose ordered l-ideals C_γ^1 ($\gamma \in \Gamma_1$) of G such that G is a small lexico-sum of l-ideals C_γ^1 and $C_\gamma^1 \neq \{0\}$ for every $\gamma \in \Gamma_1$.*

This follows from the construction given in the proof of the theorem 6.1, [2].

In the whole section 5 we assume that G is a small lexico-sum of l-groups $C_\gamma^1 \neq \{0\}$ ($\gamma \in \Gamma_1$). Clearly G is a small lexico-sum of l-groups C_γ^n ($\gamma \in \Gamma_n$) for any $n \in N$. If A is a set, we shall denote by card A the cardinality of A.

5.2. *Let C_γ^1 ($\gamma \in \Gamma_1$) be ordered groups. Then G is ordered if and only if* card $\Gamma_n = 1$ *for any* $n \in N$.

Proof. (α) Since $C_\gamma^1 \neq \{0\}$, it follows by induction on n that $C_\gamma^n \neq \{0\}$ for every $n \in N$ and $\gamma \in \Gamma_n$. If there exists an $n \in N$ such that card $\Gamma_n \geqslant 2$, then C^n is a non-trivial cardinal sum; hence C^n is non-ordered; therefore, by (b), G is non-ordered.

(β) If card $\Gamma_n = 1$ for every $n \in N$, then it follows by induction on n that all C_γ^n are ordered groups; hence by (a) and (c) every C^n is ordered; therefore, by (b), G is also ordered.

The theorem (D) can now be formulated as follows:

(D$'$) *Let $C_\gamma^1 \neq \{0\}$ be ordered groups. If G satisfies* (t), *then* card $\Gamma_n = 1$ *for any* $n \in N$.

5.3. *If* $n \in N$, $n > 1$, card $\Gamma_n = 1$, *then* $C^n = \langle C^{n-1} \rangle$.

Proof. Let $n > 1$, card $\Gamma_n = 1$, $\Gamma_n = \{\gamma\}$. Hence, by (c),

$$C^n = \langle B \rangle, \quad B = \Sigma + C_a^{n-1} \quad (a \in \Gamma'_{n-1}),$$

where Γ'_{n-1} is a non-empty finite subset of Γ_{n-1}. Suppose $a_1 \in \Gamma_{n-1} \setminus \Gamma'_{n-1}$ and choose $a_2 \in \Gamma'_{n-1}$, $x_{a_1} \in C_{a_1}^{n-1}$, $x_{a_2} \in C_{a_2}^{n-1}$, $x_{a_i} > 0$ $(i = 1, 2)$. Since x_{a_1}, $x_{a_2} \in C^{n-1} = \Sigma + C_a^{n-1} (a \in \Gamma_{n-1})$, it follows from the definition of the cardinal sum (cf. section 1) that x_{a_1}, x_{a_2} are incomparable and that $x_{a_1} \notin B$. Since $x_{a_1} \in C^n \setminus B$, we get from the definition of the lexico-extension that $x_{a_1} > x_{a_2}$; this is a contradiction. Hence $\Gamma_{n-1} = \Gamma'_{n-1}$, which implies $B = C^{n-1}$.

5.4. *If $G = A \dot{+} B$, $A \neq \{0\} \neq B$, then G cannot be covered by a finite system of infinite intervals $I^i \neq G$.*

Proof. Suppose G is covered by the system M. By 3.2, $G = I_1(v') \cup I_2(0)$. Let $v' = a + b'$, $a \in A$, $b' \in B$. There exists $a_1 \in A$ such that $a_1 > a \cup 0$. Let $b \in B$, $b > 0$. Then $a_1 - b$ non $\leqslant a + b' = v'$, $a_1 - b$ non $\geqslant 0$; hence $a_1 - b \notin I_1(v')$, $a_1 - b \notin I_2(0)$, which is a contradiction.

Corollary (cf. [6], Theorem 2, [2], lemma 6.5). *If* $G = A \dot{+} B$, $A \neq \{0\} \neq B$, *then G does not satisfy* (h).

The following theorem is a generalization of (D$'$):

5.5. Theorem. *If G satisfies* (h), *then* card $\Gamma_n = 1$ *for any* $n \in N$.

Proof. Let $n \in N$, card $\Gamma_n \geqslant 2$. Since G is a small lexico-sum of l-groups C_γ^n ($\gamma \in \Gamma_n$), we may suppose $n = 1$. Then C^1 is non-ordered; choose $a, b \in C^1$, $a \neq b$. Suppose that G satisfies (h) and let M be an s-system for a, b in G. Let m be the least positive integer such that all a_i, b_j belong to C^m. Therefore the sets

$$I_1(a_i) \cap C^m = K_i, \quad I_2(b_j) \cap C^m = K'_j$$

are infinite intervals in C^m (distinct from C^m) and

$$(\bigcup K_i) \cup (\bigcup K'_j) = C^m. \tag{1}$$

From (1) and 5.4 it follows that C^m is not a non-trivial cardinal sum; hence card $\Gamma_m = 1$ and $m > 1$. Therefore, by 5.3, $C^m = \langle C^{m-1} \rangle$. Since, by (b), $a, b \in C^{m-1}$, it follows from 2.2 that all a_i, b_j belong to C^{m-1}, which contradicts the minimality of m.

6. Direct unions of ordered groups. We have

6.1. Theorem. *Let G be an l-subgroup of a direct union ΣG_i ($i \in I$) of l-groups G_i. If there exists $g \in G$ and $j \in I$ such that*

(a) $g \in N(G)$, $g(j) > 0$,

(b) *G_j is ordered,*

then (h) $\Rightarrow$ (o).

Proof. Suppose G is a non-ordered l-group fulfilling (h) and satisfying the conditions of the assertion. From 3.3 it follows that there exists a positive integer n such that (v$_n$) holds. From $g \in N(G)$ we get $g \leqslant b$; hence $g(j) \leqslant b(j) = b_1(j) + \ldots + b_n(j)$. Let $b_k(j)$ be the greatest element of the set $b_i(j)$ $(i = 1, \ldots, n)$; since $g(j) > 0$, we have $b_k(j) > 0$. Therefore $b(j) \leqslant nb_k(j) < 2nb_k(j)$; hence $2nb_k$ non $\leqslant b$ contradicting (v$_n$) (since $2nb_k \in N(G)$).

6.2. Theorem. *If G is an l-subgroup of a direct union of ordered groups G_i, then* (h) $\Rightarrow$ (o).

Proof. Let G be an l-subgroup of a direct union of ordered groups G_i ($i \in I$). If G is not ordered, then there exist incomparable elements $g_1, g_2 \in G$ such that $g_1 \cap g_2 = 0$; hence $g_1 \in N(G)$. Clearly there exists $j \in I$ such that $g_1(j) > 0$. Therefore, by 6.1, G does not satisfy condition (h).

Remark. Since any commutative l-group is a subdirect union of ordered groups (cf. [1], Theorem 13), theorem 6.2 yields an alternative proof of theorem 4.3.

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A FIXED POINT FREE MAPPING OF A CONNECTED PLANE SET

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J. L. Kelley pointed out in [2] that the question of the possible existence of a fixed point free, periodic, continuous mapping of a connected and non-cutting subset of the plane into itself was still unanswered. Contained in the proof of a special case of his Conjecture A is the suggestion that such a mapping might possibly be of period 2. Such a mapping of period 2 does, in fact, exist.

THEOREM. *If M is a plane pseudo-arc and K is a composant of M, then there exists a fixed point free homeomorphism of period 2 of the connected point set $M-K$ onto itself. Furthermore, if K contains a point accessible from the complement of M, then the complement of $M-K$ is strongly (= continuum-wise) connected.*

Proof. The construction of a pseudo-arc M from p to q given in Moise's thesis [3] as the intersection of $\overline{C}_n^*$ where, for each n, C_n is a finite collection of open plane sets whose closures form a chain of a specific type, may be carried out in a fashion symmetrical with respect to the two points p and q as indicated in his Figure 1. In particular, if for each n, $C_n = \{C_{in}\}$ $(i = 1, \ldots, i_n)$ is the natural ordering of C_n from p to q and $C_n = \{C'_{in}\}$ $(i = 1, \ldots, i_n)$ is the reverse ordering (i. e., the natural ordering from q to p), then $\overline{C}_{jm}$ intersects $\overline{C}_{kn}$ if and only if $\overline{C}'_{jm}$ intersects $\overline{C}'_{kn}$.

Now let x be a point of M and for each n, let i be the smallest integer such that $\overline{C}_{in}$ contains x. So $x = \bigcap \overline{C}_{in}$. Define the function f on M so that $f(x) = \bigcap \overline{C}'_{in}$. Obviously $f(p) = q$, $f(q) = p$ and $f(x) = x$ if and only if $x = \bigcap \overline{C}_{in} = \bigcap \overline{C}'_{in}$. Clearly f is continuous, 1-1, and of period 2 with only one fixed point, namely, the point o determined by the middle elements of the chains.

Let o' be a point of the composant K of M. Since M is homogeneous [1], there is a homeomorphism h of M onto M such that $h(o) = o'$. The homeomorphism hfh^{-1} leaves o' fixed and hence $hfh^{-1}(K) = K$. So hfh^{-1} is of period 2 on $M-K$ and has no fixed point.